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On $\mathfrak{J}$-Reconstruction Property

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ABSTRACT

Reconstruction property in Banach spaces introduced and studied by Casazza and Christensen in [1]. In this paper we introduce reconstruction property in Banach spaces which satisfy $\mathfrak{J}$-property. A characterization of reconstruction property in Banach spaces which satisfy $\mathfrak{J}$-property in terms of frames in Banach spaces is obtained. Banach frames associated with reconstruction property are discussed.

Keywords: Frames; Banach Frames; Retro Banach Frames; Reconstruction Property

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space with inner product $\langle , \rangle$. A system $\{f_k\} \subset \mathcal{H}$ called a *frame* (Hilbert) for $\mathcal{H}$ if there exists positive constants A and B such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} \left|\langle f, f_k \rangle\right|^2 \le B\|f\|^2, \; f \in \mathcal{H}.$$

The positive constants A and B are called *lower* and *upper bounds* of the frame $\{f_k\}$, respectively. They are not unique.

The operator $T : \ell^2 \to \mathcal{H}$ given by $T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k$, $\{c_k\} \in \ell^2$ is called the *synthesis operator* or *pre-frame operator*. Adjoint of T is given by $T^* : \mathcal{H} \to \ell^2$, $T^*(f) = \{\langle f, f_k \rangle\}$ and is called the *analysis operator*. Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by $S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$, $f \in \mathcal{H}$. The frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ onto $\mathcal{H}$. Every vector $f \in \mathcal{H}$ can be written as:

$$f = SS^{-1} f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k.$$

$$(\text{Reconstruction formula})$$

The series in the right hand side converge unconditionally and is called *reconstruction formula* for $\mathcal{H}$. The representation of f in reconstruction formula need not be unique. Thus, frames are *redundant systems* in a Hilbert space which yield one natural representation for every vector in the concern Hilbert space, but which may have infinitely many different representations for a given vector.

Duffin and Schaeffer in [2] while working in non-harmonic Fourier series developed an abstract framework for the idea of time-frequency atomic decomposition by Gabor [3] and defined *frames* for Hilbert spaces. Due to some reason the theory of frames was not continued until 1986 when the fundamental work of Daubechies, Grossmann and Meyer published in [4]. Gröchenig in [5] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [6,7] related to *atomic decompositions*. Atomic decompositions appeared in the field of applied mathematics providing many applications [8,9]. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call *atoms*. On the other hand Banach frame for a Banach space ensure reconstruction via a bounded linear operator or *synthesis operator*. Frames play an important role in the theory of nonuniform sampling [10], wavelet theory [11,12], signal processing [2,10], and many more. For a nice introduction of frames and their technical details one may refer to [13].

During the development of frames and expansions systems in Banach spaces Casazza and Christensen introduced *reconstruction property* for Banach spaces in [1]. Reconstruction property is an important tool in several areas of mathematics and engineering. In fact, it is related to bounded approximation property. Casazza and Christensen in [1] study perturbation theory related to reconstruction property. They develop more general perturbation theory that does not force equivalence of the sequences.

In this paper we introduce and study reconstruction property in Banach spaces which satisfy $\mathfrak{I}$-property. A characterization of $\mathfrak{I}$-reconstruction property in terms of frames in Banach spaces is obtained. Banach frames associated with reconstruction property are discussed.

2. Preliminaries

Throughout this paper $\mathcal{X}$ will denotes an infinite dimensional Banach space over a field $\mathbb{K}$ (which can be $\mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}^*$ be the conjugate space, and for a sequence $\{f_k\} \subset \mathcal{X}$, $[f_k]$ denotes closure of $\text{span}\{f_k\}$ in norm topology of $\mathcal{X}$. The map $\pi : \mathcal{X} \to \mathcal{X}^{**}$ denotes the canonical mapping from $\mathcal{X}$ into $\mathcal{X}^{**}$.

Definition 2.1 ([5]) *Let $\{f_k^*\} \subset \mathcal{X}^*$ and $\mathfrak{S} : \mathcal{X}_d \to \mathcal{X}$ be given, where $\mathcal{X}_d$ is an associated Banach space of scalar valued sequences. A system $\mathcal{G} \equiv (\{f_k^*\}, \mathfrak{S})$ is called a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if*

1) $\{f_k^*(f)\} \in \mathcal{X}_d$, for each $f \in \mathcal{X}$.

2) There exist positive constants C and D $(0 < C \le D < \infty)$ such that

$$C\|f\|_{\mathcal{X}} \le \left\|\{f_k^*(f)\}\right\|_{\mathcal{X}_d} \le D\|f\|_{\mathcal{X}}, \quad \text{for each } f \in \mathcal{X}. \tag{2.1}$$

3) $\mathfrak{S}$ is a bounded linear operator such that

$$\mathfrak{S}(\{f_k^*(f)\}) = f, \text{ for each } f \in \mathcal{X}.$$

As in case of frames for a Hilbert space, positive constants C and D are called *lower* and *upper frame bounds* of the Banach frame $\mathcal{G}$, respectively. The operator $\mathfrak{S} : \mathcal{X}_d \to \mathcal{X}$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality 2.1 is called the *frame inequality*.

