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The Expected Discounted Tax Payments on Dual Risk Model under a Dividend Threshold*

Zhang Liu[1], Aili Zhang[2#], Canhua Li[1]
[1]College of Sciences, Jiangxi Agricultural University, Nanchang, China
[2]School of Mathematics and Statistics, Nanjing Audit University, Nanjing, China
Email: liuzhang1006@163.com, #zhangailiwh@126.com

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ABSTRACT

In this paper, we consider the dual risk model in which periodic taxation are paid according to a loss-carry-forward system and dividends are paid under a threshold strategy. We give an analytical approach to derive the expression of $g_\delta(u)$ (*i.e*. the Laplace transform of the first upper exit time). We discuss the expected discounted tax payments for this model and obtain its corresponding integro-differential equations. Finally, for Erlang (2) inter-innovation distribution, closed-form expressions for the expected discounted tax payments are given.

Keywords: Dual Risk Model; Expected Discounted Tax Payments; Dividend; Threshold Strategy

1. Introduction

Consider the surplus process of an insurance portfolio

$$R(t) = u - ct + S(t) \tag{1.1}$$

which is dual to the classical Cramér-Lundberg model in risk theory that describes the surplus at time t, where $u \geq 0$ is the initial capital, the constant $c > 0$ is the rate of expenses, and $S(t) := \sum_{i=1}^{N(t)} Y_i$ is aggregate profits process with the innovation number process $N(t)$ being a renewal process whose inter-innovation times $T_i\,(i = 1, 2, \cdots)$ have common distribution F. We also assume that the innovation sizes $\{Y_i, i \geq 1\}$, independent of $\{T_i, i \geq 1\}$, forms a sequence of i.i.d. exponentially distributed random variables with exponential parameter $\beta\,(>0)$. There are many possible interpretations for this model. For example, we can treat the surplus as the amount of capital of a business engaged in research and development. The company pays expenses for research, and occasional profit of random amounts arises according to a Poisson process.

Due to its practical importance, the issue of dividend strategies has received remarkable attention in the literature. De Finetti [1] considered the surplus of the company that is a discrete process and showed that the optimal strategy to maximize the expectation of the discounted dividends must be a barrier strategy. Since then, researches on dividend strategies has been carried out extensively. For some related results, the reader may consult the following publications therein: Bühlmann [2], Gerber [3], Gerber and Shiu [4,5], Lin *et al*. [6], Lin and pavlova [7], Dickson and Waters [8], Albrecher *et al*. [9], Dong *et al*. [10] and Ng [11]. Recently, quite a few interesting papers have been discussing risk models with tax payments of loss carry forward type. Albrecher *et al*. [12] investigated how the loss-carry forward tax payments affect the behavior of the dual process (1.1) with general inter-innovation times and exponential innovation sizes. More results can be seen in Albrecher and Hipp [13], Albrecher *et al*. [14], Ming *et al*. [15], Wang and Hu [16] and Liu *et al*. [17,18].

Now, we consider the model (1.1) under the additional assumption that tax payments are deducted according to a loss-carry forward system and dividends are paid under a threshold strategy. We rewrite the objective process as $\{R_{\gamma,b}(t), t \geq 0\}$. that is, the insurance company pays tax at rate $\gamma \in [0,1)$ on the excess of each new record high of the surplus over the previous one; at the same time, dividends are paid at a constant rate α whenever the surplus of an insurance portfolio is more than b and otherwise no dividends are paid. Then the surplus process of our model $\{R_{\gamma,b}(t), t \geq 0\}$ can be expressed as

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#Corresponding author.

$$\mathrm{d}R_{\gamma,b}(t)=\begin{cases}-c\mathrm{d}t+\mathrm{d}S(t)\mathbf{1}_{\left\{R_{\gamma,b}(t^-)+\mathrm{d}S(t)<\max\limits_{0\le s<t}R_{\gamma,b}(s)\right\}}+(1-\gamma)\left(R_{\gamma,b}(t^-)+\mathrm{d}S(t)-\max\limits_{0\le s<t}R_{\gamma,b}(s)\right)\\ \times\mathbf{1}_{\left\{R_{\gamma,b}(t^-)+\mathrm{d}S(t)\ge\max\limits_{0\le s<t}R_{\gamma,b}(s)\right\}}, & R_{\gamma,b}(t)\ge b\\ -(c-\alpha)\mathrm{d}t+\mathrm{d}S(t)\mathbf{1}_{\left\{R_{\gamma,b}(t^-)+\mathrm{d}S(t)<\max\limits_{0\le s<t}R_{\gamma,b}(s)\right\}}+(1-\gamma)\left(R_{\gamma,b}(t^-)+\mathrm{d}S(t)-\max\limits_{0\le s<t}R_{\gamma,b}(s)\right)\\ \times\mathbf{1}_{\left\{R_{\gamma,b}(t^-)+\mathrm{d}S(t)\ge\max\limits_{0\le s<t}R_{\gamma,b}(s)\right\}}, & R_{\gamma,b}(t)<b\end{cases}\tag{1.2}$$

for $t\ge 0$, with $R_{\gamma,b}(0)=u$. where $\mathbf{1}_{\{A\}}$ is the indicator function of event A and $R_{\gamma,b}(t^-)$ is the surplus immediately before time t.

