American Journal of Operations Research, **2013, 3, 380-386**
http://dx.doi.org/10.4236/ajor.2013.33035 Published Online May 2013 (http://www.scirp.org/journal/ajor)

Computing the Distribution Function of the Number of Renewals

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Received April 20, 2013; revised May 25, 2013; accepted June 15, 2013

ABSTRACT

The method of Laplace transforms is used to find the distribution function, mean, and variance of the number of renewals of a renewal process whose inter-arrival time distribution has a rational Laplace transform. Where the Laplace transform is not rational, we use the Padé approximation method. We apply our method to certain examples and the results are compared to those reported by other researchers.

Keywords: Number of Renewals; Rational Approximation; Padé

1. Introduction

Renewal and reliability theories are powerful modeling tools in many applications, considering, for example, a series of renewals during a time interval $(0,t]$ with the inter-renewal times having certain distributions. The interest, usually, is to find the distribution or the moments of the number of renewals during the interval $(0,t]$. Although the theoretical aspects of renewal theory have been discussed in various books such as Cox [1] and Feller [2,3], not much seem to have been done in applying the theoretical results to practice for lack of availability of easily computable results. Using the cubic splining algorithm to compute the recursively-defined convolution integrals that appear in renewal theory, Baxter *et al*. [4] is able to construct some tables for the mean and variance of the number of renewals for different inter-renewal time distributions with varying parameters. A straightforward method to compute the convolution integral is to use the Laplace transform (LT) method. Using the root-finding method, Chaudhry [5] develops a unified method to compute the mean and variance of the number of renewals. Though the means and variances are useful for many applications, this paper goes a step further and deals with computing the distribution of the number of renewals from which one can get more information than from the mean and variance.

There are several methods that can be used for the numerical inversion of generating functions (GFs); see the excellent reviews on the numerical inversion (of GFs as well as of LTs) by Abate *et al*. [6-9]. Some researchers have been critical of using the method of roots, see, e.g. Cox and Smith [10] and Kleinrock [11]. In view of this, this paper serves another useful purpose and shows that with the availability of high precision in current software packages, the roots can be found successfully and used to invert the transforms. Further, using the roots method, the results can be first given in an analytically explicit form and then used to find the final results.

2. Theory

2.1. Problem Description and Method of Laplace Transforms

Let $X_1, X_2, \cdots$ be a governing sequence of independently identically distributed (i.i.d.) inter-renewal times for the renewal process $\{N(t), t \ge 0\}$, where $N(t)$ denotes the number of renewals in $(0,t]$. Let X represent the generic inter-renewal time with cumulative distribution function (CDF)

$$F(t) = \begin{cases} P(X \le t), & t \ge 0 \\ 0, & t < 0 \end{cases}$$

and probability density function (pdf) $f(t)$.

Define the probability mass function (pmf) $P_n(t) = P(N(t) = n),\ n = 0,1,2,\cdots$. Let $S_n = \sum_{i=1}^{n} X_i, n \ge 1, S_0 = 0$. For $n \ge 0$, S_n represents the time when the n-th renewal occurs. Define $F_n(t) = P(S_n \le t)$, with $F_0(t) = 1$. Then

$$F_n(t) = F^{*n}(t), n = 0, 1, 2, \cdots$$

is the n-fold convolution of $F(t)$ with itself, and (see Chaudhry and Templeton [12])

$$P_n(t) = F_n(t) - F_{n+1}(t),\ t \geq 0, n \geq 0 \tag{1}$$

Let $\tilde{f}(s)$ and $\tilde{p}_n(s)$ be the Laplace-Stieltjes transform (LST) of $F(t)$ and $P_n(t)$, respectively, defined by $\tilde{f}(s) = \int_0^{+\infty} \mathrm{e}^{-st} \mathrm{d}F(t)$ and $\tilde{p}_n(s) = \int_0^{+\infty} \mathrm{e}^{-st} \mathrm{d}P_n(t)$. Taking the LST on both sides of Equation (1), we get

$$\tilde{p}_n(s) = \tilde{f}^n(s) - \tilde{f}^{n+1}(s) = \tilde{f}^n(s)\left(1 - \tilde{f}(s)\right) \tag{2}$$

and hence

$$P_n(t) = \mathcal{L}^{-1}\left[\frac{\tilde{p}_n(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{\tilde{f}^n(s)\left(1 - \tilde{f}(s)\right)}{s}\right]. \tag{3}$$