The Banach frame $\mathcal{G}$ is called *tight* if $C = D$ and *normalized tight* if $C = D = 1$. If there exists no reconstruction operator $\mathfrak{S}_m$ such that $(\{f_k^*\}, \mathfrak{S}_m)_{k \ne m}$ $(m \in \mathbb{N})$ is Banach frame for $\mathcal{X}$, then $\mathcal{G}$ will be called an *exact* Banach frame.

The notion of retro Banach frames introduced and studied in [14].

Definition 2.2 ([14]) *A system $\mathcal{F} \equiv (\{f_k\}, \Theta)$ $(\{f_k\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to X^*)$ is called a retro Banach frame for $\mathcal{X}^*$ with respect to an associated sequence space $\mathcal{Z}_d$ if*

1) $\{f^*(f_k)\} \in \mathcal{Z}_d$, for each $f^* \in \mathcal{X}^*$.

2) There exist positive constants $(0 < A_0 \le B_0 < \infty)$ such that

$$A_0\|f^*\| \le \left\|\{f^*(f_k)\}\right\|_{\mathcal{Z}_d} \le B_0\|f^*\|, \text{ for each } f^* \in \mathcal{X}^*.$$

3) $\Theta : \{f^*(f_k)\} \to f^*$ is a bounded linear operator from $\mathcal{Z}_d$ onto $\mathcal{X}^*$.

The positive constant A_0, B_0 are called *retro frame bounds* of $\mathcal{F}$ and operator $\Theta : \mathcal{Z}_d \to X^*$ is called *retro pre-frame operator* (or simply *reconstruction operator*) associated with $\mathcal{F}$.

Lemma 2.3. *Let $\mathcal{X}$ be a Banach space and $\{f_n^*\} \subset \mathcal{X}^*$ be a sequence such that $\{f \in \mathcal{X} : f_n^*(f) = 0,$ for all $n \in \mathbb{N}\} = \{0\}$. Then, $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{Z} = \{\{f_n^*(f)\} : f \in \mathcal{X}\}$, where the norm is given by $\left\|\{f_n^*(f)\}\right\|_{\mathcal{Z}} = \|f\|_{\mathcal{X}}$, $f \in \mathcal{X}$.*

Casazza and Christensen in [1] introduced reconstruction property in Banach spaces.

Definition 2.4 ([1]) *Let $\mathcal{X}$ be a separable Banach space. We say that a sequence $\{f_k^*\} \subset \mathcal{X}^*$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_k\} \subset \mathcal{X}$, if*

$$f = \sum_{k=1}^{\infty} f_k^*(f) f_k, \quad \text{for all } f \in \mathcal{X}.$$

In short, we will also say $(\{f_k\}, \{f_k^*\})$ has reconstruction property for $\mathcal{X}$. More precisely, we say that $(\{f_k\}, \{f_k^*\})$ is a *reconstruction system* for $\mathcal{X}$.

Remark 2.5 *An interesting example for a reconstruction property is given in* [1]: *Let $\{f_k^*\} \subset \ell^\infty$ and $\{f_k^*\}$ is unitarily equivalent to the unit vector basis of ℓ^2. Then, $\{f_k^*\}$ has a reconstruction property with respect to its own pre-dual* (*that is, expansions with respect to the orthonormal basis*). *Further examples on reconstruction property are discussed in Example* 3.4.

Definition 2.6 *A reconstruction system $(\{f_k\}, \{f_k^*\})$ for $\mathcal{X}$ is said to be*

1) *pre-shrinking* if $[f_k^*] = \mathcal{X}^*$.

2) *shrinking* if $(\{f_k^*\}, \{f_k\})$ is a reconstruction system for $\mathcal{X}^*$.

Regarding existence of Banach spaces which have reconstruction system, Casazza and Christensen proved the following result.

Proposition 2.7 ([1]) *There exists a Banach space $\mathcal{X}$ with the following properties*:

1) There is a sequence $\{f_k\}$ such that each $f \in \mathcal{X}$ has a expansion $f = \sum_{k=1}^{\infty} f_k^*(f) f_k$.

2) $\mathcal{X}$ does not have the reconstruction property with respect to any pair $(\{h_k\}, \{h_k^*\})$.