For practical consideration, we assume that the positive safety loading condition

$$c<E(Y_1)/E(T_1),\tag{1.3}$$

holds all through this paper. The time of ruin is defined as $T_{\gamma,b}=\inf\{t\ge 0:R_{\gamma,b}(t)\le 0\}$ with $T_{\gamma,b}=\infty$ if $R_{\gamma,b}(t)>0$ for all $t\ge 0$.

For initial surplus $u>0$, let $D=\int_0^{T_{\gamma,b}}\mathrm{e}^{-\delta t}\mathrm{d}D(t)$ be the present value of all dividends until ruin, and $\delta>0$ is the discount factor. Denote by $V_\gamma(u,b)$ the expectation of D, that is,

$$V_\gamma(u,b)=E\left[D\middle|R_{\gamma,b}(0)=u\right].\tag{1.4}$$

It needs to be mentioned that we shall drop the subscript γ whenever γ is zero.

The rest of this paper is organized as follows. In Section 2, We derive the expression of $g_\delta(u)$ (*i.e.* the Laplace transform of the first upper exit time). We also discuss the expected discounted tax payments for this model and obtain its satisfied integro-differential equations. Finally, for Erlang (2) inter-innovation distribution, closed-form expressions for the the expected discounted tax payments are given.

2. Main Results and Proofs

Let $g(u,u_0):=E_u\left[\mathrm{e}^{-\delta\tau(u,0,u_0)}\right]$ denote the Laplace transform of the upper exit time $\tau(u,0,u_0)$, which is the time until the risk process $\{R_b(t),t>0\}$ starting with initial capital $u(\le u_0)$ up-crosses the level $u_0(\ge b)$ for the first time without leading to ruin before that event. In particular, $g_0(u,u_0):=\lim\limits_{\delta\downarrow 0}(u,u_0)$ is the probability that the process $\{R_b(t),t>0\}$ up-crosses the level $u_0(\ge b)$ before ruin.

For general innovation waiting times distribution, one can derive the integral equations for $g(u,u_0)$. When $u<b$,

$$g(u,u_0)=\int_0^{u/(c-\alpha)}\mathrm{e}^{-\delta t}f_{T_1}(t)\mathrm{d}t\left\{\int_0^{u_0-u+(c-\alpha)t}g(u-(c-\alpha)t+y,u_0)\beta\mathrm{e}^{-\beta y}\mathrm{d}y+\int_{u_0-u+(c-\alpha)t}^{\infty}\beta\mathrm{e}^{-\beta y}\mathrm{d}y\right\}.\tag{2.1}$$

When $b\le u\le u_0$,

$$\begin{aligned}&g(u,u_0)\\&=\int_0^{(u-b)/c}\mathrm{e}^{-\delta t}f_{T_1}(t)\mathrm{d}t\left\{\int_0^{u_0-u+ct}g(u-ct+y,u_0)\beta\mathrm{e}^{-\beta y}\mathrm{d}y+\mathrm{e}^{-\beta(u_0-u+ct)}\right\}+\int_{(u-b)/c}^{(u-b)/c+b/(c-\alpha)}\mathrm{e}^{-\delta t}f_{T_1}(t)\mathrm{d}t\\&\times\left\{\int_0^{u_0-b+(c-\alpha)(t-(u-b)/c)}g(u-ct+y,u_0)\beta\mathrm{e}^{-\beta y}\mathrm{d}y+\mathrm{e}^{-\beta(u_0-b+(c-\alpha)(t-(u-b)/c))}\right\}.\end{aligned}\tag{2.2}$$

It follows from Equation (2.1) and from Equation (2.2) that $g(u,u_0)$ is continuous on $(0,u_0)$ as a function of u and that

$$g(0^+,u_0)=0,g(b^+,u_0)=g(b^-,u_0).\tag{2.3}$$

For certain distributions F_{T_1}, one can derive integro-differential equations for $g(u,u_0)$ and $V(u,b)$. Let us assume that the i.i.d innovation waiting times have a common generalized Erlang (n) distribution, *i.e.* the T_i's are distributed as the sum of n independent and exponentially distributed r.v.'s $S_n:=\eta_1+\eta_2+\cdots+\eta_n$ with η_i having exponential parameters $\lambda_i>0$.

The following theorem 2.1 gives the integro-differential equations for $g(u,u_0)$ when T_i's have a generalized Erlang(n) distribution.