2.2. Inversion of Laplace Transforms

2.2.1. $\tilde{f}(s)$ Is a Rational Function

The inverse LT Equation (3) can be obtained analytically using partial fractions. Let

$$\tilde{f}(s) = \frac{N(s)}{D(s)}$$

where $D(s)$ and $N(s)$ are polynomials of degree k and at most k, respectively. Then by Equation (2), we have

$$\frac{\tilde{p}_n(s)}{s} = \frac{N^n(s)\left[D(s) - N(s)\right]}{sD^{n+1}(s)} \tag{4}$$

which is a rational function.

Without loss of generality, we assume that the equation $D(s) = 0$ has k distinct roots $s_1, s_2, \cdots, s_k$. Since $\tilde{f}(0) = 1$, $N(s)$ and $D(s)$ have the same constant terms, $D^{n+1}(s) = (s - s_1)^{n+1}(s - s_2)^{n+1} \cdots (s - s_k)^{n+1}$. And Equation (4) can be expressed in partial fractions as

$$\frac{\tilde{p}_n(s)}{s} = \sum_{j=1}^{k}\sum_{i=1}^{n+1} \frac{A_{j,i}}{\left(s - s_j\right)^i}$$

where the constant coefficient $A_{j,i}$ is given by

$$A_{j,i} = \frac{1}{(i-1)!}\frac{\mathrm{d}^{(i-1)}}{\mathrm{d}s^{i-1}}\left[\frac{\tilde{p}_n(s)}{s}\left(s - s_j\right)^i\right]$$

The final inversion can be written as

$$P_n(t) = \sum_{j=1}^{k}\sum_{i=1}^{n+1} \frac{A_{j,i}}{(i-1)!} t^{i-1} \mathrm{e}^{s_j t} \tag{5}$$

The case when $D(s) = 0$ has repeated roots can be dealt with similarly.

2.2.2. $\tilde{f}(s)$ Is Not a Rational Function

We use the Padé approximation method. Assume

$$\tilde{f}(s) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!} M_n s^n$$

where

$$M_n = \int_0^{+\infty} x^n \mathrm{d}F(x)$$

is the n-th moment of the inter-renewal time. We can find a rational approximation function

$$\hat{f}(s) = \frac{N(s)}{D(s)} = \frac{\sum_{n=0}^{K} b_n s^n}{\sum_{n=0}^{L} a_n s^n}$$

where $N(s)$ and $D(s)$ are polynomials of degree K and L, respectively with undetermined coefficients b_n and a_n, such that the first $K + L$ moments of $\hat{f}(s)$ are equal to those of $\tilde{f}(s)$. We denote the above Padé approximation as $[K/L]$ (Baker *et al*. [13]). In practice, a_0 is set to one and K and L are chosen by trial and error. Equating the moments and formulating the simultaneous equations, the coefficients b_n and a_n are uniquely determined (Baker *et al*. [13], Harris [14]).

Use of continued fractions is another way to obtain an approximate rational function $\tilde{f}(s)$. The method is, in fact, a special case of the Padé method (Baker *et al*. [13]).

There are times when using Padé method directly is not possible or does not give the desired results. For example, the Pareto distribution has an infinite second moment, and directly equating the moments cannot be done. For the lognormal distribution, the waveform of the Padé approximated distribution function shows quite large errors. The solution is to have a two-step approximation. The first step is to use line segments to approximate the distribution function. In the second step, the Padé method is used to generate a rational LT. By adjusting the parameters in these two steps, $\tilde{f}(s)$ can be obtained with the desired properties. We will illustrate this technique in the examples.