The notion of reconstruction property is related to Bounded Approximation Property (BAP). If $(\{f_k\}, \{f_k^*\})$ has reconstruction property for $\mathcal{X}$, then $\mathcal{X}$ has the bounded approximation property. So, $\mathcal{X}$ is isomorphic to a complemented subspace of a Banach space with a basis. It is also used to study geometry of Banach spaces. For more results and basics on reconstruction property and bounded approximation property one may refer to [15] and references therein.

3. $\mathfrak{J}$-Reconstruction Property

Definition 3.1 *Suppose* $\{f_k^*\} \subset \mathcal{X}^*$ *has the reconstruction property for* $\mathcal{X}$ *with respect to* $\{f_k\} \subset \mathcal{X}$. *Then, we say that* $(\{f_k\}, \{f_k^*\})$ *satisfy property* $\mathfrak{J}$ *if* $\inf_{1\le k<\infty} \|f_k\| > 0$ *and there exists a functional* $\Psi \in \mathcal{X}^{**}$ *such that* $\Psi(f_k^*) = 1$, *for all* $k \in \mathbb{N}$. *In this case we say that* $(\{f_k\}, \{f_k^*\})$ *is a* $\mathfrak{J}$*-reconstruction system for* $\mathcal{X}$.

Remark 3.2 *If* $\inf_{1\le k<\infty} \|f_k\| = 0$ *and there exists a functional* $\Psi \in \mathcal{X}^{**}$ *such that* $\Psi(f_k^*) = 1$, *for all* $k \in \mathbb{N}$, *then we say that* $(\{f_k\}, \{f_k^*\})$ *is a* $\mathfrak{J}_\omega$*-reconstruction system* (*or weak* $\mathfrak{J}$*-reconstruction system for* $\mathcal{X}$).

Remark 3.3 *A* $\mathfrak{J}$*-reconstruction system is actually a dual system of a* Φ*-Schauder frame* [16] *in the context of reconstruction property*.

Example 3.4 *Let* $\mathcal{X} = c_0$ *and* $\{e_k\} \subset \mathcal{X}$ *be a sequence of canonical unit vectors. Define* $\{f_k^*\} \subset \mathcal{X}^*$ *by*

$$f_1^*(f) = \frac{1}{2}\xi_1,\ f_2^*(f) = \frac{1}{2}\xi_1,\ f_k^*(f) = \xi_{k-1},\ f = \{\xi_k\} \in \mathcal{X}.$$

Then, $\{f_k^*\}$ *has a reconstruction property with respect to* $\{f_k\} \subset \mathcal{X}$, *where* $f_1 = e_1, f_2 = e_1, f_k = e_{k-1}$. *Hence* $(\{f_k\}, \{f_k^*\})$ *is a* $\mathfrak{J}$*-reconstruction system for* $\mathcal{X}$ [*See Proposition* 3.5]. *Note that the reconstruction system* $(\{f_k\}, \{f_k^*\})$ *is shrinking.*

Now define $\{g_k^*\} \subset \mathcal{X}^*$ by $g_1^*(f) = \xi_1,\ g_k^*(f) = \xi_{k-1},\ f = \{\xi_k\} \in \mathcal{X}$. Then, $\{g_k^*\}$ has a reconstruction property with respect to $\{g_k\} \subset \mathcal{X}$, where $g_1 = 0, g_k = e_{k-1}$. By Proposition 3.5, $(\{g_k\}, \{g_k^*\})$ is not a $\mathfrak{J}$-reconstruction system for $\mathcal{X}$. Note that $(\{g_k\}, \{g_k^*\})$ is $\mathfrak{J}_\omega$-reconstruction system which is shrinking. Thus, a shrinking reconstruction system for $\mathcal{X}$ need not be a $\mathfrak{J}$-reconstruction system.

We now give a characterization of a $\mathfrak{J}$-reconstruction system for $\mathcal{X}$ as claimed in section 1, in terms of frames.

Proposition 3.5 *Let* $(\{f_k\}, \{f_k^*\})$ *be a reconstruction system for* $\mathcal{X}$ *with* $\inf_{1\le k\le\infty} \|f_k\| > 0$. *Then,* $(\{f_k\}, \{f_k^*\})$ *satisfy property* $\mathfrak{J}$ *if and only if there is no retro pre-frame operator* Θ_0 *such that* $(\{f_k^* - f_{k+1}^*\}, \Theta_0)$ *is retro Banach frame for* $[f_k^*]^*$.

This is an immediate consequence of the following lemma.

Lemma 3.6 *Let* $(\{f_k\}, \{f_k^*\})$ *be a pre-shrinking reconstruction system for* $\mathcal{X}$. *Then,* $(\{f_k\}, \{f_k^*\})$ *is a* $\mathfrak{J}_\omega$*-reconstruction system if and only if there exists no retro pre-frame operator* $\hat{\Theta}$ *such that* $(\{f_k^* - f_{k+1}^*\}, \hat{\Theta})$ *is retro Banach frame for* $\mathcal{X}^{**}$.

