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Trigonometric Approximation of Signals (Functions) Belonging to the $Lip(\xi(t), r), (r > 1)$-Class by $(E, q)(q > 0)$-Means of the Conjugate Series of Its Fourier Series*

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ABSTRACT

Various investigators such as Khan ([1-4]), Khan and Ram [5], Chandra [6,7], Leindler [8], Mishra *et al*. [9], Mishra [10], Mittal *et al*. [11], Mittal, Rhoades and Mishra [12], Mittal and Mishra [13], Rhoades *et al*. [14] have determined the degree of approximation of 2π-periodic signals (functions) belonging to various classes $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$ of functions through trigonometric Fourier approximation (TFA) using different summability matrices with monotone rows. Recently, Mittal *et al*. [15], Mishra and Mishra [16], Mishra [17] have obtained the degree of approximation of signals belonging to $Lip(\alpha, r)$-class by general summability matrix, which generalizes the results of Leindler [8] and some of the results of Chandra [7] by dropping monotonicity on the elements of the matrix rows (that is, weakening the conditions on the filter, we improve the quality of digital filter). In this paper, a theorem concerning the degree of approximation of the conjugate of a signal (function) f belonging to $Lip(\xi(t), r)$ class by (E, q) summability of conjugate series of its Fourier series has been established which in turn generalizes the results of Chandra [7] and Shukla [18].

Keywords: Signals; Conjugate Fourier Series; Trigonometric Fourier Approximation; Degree of Approximation; $Lip(\xi(t), r)$-Class; (E, q) Summability

1. Introduction

The theory of approximation is a very extensive field and the study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. Broadly speaking, Signals are treated as functions of one variable and images are represented by functions of two variables. The study of these concepts is directly related to the emerging area of information technology. Khan [1-4] and Mittal, Rhoades and Mishra [12] have initiated the studies of error estimates $E_n(f)$ through trigonometric Fourier approximation (TFA) using different summability matrices. Chandra [7] has studied the degree of approximation of a signal (function) belonging to Lip α-class by (E,q) means, $q > 0$.

Generalizing the result of Chandra [7], very interesting result has been proved by Shukla [18] for the signals (functions) of $Lip(\alpha, r)$-class through trigonometric Fourier approximation by applying (E,q) $(q > 0)$ summa-

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bility matrix.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its partial sums $\{s_n\}$.

The (E,q) transform is defined as the n^{th} partial sum of (E,q) summability and we denote it by E_n^q.

If

$$E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s \quad \text{as } n \to \infty, \quad (1.1)$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E,q) summable to a definite number "s" [19].

A signal (function) $f(x) \in Lip\alpha$ if

$$f(x+t) - f(x) = O(|t^\alpha|) \quad \text{for } 0 < \alpha \le 1,\ t > 0 \quad (1.2)$$

and $f(x) \in Lip(\alpha, r)$, for $0 \le x \le 2\pi$ [1], if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{1/r} = O(|t|^\alpha), \quad (1.3)$$

$0 < \alpha \le 1, r \ge 1, t > 0.$

Given a positive increasing function $\xi(t)$, $f(x) \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{1/r} = O(\xi(t)), \quad (1.4)$$

$r \ge 1, t > 0.$

We observe that

$$Lip(\xi(t), r) \xrightarrow{\xi(t)=t^\alpha} Lip(\alpha, r) \xrightarrow{r\to\infty} Lip\alpha, \quad (1.5)$$

for $0 < \alpha \le 1, r \ge 1, t > 0.$

The L_∞-norm of a signal $f: R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$.

The L_r-norm of a signal is defined by

$$\|f\|_r = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^r \, dx \right)^{1/r}, \quad 1 \le r < \infty. \quad (1.6)$$

The degree of approximation of a function $f: R \to R$ by trigonometric polynomial t_n of order "n" under sup norm $\| \ \|_\infty$ is defined by Zygmund [20].