2.3. Verification of the Distribution

The expected number of renewals of $N(t)$ in $(0,1]$ is given by

$$E\left[N(t)\right] = \sum_{n=0}^{+\infty} nP_n(t)$$

and its variance by

$$V\left[N(t)\right] = \sum_{n=0}^{+\infty} n^2 P_n(t) - E\left[N(t)\right]^2$$

In all our examples, the $P_n(t)$ obtained in Equation (5) is used to compute $E\left[N(t)\right]$ and $V\left[N(t)\right]$, and are checked against the results obtained by (Chaudhry [5]) using

$$E\left[N(t)\right]=\mathcal{L}^{-1}\left[\frac{\tilde f(s)}{s\left(1-\tilde f(s)\right)}\right]$$

and

$$V\left[N(t)\right]=\mathcal{L}^{-1}\left[\frac{\tilde f(s)\left(1+\tilde f(s)\right)}{s\left(1-\tilde f(s)\right)^2}\right].$$

In addition, we also match the resulting mean and variance with those of Baxter *et al*. [4].

3. Examples

3.1. Erlang Distribution

We first consider the simple case where $f(t)$ follows the Erlang distribution with the shape parameter equal to 2, and we can get exact analytical results.

The pdf and CDF of $f(t)$ are given, respectively, by

$$f(t)=\lambda^2 t\mathrm{e}^{-\lambda t}$$

and

$$F(t)=1-\mathrm{e}^{-\lambda t}\left(1+\lambda t\right)$$

Thus,

$$\tilde f(s)=\frac{\lambda^2}{(s+\lambda)^2}.$$

We get

$$\tilde p_n(s)=\frac{\lambda^{2n}s^2}{(s+\lambda)^{2n+2}}+\frac{2\lambda^{2n+1}s}{(s+\lambda)^{2n+2}}$$

$$P_n(t)=\frac{\mathrm{e}^{-\lambda t}(\lambda t)^{2n}}{(2n+1)!}\left(\lambda t+(2n+1)\right),$$

$$E\left[N(t)\right]=\mathcal{L}^{-1}\left[\frac{\lambda^2}{s^2(s+2\lambda)}\right]=-\frac{1}{4}+\frac{1}{2}\lambda t+\frac{1}{4}\mathrm{e}^{-2\lambda t},$$

and

$$V\left[N(t)\right]=\mathcal{L}^{-1}\left[\frac{\lambda^2\left(s^2+2\lambda s+2\lambda^2\right)}{s^2(s+2\lambda)^2}\right]$$

$$=\frac{1}{16}+\frac{1}{4}\lambda t-\frac{1}{2}\mathrm{e}^{-2\lambda t}\lambda t-\frac{1}{16}\mathrm{e}^{-4\lambda t}.$$

It may be remarked that the analytic expression for the mean has been incorrectly reported by Parzen [15, p. 177].

3.2. Mixed Generalized Erlang Distribution

The pdf and LST of a Mixed Generalized Erlang (MGE) distribution are given, respectively, by

$$f(t)=\sum_{j=1}^{k}c_j\frac{\lambda^j t^{j-1}\mathrm{e}^{-\lambda t}}{(j-1)!},\ t\ge 0$$

and

$$\tilde f(s)=\sum_{j=1}^{k}c_j\left[\frac{\lambda}{s+\lambda}\right]^j.$$

with $\sum_{j=1}^{k}c_j=1$. We consider the case of MGE with $c_1=c_5=0.25, c_{10}=0.5$, and $\lambda=1$. The $P_n(t)$, mean and variance are obtained by the LT method discussed in Section 2.2. **Table 1** shows the results for several values of t. They are checked against the known results obtained without using roots by Chaudhry [5].

The graph in **Figure 1** shows that the $E\left[N(t)\right]$ calculated from the obtained distribution functions asymptotically converges to the line