Proof. Forward part is obvious. Indeed, by using lower retro frame inequality of $(\{f_k^* - f_{k+1}^*\}, \hat{\Theta})$ and existence of $\Psi \in \mathcal{X}^{**}$ such that $\Psi(f_k^* - f_{k+1}^*) = 0$, for all $k \in \mathbb{N}$, we obtain $\Psi = 0$. This is a contradiction.

For reverse part, let if possible, there is no reconstruction operator $\hat{\Theta}$ such that $(\{f_k^* - f_{k+1}^*\}, \hat{\Theta})$ is a retro Banach frame for $\mathcal{X}^{**}$. Then, Hahn Banach Theorem force to admit a non zero functional $\phi \in E^{**}$ such that $\phi(f_k^* - f_{k+1}^*) = 0$, for all $k \in \mathbb{N}$. That is, $\phi(f_k^*) = \phi(f_{k+1}^*)$, for all $k \in \mathbb{N}$. Put $\phi(f_k^*) = \alpha$, for all $k \in \mathbb{N}$. If $\alpha = 0$, then $\phi(f_k^*) = 0$ for all $k \in \mathbb{N}$. But $(\{f_k\}, \{f_k^*\})$ is pre-shrinking, therefore $\phi = 0$, a contradiction. Thus $\alpha \ne 0$. Put $\Psi = \dfrac{\phi}{\alpha}$. Then, $\Psi \in \mathcal{X}^{**}$ is such that $\Psi(f_k^*) = 1$ for all $k \in \mathbb{N}$. Thus, $(\{f_k\}, \{f_k^*\})$ is a $\mathfrak{J}_\omega$-reconstruction system. □

Remark 3.7 *Note that Lemma* 3.6 *is no longer true if* $(\{f_k\}, \{f_k^*\})$ *is not pre-shrinking.*

Application: Let $\mathcal{X} = L^2(a,b)$. Consider a boundary value problem(BVP) with a set of n boundary conditions:

$$\text{BVP: } \nabla(f) = \lambda f,\ \Xi(f) = 0,$$

where $\nabla(\cdot) = (\cdot)^n + \Phi_1(\xi)(\cdot)^{n-1} + \cdots + \Phi_n(\xi)(\cdot)$ is a linear differential operator with $\Phi_j \in C^{n-k}[a,b]$, and

$\Xi(f)=0$ denotes the set of n boundary conditions:

$$\Xi_j(f)=\sum_{k=1}^{n}\left[\alpha_{j,k}\Phi^{k-1}(a)+\beta_{j,k}\Phi^{k-1}(b)\right]=0.$$

It is given in [17] (at page 66) that for a large class of boundary conditions (which are known as regular boundary conditions), the BVP admits a system $\{\Phi_n(\xi)\}$ and $\{\Psi_n(\xi)\}$ consisting of eigenfunction associated with given BVP such that

$$\Phi_n(\xi)=A_n\left[\cos\frac{2\pi n\xi}{b-a}+O\left(\frac{1}{n}\right)\right];$$

$$\Psi_n(\xi)=B_n\left[\sin\frac{2\pi n\xi}{b-a}+O\left(\frac{1}{n}\right)\right], n=0,1,2,3,\cdots.$$

It is well known that the corresponding to $\{f_n\}\equiv\left\{\cos\frac{2\pi n\xi}{b-a}\right\}\sqcup\left\{\sin\frac{2\pi n\xi}{b-a}\right\}$ there exists a $\{f_n^*\}\in\mathcal{X}^*$ such that $(\{f_n\},\{f_n^*\})$ is a reconstruction system for $\mathcal{X}=L^2(a,b)$. Now

$$\left\|\Phi_n(\xi)-A_n\cos\frac{2\pi nt}{b-a}\right\|^2=O\left(\frac{1}{n^2}\right) \text{ and}$$

$$\left\|\Psi_n(\xi)-B_n\sin\frac{2\pi nt}{b-a}\right\|^2=O\left(\frac{1}{n^2}\right).$$

Therefore, by using Paley and Wiener theorem in [18, p. 208], there exists a sequence $\{f_n^*\}\subset\mathcal{X}^*$ such that $\{f_n^*\}$ admits a reconstruction system with respect to $\{\Phi_n\}\sqcup\{\Psi_n\}$. This reconstruction system is not of type $\Im_\omega$. Therefore, by using Lemma 3.6, there exists a retro pre-frame operator $\hat{\Theta}$ such that $(\{f_n^*-f_{n+1}^*\},\hat{\Theta})$ is retro Banach frame for $\mathcal{X}^{**}$. Recall that if we write a function in terms of reconstruction system, then computation of all the coefficients is required. If calculation of coefficients which appear in the series expansion of a given reconstruction system are complicated, then we reconstruct the function by pre-frame operator of $(\{f_n^*-f_{n+1}^*\},\hat{\Theta})$.