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_n \|f(x) - t_n(f;x)\|_r \quad (1.7)$$

in terms of n, where $t_n(f;x)$ is a trigonometric polynomials of order "n".

This method of approximation is called Trigonometric Fourier Approximation (TFA) [12].

Let $f(x)$ be a 2π-periodic signal (function) and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (1.8)$$

with n^{th} partial sum $s_n(f;x)$ called trigonometric polynomial of degree (order) n of the first $(n+1)$ terms of the Fourier series of f.

The conjugate series of Fourier series (1.8) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \quad (1.9)$$

We note that E_n^q is also trigonometric polynomial of degree (or order) "n".

We use the following notations throughout this paper

$$\psi_x(t) = \psi(t) = f(x+t) - f(x-t),$$

$$\tilde{G}_n(t) = \frac{1}{2\pi(1+q)^n \sin\frac{t}{2}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \cos\left(k + \frac{1}{2}\right)t.$$

2. Known Results

Chandra [7] has studied the degree of approximation to a function $f \in Lip\alpha$ $(0 < \alpha \le 1)$ by $(E,q), q > 0$ of Fourier series (1.8) by proving the following theorem. He proved:

Theorem 2.1 The degree of approximation of a periodic function $f(x)$ with period 2π and belonging to the class $Lip\alpha$ by Euler's mean of its Fourier series is given by

$$\max |f(x) - T_n^q(x)| = O(n^{-\alpha/2}) \quad (2.1)$$

where $T_n^q(x)$ is the n^{th} Euler mean of order $q > 0$ of the sequence $\{s_n\}$ of partial sums of the Fourier series (1.8) of the function f at a point x in $[-\pi, \pi]$.

Shukla [18] improved Theorem 2.1 by extending to a function $f \in Lip(\alpha, r)$ by (E,q) matrix means of the conjugate series (1.9) of its Fourier series (1.8). He proved:

Theorem 2.2 Let $f \in Lip(\alpha, r)$, $0 < \alpha \le 1$, $r \ge 1$ be a 2π-periodic and Lebesgue integrable function of "t" in the interval $[-\pi, \pi]$. If

$$\left\{ \int_0^t u^{(1-\alpha)} |\psi(u)|^r \, du \right\}^{1/r} = O(t) \quad (2.2)$$

and

$$\left\{ \int_1^\pi u^{(-\delta-\alpha)} |\psi(u)|^r \, du \right\}^{1/r} = O(t^{-\alpha-1}), \quad (2.3)$$

where δ is an arbitrary number such that $s(2-\delta) > 1$, s being conjugate to $r \ge 1$ with $r^{-1} + s^{-1} = 1$, then the degree of approximation of the conjugate to a function

$f \in Lip(\alpha, r)$, by (E, q) means, $q > 0$, of the conjugate series (1.9) of its Fourier series (1.8) will be given by

$$\max\left\|\tilde{f}(x) - \tilde{E}_n^q(x)\right\| = O\left\{n^{-\frac{\alpha}{2}+\frac{1}{2r}}\right\}, \tag{2.4}$$

where $\tilde{E}_n^q(x)$ is n^{th} (E, q) mean of the sequence $\tilde{s}_n(x)$ of partial sums of the conjugate series (1.9) of the Fourier series (1.8) of the function f at every point x in $[-\pi, \pi]$ at which

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\cos\frac{t}{2}\mathrm{d}t \tag{2.5}$$

exists.