$$M(t)=\frac{1}{M_1}t+\frac{M_2-2M_1^2}{2M_1^2}+o(1) \tag{6}$$

derived in [16], where M_1 and M_2 are the first and

Table 1. $P_n(t)$ for MGE distribution with $c_1=c_5=0.25, c_{10}=0.5$, and $\lambda=1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Chaudhry mean	var	Chaudhry var
1.0	0.8411	0.1424	0.0153	0.0012	0.0001	…	0.1769	0.1769	0.1843	0.1843
2.0	0.7706	0.1902	0.0339	0.0047	0.0005	…	0.2744	0.2744	0.3025	0.3025
3.0	0.7157	0.2236	0.0501	0.0090	0.0014	…	0.3574	0.3574	0.4047	0.4047
5.0	0.5959	0.2907	0.0875	0.0208	0.0042	…	0.5498	0.5948	0.6183	0.6183
10.0	0.2363	0.4248	0.2235	0.0822	0.0248	…	1.2622	1.2622	1.1013	1.1013
50.0	0.0000	0.0000	0.0001	0.0056	0.0418	…	7.4379	7.4379	4.0784	4.0784

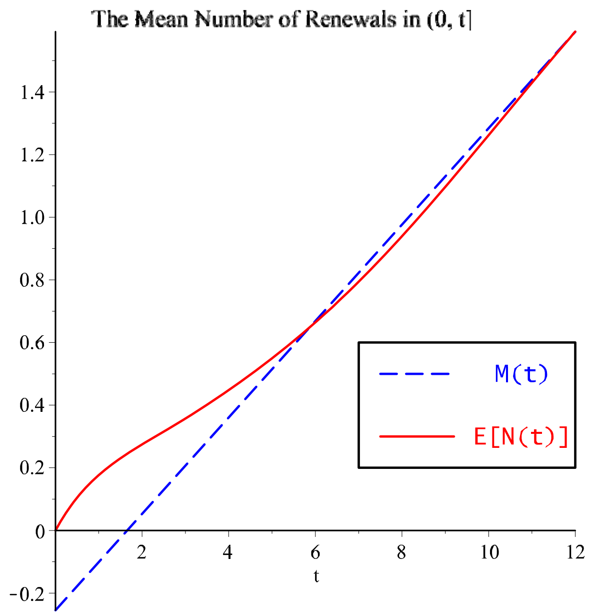

Figure 1. $E\left[N(t)\right]$ **asymptotically converges to the line** $M(t)$.

second moments of $f(t)$. As expected, one can see from this graph that the results given in Equation (6) are good only for large *t*.

3.3. Matrix Exponential Distribution

Consider the matrix exponential and non-phase-type distribution (see Bladt [17]) given below. The pdf and its LT are, respectively,

$$f(t)=\left(1+\frac{1}{4\pi^2}\right)\left(1-\cos(2\pi t)\right)e^{-t}$$

and

$$\tilde{f}(s)=\frac{4\pi^2+1}{(s+1)\left((s+1)^2+4\pi^2\right)}$$

Using partial fractions for transform inversion, we obtain $P_n(t)$ for some selected t values. The results are listed in **Table 2**.

3.4. Gamma Distribution

The pdf and LST of the Gamma distribution are given, respectively, by

$$f(t)=\frac{t^{\alpha-1}e^{\frac{-t}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$

and

$$\tilde{f}(s)=\frac{1}{(1+\beta s)^{\alpha}}$$

where α and β are the shape and scale parameters, respectively. In this example, we set $\beta=1$ and $\alpha=0.55$. The Padé [4/5] approximation of $\tilde{f}(s)$ is given by

$$\hat{f}(s)=\frac{1+1.97778s+1.27938s^2+0.29852s^3+0.01804s^4}{1+2.52778s+2.24340s^2+0.81724s^3+0.10556s^4+0.00232s^5}$$

The $P_n(t)$ are obtained and some selected pmf's, means, and variances are tabulated together with those from Baxter *et al*. [4] in **Table 3**.