The following proposition provides a sufficient condition for a reconstruction system to satisfy property $\Im_\omega$.

Proposition 3.8 *Let $(\{f_k\},\{f_k^*\})$ be a reconstruction system for $\mathcal{X}$. If there exists a vector f_0 in $\mathcal{X}$ such that $f_k^*(f_0)=1$ for all $k\in\mathbb{N}$, then $(\{f_k\},\{f_k^*\})$ is a $\Im_\omega$-reconstruction system.*

Proof. Let $\pi:\mathcal{X}\to\mathcal{X}^{**}$ be the canonical embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$. Then $\Psi=\pi(f_0)\in\mathcal{X}^{**}$ is such that $\Psi(f_k^*)=1$, for all $k\in\mathbb{N}$. Thus, $(\{f_k\},\{f_k^*\})$ is a $\Im_\omega$-reconstruction system for $\mathcal{X}$.

Remark 3.9 *The condition in Proposition 3.8 is not necessary. However, if $\mathcal{X}$ is reflexive, then the condition given in Proposition 3.8 turns out to be necessary. Moreover, this is equivalent to the condition: There exists no pre-frame operator $\mathfrak{S}_\circ$ such that $(\{f_k^*-f_{k+1}^*\},\mathfrak{S}_\circ)$ is a Banach frame for $\mathcal{X}$.*

To conclude the section we show that a given $\Im$-reconstruction system in Banach spaces produce another $\Im$-reconstruction system: Consider a $\Im$-reconstruction system $(\{f_k\},\{f_k^*\})$ for $\mathcal{X}$.

Let $\mathcal{U}=\left\{\{\gamma_i\}\subset\mathbb{K}:\sum_{i=1}^{\infty}\gamma_i f_i \text{ converges}\right\}$.

Then $\mathcal{U}$ is a Banach space with norm given by

$$\left\|\{\gamma_i\}\right\|_{\mathcal{U}}=\sup_{1\le k\le\infty}\left\|\sum_{i=0}^{k}\gamma_i f_i\right\|_{\mathcal{X}}.$$

Define $\Gamma:\mathcal{X}\to\mathcal{U}$ by $\Gamma(f)=\{f_k^*(f)\}, f\in\mathcal{X}$.

Then Γ is an isomorphism of $\mathcal{X}$ into $\mathcal{U}$.

Also $\tilde{\Theta}:\mathcal{U}\to\mathcal{X}$ defined by $\tilde{\Theta}(\{\gamma_i\})=\sum_{i=1}^{\infty}\gamma_i f_i$ is also a bounded linear operator from $\mathcal{U}$ onto $\mathcal{X}$.

Put $\Xi=Ker\tilde{\Theta}$. Then Ξ is a closed subspace of $\mathcal{U}$ such that $\Gamma(x)\cap\Xi=\{0\}$. Moreover, if $\{\gamma_i\}\in\mathcal{U}$ is any element such that $f=\sum_{i=1}^{\infty}\gamma_i f_i$, then $\{f_i^*(f)\}\in\Gamma(\mathcal{X})$ and

$$\sum_{i=1}^{\infty}\left(\gamma_i-\{f_i^*(f)\}\right)f_i=\sum_{i=1}^{\infty}\gamma_i f_i-\sum_{i=1}^{\infty}f_i(f)f_i=0.$$

Therefore, $(\gamma_i-\{f_i^*(f)\})\in\Xi$ is such that

$\{\gamma_i\}=\{f_i(f)\}+\{\gamma_i-f_i(f)\}$.

Hence $\mathcal{U}=\Gamma(\mathcal{X})\oplus\Xi$.

Let V be projection on $\mathcal{U}$ onto $\Gamma(\mathcal{X})$.

Then, $V(\{\gamma_i\})=\left\{f_k^*\left(\sum_{i=1}^{\infty}\gamma_i f_i\right)\right\}, \{\gamma_i\}\in\mathcal{U}$. Therefore, for each $k\in\mathbb{N}$, we have

$$V(e_k)=\left\{f_k^*\left(\sum_{i=1}^{\infty}\delta_{i,k}f_i\right)\right\},$$

where $\delta_{i,k}=(1\ \ i=k \text{ and } 0\ \ i\ne k)$.

That is: $V(e_k)=\Gamma(f_k)$ for all k. So, $f_k=\Gamma^{-1}(V(e_k))$ for all $k\in\mathbb{N}$, where $\{e_k\}$ is sequence of canonical unit vectors in $\mathcal{U}$. Hence $(\{\Gamma^{-1}(V(e_k))\},\{f_k^*\})$ is a reconstruction system for $\mathcal{X}$ which satisfy property $\Im$.