3. Main Result

The purpose of the present paper is to extend Theorems 2.1 and 2.2 on the degree of approximation of signal $\tilde{f}(x)$, conjugate to a 2π-periodic signal $f \in Lip(\xi(t), r)$ class by (E, q) summability means with a proper set of conditions. More precisely, we prove:

Theorem 3.1

If $\tilde{f}(x)$, conjugate to a 2π-periodic signal (function) f belonging to $Lip(\xi(t), r)$-class, then its degree of approximation by (E, q) means of conjugate series of Fourier series (1.9) is given by

$$\left\|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right\|_r = O\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\} \tag{3.1}$$

provided positive increasing $\xi(t)$ satisfies the following conditions

$$\left\{\int_0^{\pi/(n+1)}\left(\frac{|\psi_x(t)|}{\xi(t)}\right)^r \mathrm{d}t\right\}^{1/r} = O(1), \tag{3.2}$$

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}\right)^r \mathrm{d}t\right\}^{1/r} = O\left((n+1)^\delta\right) \tag{3.3}$$

and

$$\{\xi(t)/t\} \text{ is non-increasing in "}t\text{"}, \tag{3.4}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $r^{-1} + s^{-1} = 1, 1 \le r \le \infty$, condition (3.2) and (3.3) hold uniformly in x and $\tilde{E}_n^q$ is the nth (E, q) means of the series (1.9) and the conjugate function $\tilde{f}(x)$ is defined for almost every x by

$$\begin{aligned}\tilde{f}(x) &= -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot t/2\,\mathrm{d}t \\ &= \lim_{h\to 0}\left(-\frac{1}{2\pi}\int_h^\pi \psi(t)\cot t/2\,\mathrm{d}t\right).\end{aligned} \tag{3.5}$$

Note 3.2 Using condition (3.4), we get

$$\frac{(n+1)}{\pi}\xi\left(\frac{\pi}{(n+1)}\right) \le (n+1)\xi\left(\frac{1}{n+1}\right).$$

Note 3.3 Also, if $\xi\left(\frac{1}{n+1}\right) = \left(\frac{1}{n+1}\right)^\alpha$, then our main Theorem (3.1) reduces to Theorem 2.2, and thus generalizes the theorem of Shukla [18].

Note 3.4 The transform (E, q) plays an important role in signal theory and the theory of Machines in Mechanical Engineering.

4. Lemma

For the proof of our theorem, we need the following lemma.

Lemma 4.1 [18]: For $0 \le t \le \pi$, we have

$$\left|\tilde{G}_n(t)\right| = O\left(t^{-1}\mathrm{e}^{-2snt^2/\{\pi(1+s)\}^2}\right).$$

5. Proof of Theorem 3.1

Let $\tilde{s}_n(x)$ denote the partial sum of series (1.9), then we have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}\mathrm{d}t.$$

Therefore the (E, q) transform $\tilde{E}_n^q(x)$ of $\tilde{s}_n(x)$ is given by

$$\begin{aligned}&\tilde{E}_n^q(f;x) - \tilde{f}(x) \\ &= \frac{1}{2\pi(1+q)^n}\int_0^\pi \frac{\psi_x(t)}{\sin(t/2)}\left\{\sum_{k=0}^n \binom{n}{k} q^{n-k}\cos(k+1/2)t\right\}\mathrm{d}t \\ &= \int_0^\pi \psi_x(t)\tilde{G}_n(t)\mathrm{d}t \\ &= \int_0^{\pi/(n+1)} \psi_x(t)\tilde{G}_n(t)\mathrm{d}t + \int_{\pi/(n+1)}^{\pi} \psi_x(t)\tilde{G}_n(t)\mathrm{d}t \\ &= I_1 + I_2,\end{aligned} \tag{5.1}$$

Now, we consider

$$|I_1| \le \int_0^{\pi/(n+1)} |\psi_x(t)|\left|\tilde{G}_n(t)\right|\mathrm{d}t$$