3.5. Weibull Distribution

The pdf

$$f(t)=\frac{\alpha\left(\frac{t}{\beta}\right)^{\alpha-1}e^{-\left(\frac{t}{\beta}\right)^{\alpha}}}{\beta},\quad t>0$$

and the CDF

$$F(t)=1-e^{-\left(\frac{t}{\beta}\right)^{\alpha}},\quad t>0$$

of the Weibull distribution do not have the closed-form LT. However, the moments of the Weibull distribution can be obtained from its CDF to use in the Padé method. For $\alpha=3$ and $\beta=1$, we equate moments up to the 8-th moment in the Padé [2/6] approximation, and get

$$\hat{f}(s)=\frac{1-0.0834s+0.01200s^2}{1+0.88464s+0.35058s^2+0.08043s^3+0.01141s^4+0.00097s^5+0.00004s^6}$$

Table 2. $P_n(t)$ for the non-phase-type distribution discussed in Section 3.3.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	var
1.0	0.3679	0.3915	0.2154	0.0242	0.0009	…	0.8988	0.6785
2.0	0.1353	0.2881	0.3035	0.1693	0.0832	…	1.8411	1.5077
3.0	0.0498	0.1590	0.2475	0.2322	0.1779	…	2.7919	2.3400
5.0	0.0067	0.0359	0.0919	0.1523	0.1963	…	4.6973	3.9929
10.0	0.0001	0.0005	0.0025	0.0084	0.0215	…	9.4619	8.1164

Table 3. $P_n(t)$ for the Gamma distribution with $\alpha = 0.55$ and $\beta = 1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Baxter mean	var	Baxter var
0.10	0.6871	0.2385	0.0602	0.0119	0.0019	…	0.4040	0.3933	0.4623	0.4485
0.40	0.4071	0.3088	0.1677	0.0743	0.0284	…	1.0545	1.0550	1.3970	1.3901
1.25	0.1291	0.1951	0.2050	0.1730	0.1249	…	2.6663	2.6653	4.0487	4.0441

Some selected results are listed in **Table 4** together with Baxter's (see Baxter *et al*. [4]). It is noted that when *t* becomes large, our resulting means and variances match those of Chaudhry [5].

3.6. Truncated Normal Distribution

The pdf of the truncated normal distribution is given by

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, \quad t > 0$$

where $a = 1 - \Phi\left(\frac{-\mu}{\sigma}\right)$, with $\Phi(*)$ being the standard normal distribution function. By the Padé [5/6] approximation

$$\hat{f}(s) = \frac{1 - 0.77069s + 0.31300s^2 + 0.07039s^3 + 0.00886s^4 + 0.00048s^5}{1 + 1.56875s + 1.06454s^2 + 0.40145s^3 + 0.08908s^4 + 0.01108s^5 + 0.00061s^6}$$

we are able to obtain $P_n(t)$. Some results and comparisons are shown in **Table 5**.

3.7. Inverse Gaussian Distribution

The pdf for the inverse Guassian distribution is

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} e^{-\frac{\lambda(t-\mu)^2}{2\mu^2 t}}, \quad t \ge 0.$$

Setting $\mu = 0.75$ and $\lambda = 0.5625$, its LST is given by

$$\tilde{f}(s) = e^{0.75 - 0.75(1+2s)^{0.5}}$$

where λ and μ can also be expressed in terms of the shape and scale parameters $\phi = \frac{\lambda}{\mu}$ and $\gamma = \frac{\mu^2}{\lambda}$, respectively, (see Baxter *et al*. [4]). By the Padé [4/7] approximation, we get

$$\hat{f}(s) = \frac{1 - 4.95748s + 9.70859s^2 + 6.32864s^3 + 1.57069s^4}{1 + 5.70748s + 12.33295s^2 + 12.55938s^3 + 6.10460s^4 + 1.27288s^5 + 0.09280s^6 + 0.00385s^7}$$

Some results are shown in **Table 6** along with the results of Baxter *et al*. [4].

3.8. Lognormal Distribution

The pdf

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\log(t)-\mu)^2}{2\sigma^2}}, \quad t \ge 0$$

and CDF

$$F(t) = \Phi\left(\frac{\ln t - \mu}{\sigma}\right), \quad t > 0$$

of the lognormal distribution have no closed-form LT. Using the Padé approximation method directly on the lognormal distribution does not lead to the desired results. Our solution is a two-step approximation. In the first step, we sample uniformly N points of $F(t)$, and then connect the adjacent points to form N line segments as the first approximation of the lognormal distribution. The