This is summarized in the following proposition.

Proposition 3.10 *Let $(\{f_k\},\{f_k^*\})$ be a $\Im$-reconstruction system for $\mathcal{X}$. Then, there exists $\Gamma^{-1}(V(e_k))\in\mathcal{X}$ such that $(\{\Gamma^{-1}(V(e_k))\},\{f_k^*\})$ is a $\Im$-reconstruction system for $\mathcal{X}$, where Γ and V are same as in above discussion.*

4. Associated Banach Frames

Definition 4.1 *Suppose that* $\{f_k^*\}$ *has the reconstruction property for* $\mathcal{X}$ *with respect to* $\{f_k\} \subset \mathcal{X}$. *Then, there exists a reconstruction operator* $\mathfrak{S} : \mathcal{X}_d \to \mathcal{X}$ *such that* $(\{f_k^*\}, \mathfrak{S})$ *is a Banach frame for* $\mathcal{X}$ *with respect to some* $\mathcal{X}_d$. *We say that* $(\{f_k^*\}, \mathfrak{S})$ *is an associated Banach frame of* $(\{f_k\}, \{f_k^*\})$.

Consider a reconstruction system $(\{f_k\}, \{f_k^*\})$ for a Banach space $\mathcal{X}$. We can write each element of $\mathcal{X}$ (we can reconstruct $\mathcal{X}$) by mean of an infinite series formed by $\{f_k\}$ over scalars $\{f_k^*(f)\}$. For a non zero functional h^* (say), in general, there is

- no $\{h_k\} \subset \mathcal{X}$ such that $\{f_k^* + h^*\}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{h_k\}$.
- no reconstruction operator $\mathfrak{S}_0$ such that $(\{f_k^* + h^*\}, \mathfrak{S}_0)$ is a Banach frame for $\mathcal{X}$.

More precisely, two natural and important problem arise, namely, existence of $\{h_k\} \subset \mathcal{X}$ such that $\{f_k^* + h^*\}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{h_k\}$ and other is the existence of a reconstruction operator $\mathfrak{S}_0$ associated with $\{f_k^* + h^*\}$. Cassaza and Christensen in [1] study some stability of reconstruction property in Banach spaces in terms of closeness of certain sequence to a given reconstruction system. In the present section we focus on pre-frame operator associated with $\{f_k^* + h^*\}$.

Motivation: Consider a signal space $\mathcal{H}_0$. If $\{f_k\}$ is a frame (Hilbert) for $\mathcal{H}_0$, then each element of $\mathcal{H}_0$ can be recovered by an infinite combinations of frame elements. That is, by the reconstruction formula. If a signal *f* is transmitted to a receiver, then there are some kind of disturbances in the received signal. To overcome these disturbances from the receiver, frames plays an important role. Actually, a signal in the space (after its transmission) is in the form of the frame coefficients $\{\langle f, S^{-1} f_k \rangle\}$, $f \in \mathcal{H}_0$. An error $\tilde{e}$ is always is expected with concern signal in the space. That is, actual signal in the space is of the form $\{\langle f, S^{-1} f_k \rangle + \tilde{e}\}$, where $\tilde{e}$ is an error associated with *f*. An interesting discussion in this direction is given in [13]. We extend the said problem to Banach frames in general Banach spaces.

The following proposition provides sufficient condition for a reconstruction system to satisfy property $\Im_\omega$ in terms of non-existence of pre-frame operator associated with certain error.

Proposition 4.2 *Suppose that* $\{f_k^*\}$ *has the reconstruction property for a signal space (Banach)* $\mathcal{X}$ *with respect to* $\{f_k\}$. *Let* h^* *(error) be in* $\mathcal{X}^*$ *for which there is no pre-frame operator* $\mathfrak{S}_0$ *such that* $(\{f_k^* + h^*\}, \mathfrak{S}_0)$ *is a Banach frame for* $\mathcal{X}$, *then* $(\{f_k\}, \{f_k^*\})$ *is a* $\Im_\omega$*-reconstruction system for* $\mathcal{X}$.

Proof. Let $(\{f_k^*\}, \mathfrak{S})$ be an associated Banach frame of $(\{f_k\}, \{f_k^*\})$. If there exists no pre-frame operator $\mathfrak{S}_0$ such that $(\{f_k^* + h^*\}, \mathfrak{S}_0)$ is a Banach frame for $\mathcal{X}$, then, there is a non-zero vector $f_0 \in \mathcal{X}$ such that $(f_k^* + h^*)(f_0) = 0$, for all $k \in \mathbb{N}$. By frame inequality of $(\{f_k^*\}, \mathfrak{S})$, we conclude that $h^*(f_0) \neq 0$. Put $\Psi = -\pi\left(\frac{1}{h^*(f_0)} f_0\right)$. Then, $\Psi \in \mathcal{X}^{**}$ is such that $\Psi(f_k) = 1$, for all $k \in \mathbb{N}$. Hence $(\{f_k\}, \{f_k^*\})$ is a $\Im_\omega$-reconstruction system for $\mathcal{X}$.