Applying Hölder's inequality, using the fact that $\psi_x(t) \in Lip(\xi(t), r)$ due to $f \in Lip(\xi(t), r)$, condition (3.2) and Lemma 4.1, we have

$$|I_1| \le \left[\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\xi(t)}\right)^r \mathrm{d}t\right]^{1/r} \left[\int_0^{\pi/(n+1)} \left(\xi(t)\left|\tilde{G}_n(t)\right|\right)^s\right]^{1/s}$$

$$= O(1)\left[\int_{0^+}^{\pi/(n+1)} \left(\frac{\xi(t)}{t} \mathrm{e}^{-2snt^2/\{\pi(1+s)\}^2}\right)^s \mathrm{d}t\right]^{1/s}$$

$$= O(1)\left[\int_{0^+}^{\pi/(n+1)} \left(\frac{\xi(t)}{t}\right)^s \mathrm{d}t\right]^{1/s}$$

Since $\xi(t)$ is positive increasing function so using condition (3.4), we have $\xi(\pi/(n+1)) \le \pi\xi(1/(n+1))$ for $\pi/(n+1) \ge 1/(n+1)$, $r^{-1}+s^{-1}=1$ and Second Mean Value Theorem for integrals, we get

$$|I_1| = O\left(\xi\left(\frac{\pi}{n+1}\right)\right)\left[\lim_{\in\to 0}\int_{\in}^{\pi/(n+1)} t^{-s}\mathrm{d}t\right]^{1/s}$$

$$= O\left((n+1)^{1-1/s}\xi(1/(n+1))\right) = O\left((n+1)^{1/r}\xi(1/(n+1))\right). \tag{5.2}$$

Now, we consider

$$|I_2| \le \int_{\pi/(n+1)}^{\pi} |\psi_x(t)|\left|\tilde{G}_n(t)\right|\mathrm{d}t.$$

Again applying Hölder's inequality, using the fact that $\psi_x(t) \in Lip(\xi(t), r)$ due to $f \in Lip(\xi(t), r)$, condition (3.3) and Lemma 4.1, we obtain

$$|I_2| \le \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}\right)^r \mathrm{d}t\right]^{1/r} \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)\left|\tilde{G}_n(t)\right|}{t^{-\delta}}\right)^s\right]^{1/s}$$

$$= O\left((n+1)^{\delta}\right)\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+1}} \mathrm{e}^{-2snt^2/\{\pi(1+s)\}^2}\right)^s \mathrm{d}t\right\}^{1/s}$$

$$= O\left((n+1)^{\delta}\right)\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+1}}\right)^s \mathrm{d}t\right\}^{1/s}$$

$$= O\left((n+1)^{\delta}\right)\left\{\int_{1/\pi}^{\pi/(n+1)} \left(\frac{\xi(1/y)}{y^{\delta-1}}\right)^s \frac{\mathrm{d}y}{y^2}\right\}^{1/s}$$

$$= O\left\{(n+1)^{\delta}\left(\frac{n+1}{\pi}\right)\xi\left(\frac{\pi}{n+1}\right)\right\}\left\{\int_{\in_1}^{(n+1)/\pi} y^{-\delta s-2}\mathrm{d}y\right\}^{1/s},$$

for some $1/\pi < \in_1 < (n+1)/\pi$,

$$= O\left((n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right)\left(\frac{(n+1)^{-\delta s-1} - (\in_1)^{-\delta s-1}}{-\delta s - 1}\right)^{1/s}$$

$$= O\left((n+1)^{1-1/s}\xi\left(\frac{1}{n+1}\right)\right) = O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right), \tag{5.3}$$

in view of increasing nature of $y\xi(1/y)$, $r^{-1}+s^{-1}=1$, where $\in_1$ lie in $\left[\pi^{-1}, (n+1)\pi^{-1}\right]$, Second Mean Value Theorem for integrals and Note 3.2.

Collecting (5.1) - (5.3), we get

$$\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right| = O\left((n+1)^{1/r}\xi(1/(n+1))\right).$$

Now, using the L_r-norm of a function, we get

$$\left\|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right\|_r$$

$$= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right|^r \mathrm{d}x\right\}^{1/r}$$

$$= O\left\{\int_0^{2\pi}\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right)^r \mathrm{d}x\right\}^{1/r}$$

$$= O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}\mathrm{d}x\right)^{1/r}\right)$$

$$= O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of Theorem 3.1.