Table 4. $P_n(t)$ **for the Weibull distribution with** $\alpha = 3$ **and** $\beta = 1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Baxter mean	var	Baxter var
0.25	0.9841	0.0159	0.0000			…	0.0159	0.0156	0.0156	0.0154
0.60	0.8070	0.1908	0.0022	0.0000		…	0.1953	0.1965	0.1616	0.1624
1.00	0.3664	0.5939	0.0391	0.0005	0.0000	…	0.6738	0.6724	0.3012	0.3018

Table 5. $P_n(t)$ **for the Truncated Normal distribution with** $\mu = 0$ **and** $\sigma = 1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Baxter mean	var	Baxter var
0.15	0.8807	0.1121	0.0068	0.0003	0.0000	…	0.1267	0.1267	0.1261	0.1261
0.45	0.6527	0.2849	0.0548	0.0068	0.0006	…	0.4178	0.4179	0.4023	0.4024
1.25	0.2113	0.4002	0.2567	0.0981	0.0268	…	1.3507	1.3508	1.0997	1.1007

Table 6. $P_n(t)$ **for the Inverse Gaussian distribution with** $\lambda = 0.5625$ **and** $\mu = 0.75$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Baxter mean	var	Baxter var
0.25	0.7445	0.2442	0.0112	0.0001	0.0000	…	0.2669	0.2715	0.2188	0.2200
0.70	0.3390	0.4042	0.2062	0.0457	0.0046	…	0.9736	0.9739	0.7732	0.7718
1.25	0.1623	0.2869	0.2867	0.1762	0.0683	…	1.7635	1.7638	1.5294	1.5293

Table 7. $P_n(t)$ **for the Lognormal distribution with** $\mu = 0$ **and** $\sigma = 1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	…	mean	Baxter mean	var	Baxter var
0.10	0.9855	0.0145	0.0000			…	0.0145	0.0107	0.0143	0.0106
0.40	0.8226	0.1698	0.0074	0.0001	0.0000	…	0.1851	0.1867	0.1664	0.1658
1.25	0.4113	0.4052	0.1532	0.0274	0.0027	…	0.8056	0.8043	0.6636	0.6616

uniform sampling step size used is $h = \frac{t_N}{N}$, where t_N is chosen arbitrarily such that $F(t_N)$ is close to 1. In addition, the last line segment is modified to reach 1 at t_N. The resulting first approximation function can be written as

$$\sum_{i=1}^{N}\left[\frac{F(t_i)-F(t_{i-1})}{h}(t-t_{i-1})+F(t_{i-1})\right]\left[u(t-t_{i-1})-u(t-t_i)\right],$$

where $u(t-t_i)$, $i = 1,2,\cdots,N$ are unit step functions. In this example, μ is set to 0 and σ^2 to 1. Accordingly, we set $N = 100$ and $t_N = 10$.

In the second step, the LST of the first approximation function is obtained and expanded as a Taylor series. Using the Padé [7/8] method, we have the second approximation of $\tilde{f}(s)$ to be

$$\hat{f}(s) = \frac{1+2.7966s+4.7293s^2+4.0493s^3+2.8807s^4+0.9596s^5+0.3004s^6+0.0036s^7}{1+4.3795s+8.9689s^2+11.2718s^3+9.5600s^4+5.6388s^5+2.2756s^6+0.5795s^7+0.0721s^8}$$

Table 7 lists selected $P_n(t)$, means and variances along with the results of Baxter *et al*. [4].

4. Conclusions

Using Laplace transforms, it is shown that computing the distribution of the number of renewals is straightforward when the LT of the inter-renewal time distribution is rational. For inter-renewal time distributions having non-rational LTs, the Padé method provides good approximations. For the case where the inter-renewal time distributions do not have Laplace transforms, we provide the line segment approximation for the distribution function and apply the Padé method thereafter.

The proposed numerical method is not limited to compute the distribution of number of renewals. It can be applied to other similar processes such as alternating renewal process, the superposition of renewal processes, and cumulative processes, for which Laplace transforms are used to solve such problems. Further, the method discussed here can be applied to more complex models such as bulk-renewal processes both in discrete- and continuous-times once the analytic results are known.

5. Acknowledgements

The first author acknowledges with thanks for the partial support he received through NSERC #17287 for carrying out this research.

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