Remark 4.3 *The condition in Proposition* 4.2 *is not necessary unless* Ψ *correspond to a vector in* $\mathcal{X}$. *More precisely, we can find a certain error* $h^* \in \mathcal{X}^*$ *such that there exists no pre-frame operator* $\mathfrak{S}_0$ *associated with* $\{f_k^* + h^*\}$ *provided* $\Psi \leftrightarrow f \in \mathcal{X}$.

Remark 4.4 *Let us continue with the outcomes in Proposition* 4.2, *where* $(\{f_k\}, \{f_k^*\})$ *is found to be a* $\Im_\omega$*-reconstruction system for* $\mathcal{X}$ *provided there is no pre-frame operator* $\mathfrak{S}_0$ *such that* $(\{f_k^* + h^*\}, \mathfrak{S}_0)$ *is a Banach frame for* $\mathcal{X}$, *where* h^* *is certain choice of error (functional). A natural problem arises, which is of determining a Banach space* $\mathcal{B}$ *for which the system* $\{f_k^* + h^*\}$ *admits a pre-frame operator. Answer to this problem is positive, provided* $(\{f_k\}, \{f_k^*\})$ *is pre-shrinking. The outline of construction of such a Banach space can be understood as follows: Put* $\zeta = -\frac{1}{h^*(f_0)} f_0$ *(where* f_0 *is same as in the proof of Proposition* 4.2*). Now, there is no pre-frame operator* $\mathfrak{S}_0$ *associated with* $\{f_k^* + h^*\}$, *so there exists a non-zero vector* g *such that* $(f_k^* + h^*)(g) = 0$, *for all* $k \in \mathbb{N}$. *By using frame inequality of the associated Banach frame* $(\{f_k^*\}, \mathfrak{S})$ *we have* $h^*(g) \neq 0$. *Put* $\varphi = \frac{-1}{h^*(g)} g$. *Then,* φ *is a non-zero vector in* $\mathcal{X}$ *such that* $f_k^*(\varphi) = 1$, *for all* $k \in \mathbb{N}$. *Therefore,* $f_k^*(\zeta - \varphi) = 0$ *for all* $k \in \mathbb{N}$. *Now* $(\{f_k\}, \{f_k^*\})$ *is pre-shrinking, so we have* $\zeta = \varphi$. *Hence* $g = p\zeta$, *where* $p = -h^*(g)$. *By using Lemma 2.3 there exists a pre-frame operator* $\mathfrak{S}_1$ *such that* $(\{f_k^* + h^*\}, \mathfrak{S}_1)$ *is a*

Banach frame(*normalized tight*) *for the Banach space* $\mathcal{B}$, *where* $\mathcal{B}^* = [\zeta]^{\perp}$;
$[\zeta]^{\perp} = \{ f^* \in \mathcal{X}^* : f^*(f) = 0, f \in [\zeta] \}$.

An application of Proposition 4.2 is given below:

Example 4.5 *Let* $(\{f_k\}, \{f_k^*\})$ *be a reconstruction system given in Example* 3.4 *for* $\mathcal{X} = c_0$. *Then*, $\mathfrak{S} : \mathcal{X}_d = \{\{f_k^*(f)\} : f \in \mathcal{X}\} \to \mathcal{X}$ *is a bounded linear operator such that* $(\{f_k^*\}, \mathfrak{S})$ *is a Banach frame* (*associated*) *for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$ *and with bounds* $A = B = 1$. *Put* $h^* = -f_4^*$ (*this choice makes sense*, *because disturbances are not constant*!). *Then*, h^* *is an error in* $\mathcal{X}^*$ *for which there is no reconstruction operator* $\mathfrak{S}_o$ *such that* $(\{f_k^* + h^*, \mathfrak{S}_o\})$ *is a Banach frame for* $\mathcal{X}$. *Hence by Proposition* 4.2, $(\{f_k\}, \{f_k^*\})$ *is a* $\mathfrak{I}_\omega$*-reconstruction system for* $\mathcal{X}$. □

Definition 4.6 *Fix* $f \in \mathcal{X}$. *A pair* $(\{f_k\}, \{f_k^*\})$, (*where* $\{f_k\} \subset \mathcal{X}, \{f_k^*\} \subset \mathcal{X}^*$) *is said to be localized at* f, *if* $f = \sum_{k=1}^{\infty} \epsilon_k f_k^*(f) f_k$, *where* $\{\epsilon_k\}$ *is a sequence of scalars*.