6. Corollaries

The following corollaries can be derived form Theorem 3.1.

Corollary 6.1: If $\xi(t) = t^{\alpha}$, $0 \le \alpha \le 1$ then the class $Lip(\xi(t), r)$, $r \ge 1$ reduces to the class $Lip(\alpha, r)$, $1/r < \alpha \le 1$ and the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the $Lip(\alpha, r)$ class is given by

$$\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right). \tag{6.1}$$

Proof. Putting $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$ in Theorem 3.1, we have

$$\left\|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right|^r \mathrm{d}x\right\}^{1/r}$$

or,

$$O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right) = \left\{\int_0^{2\pi}\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right|^r \mathrm{d}x\right\}^{1/r}$$

or,

$$O(1)$$

$$= \left\{\int_0^{2\pi}\left|\tilde{E}_n^q(f;x) - \tilde{f}(x)\right|^r \mathrm{d}x\right\}^{1/r} O\left(\frac{1}{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)}\right),$$

For if not the right hand side of the above equation will be $O(1)$, therefore, we have

$$\left|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right|$$
$$=O\left(\left(\frac{1}{n+1}\right)^{\alpha}(n+1)^{1/r}\right)=O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right).$$

This completes the proof of Corollary 6.1.

Corollary 6.2 If $\xi(t)=t^{\alpha}$ for $0<\alpha<1$ and $r\to\infty$ in Theorem 3.1, then $f\in Lip\alpha$. In this case, the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the class $Lip\alpha\ (0<\alpha<1)$ is given by

$$\left|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right|=O\left((n+1)^{-\alpha}\right).$$

Proof. For $r\to\infty$ in Corollary 6.1, we get

$$\left\|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right\|_{\infty}$$
$$=\sup_{0\le x\le 2\pi}\left|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right|=O\left((n+1)^{-\alpha}\right).$$

Thus, we have

$$\left|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right|\le\left\|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right\|_{\infty}$$
$$=\sup_{0\le x\le 2\pi}\left|\tilde{E}_n^q(f;x)-\tilde{f}(x)\right|=O\left((n+1)^{-\alpha}\right).$$

This completes the proof of Corollary 6.2.

7. An Example

Consider an infinite series

$$\sum_{n=0}^{\infty}(-1)^n(2q+1)^n\cos nx \tag{7.1}$$

The n^{th} partial sums s_n of series (7.1) at $x=0$ is given by

$$s_n=\sum_{r=0}^{n}(-1)^r(2q+1)^r\cos rx\le\frac{1-(-1)^{n+1}(2q+1)^{n+1}}{2(1+q)}.$$

Since $\lim_{n\to\infty}s_n$ does not exist. Therefore the series (7.1) is non-convergent.

Now, we have the (E,q) transform of (7.1) is given by

$$E_n^q=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k,\ q>0$$
$$\le\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left(\frac{1-(-1)^{k+1}(2q+1)^{k+1}}{2(1+q)}\right)$$
$$=\frac{1}{2(1+q)}+\frac{(1+2q)(q-1-2q)^n}{2(1+q)(q+1)^n}$$
$$=\frac{1}{2(1+q)}+\frac{(1+2q)(-1)^n}{2(1+q)}.$$

Here, $\lim_{n\to\infty}E_n^q$ does not exist. Hence the series (7.1) is not summable, while the series (7.1) is product summable.

8. Conclusion

Several results concerning to the degree of approximation of periodic signals (functions) belonging to the Lipschitz class by matrix (E,q) operator have been reviewed. Further, a proper set of conditions have been discussed to rectify the errors. Some interesting application of the operator (E,q) used in this paper pointed out in Note 3.4. An example has been discussed also.

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