If $(\{f_k\}, \{f_k^*\})$ is localized at every $f \in \mathcal{X}$ with $\epsilon_k = 1$, for all k, then $(\{f_k\}, \{f_k^*\})$ turns out to be a reconstruction system for $\mathcal{X}$. Consider a reconstruction system $(\{f_k\}, \{f_k^*\})$ for $\mathcal{X}$ and $(\{f_k^*\}, \mathfrak{S}_0)$ be its associated Banach frame with respect to $\mathcal{X}_d$. Let $0 \neq \{\psi_k^*(f)\} \in \mathcal{X}_d, f \in \mathcal{X}$. Then, in general, there is no pre-frame operator $\hat{\mathfrak{S}}_0$ associated with system $\left\{ \frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right\}$. This problem is also known as stability of $(\{f_k^*\}, \mathfrak{S}_0)$ with respect to $\mathcal{X}_d$. If $(\{f_k\}, \{\psi_k^*\})$ is not localized at certain vectors in $\mathcal{X}$, then we can find such pre-frame operator associated with $\left\{ \frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right\}$. This is what concluding proposition of this paper says.

Proposition 4.7 *Let* $(\{f_k\}, \{f_k^*\})$ *be a reconstruction system for* $\mathcal{X}$. *Assume that* $(\{f_k\}, \{\psi_k^*\})$ *is not localized at* $f \in \mathcal{X}$, *where* $\pi_f(\mathcal{E}) = 0$; $\mathcal{E} = span(f_k^*)_k$. *Then*, *there exists a pre-frame operator* $\hat{\mathfrak{S}}_0$, *such that* $\left(\left\{ \frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right\}, \hat{\mathfrak{S}}_0 \right)$ *is a Banach frame for* $\mathcal{X}$.

Proof. Let $(\{f_k^*\}, \mathfrak{S})$ be associated Banach frame of $(\{f_k\}, \{f_k^*\})$. Let, if possible, there is no reconstruction operator $\hat{\mathfrak{S}}_0$, such that $\left(\left\{ \frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right\}, \hat{\mathfrak{S}}_0 \right)$ is a Banach frame for $\mathcal{X}$. Then, there exists a non zero vector $\boldsymbol{f}_0$ such that $\left(\frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right)(\boldsymbol{f}_0) = 0$, for all $k \in \mathbb{N}$.

This gives

$$\frac{1}{\psi_k^*(f)} f_k^*(\boldsymbol{f}_0) = \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^*(\boldsymbol{f}_0), \ k \in \mathbb{N}.$$

By using frame inequality of $(\{f_k^*\}, S)$, we obtain,

$$f_k^*(\boldsymbol{f}_0) = \frac{\psi_k^*(f)}{\psi_1^*(f)} f_1^*(\boldsymbol{f}_0) \neq 0, \ k \in \mathbb{N}.$$

Since $(\{f_k\}, \{f_k^*\})$ is a reconstruction system for $\mathcal{X}$, we have

$$\boldsymbol{f}_0 = \sum_{k=1}^{\infty} f_k^*(\boldsymbol{f}_0) f_k = \sum_{k=1}^{\infty} \frac{\psi_k^*(f)}{\psi_1^*(f)} f_1^*(\boldsymbol{f}_0) f_k.$$

Thus, $(\{f_k\}, \{\psi_k^*\})$ is localized at $f_0 \in \mathcal{X}$, where $\pi_{f_0}(\mathcal{E}) = 0$, a contradiction. Hence there exists a pre-frame operator $\hat{\mathfrak{S}}_0$, such that $\left(\left\{ \frac{1}{\psi_k^*(f)} f_k^* - \frac{1}{\psi_{k+1}^*(f)} f_{k+1}^* \right\}, \hat{\mathfrak{S}}_0 \right)$ is a Banach frame for $\mathcal{X}$.

5. Conclusion

The notion of $\mathfrak{I}$-reconstruction property is proposed in section 3 and its characterization in terms of frames in Banach spaces is given. More precisely, Proposition 3.5 characterize $\mathfrak{I}$-reconstruction property in terms of existence of pre-frame operator but in a contrapositive way. This situation is same as in electrodynamics, where there is a game of movement of electron but charge given to electron is negative! Moreover, the action of a functional from $\mathcal{X}^{**}$ on a given system from $\mathcal{X}^*$ decide the existence of pre-frame operator associated with certain system. This looks like *dynamics of reconstruction property*. By motivation from the theory of frames for Hilbert spaces which control the perturbed system associated with a signal in space(after its transmission), we extend the said situation to Banach spaces.

More precisely, Proposition 4.2 control the situation in abstract setting via non-existence of pre-frame operator. Finally, the notion of local reconstruction system is proposed and its utility in complicated stability of associated Banach frames is reflected in Proposition 4.7.

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