Theorem 2.1 Let **I** and **D** denote the identity operator and differentiation operator respectively. Then $g(u,u_0)$ satisfies the following equation for $0<u<b$

$$\prod_{k=1}^{n}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_k}\mathbf{D}\right]g(u,u_0) = \int_0^{u_0-u} g(u+y,u_0)\beta e^{-\beta y}dy + e^{-\beta(u_0-u)}, \tag{2.4}$$

and

$$\prod_{k=1}^{n}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c}{\lambda_k}\mathbf{D}\right]g(u,u_0) = \int_0^{u_0-u} g(u+y,u_0)\beta e^{-\beta y}dy + e^{-\beta(u_0-u)}, \tag{2.5}$$

for $u \ge b$.

Proof First, we rewrite $g(u,u_0)$ as $g_k(u,u_0)$ when $T_i \overset{d}{=} S_n - S_{k-1}$ with $S_0 = 0$ in the surplus process (1.2) with $\gamma = 0$. Thus, we have $g_1(u,u_0) = g(u,u_0)$. When $0 < u < b$,

$$g_k(u,u_0) = \int_0^{u/(c-\alpha)} \lambda_k e^{-(\lambda_k+\delta)t} g_{k+1}(u-(c-\alpha)t,u_0)dt, \tag{2.6}$$

for $k = 1,2,\cdots,n-1$, and

$$g_n(u,u_0) = \int_0^{u/(c-\alpha)} \lambda_n e^{-(\lambda_n+\delta)t}dt\left\{\int_0^{u_0-u+(c-\alpha)t} g(u-(c-\alpha)t+y,u_0)\beta e^{-\beta y}dy + \int_{u_0-u+(c-\alpha)t}^{\infty}\beta e^{-\beta y}dy\right\}. \tag{2.7}$$

By changing variables in from Equation (2.6) and from Equation (2.7), we have for $0 < u < b$,

$$g_k(u,u_0) = \int_0^u \frac{\lambda_k}{c-\alpha} e^{-(\lambda_k+\delta)\frac{u-x}{c-\alpha}} g_{k+1}(x,u_0)dx, \tag{2.8}$$

for $k = 1,2,\cdots,n-1$, and

$$g_n(u,u_0) = \int_0^u \frac{\lambda_n}{c-\alpha} e^{-(\lambda_n+\delta)\frac{u-x}{c-\alpha}}dx \cdot\left[\int_0^{u_0-x} g(x+y,u_0)\beta e^{-\beta y}dy + \int_{u_0-x}^{\infty}\beta e^{-\beta y}dy\right]. \tag{2.9}$$

Then, differentiating both sides of from Equation (2.8) and from Equation (2.9) with respect to u, one gets

$$\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_k}\mathbf{D}\right]g_k(u,u_0) = g_{k+1}(u,u_0), \tag{2.10}$$

for $k = 1,2,\cdots,n-1$, and

$$\left[\left(1+\frac{\delta}{\lambda_n}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_n}\mathbf{D}\right]g_n(u,u_0) = \int_0^{u_0-u} g(u+y,u_0)\beta e^{-\beta y}dy + e^{-\beta(u_0-u)}. \tag{2.11}$$

Using from Equation (2.10) and from Equation (2.11), we can derive from Equation (2.4) for $g(u,u_0)$ on $(0,b)$.

Similar to from Equation (2.6) and Equation (2.7), we have for $u \ge b$

$$g_k(u,u_0) = \int_0^{(u-b)/c} \lambda_k e^{-(\lambda_k+\delta)t} g_{k+1}(u-ct,u_0)dt + \int_{(u-b)/c}^{(u-b)/c+b/(c-\alpha)} \lambda_k e^{-(\lambda_k+\delta)t} g_{k+1}\big(b-(c-\alpha)(t-(u-b)/c),u_0\big)dt, \tag{2.12}$$

for $k = 1,2,\cdots,n-1$, and

$$\begin{aligned} g_n(u,u_0) &= \int_0^{(u-b)/c} \lambda_n e^{-(\lambda_n+\delta)t}dt\left\{\int_0^{u_0-u+ct} g(u-ct+y,u_0)\beta e^{-\beta y}dy + e^{-\beta(u_0-u+ct)}\right\} + \int_{(u-b)/c}^{(u-b)/c+b/(c-\alpha)} \lambda_n e^{-(\lambda_n+\delta)t}dt \\ &\times\left\{\int_0^{u_0-b+(c-\alpha)(t-(u-b)/c)} g\big(b-(c-\alpha)(t-(u-b)/c)+y,u_0\big)\beta e^{-\beta y}dy + e^{-\beta(u_0-b+(c-\alpha)(t-(u-b)/c))}\right\}. \end{aligned} \tag{2.13}$$

Again, by changing variables in Equation (2.12) and Equation (2.13) and then differentiating them with respect to u, we obtain for $u \ge b$

$$\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c}{\lambda_k}\mathbf{D}\right]g_k(u,u_0) = g_{k+1}(u,u_0), \tag{2.14}$$

for $k = 1,2,\cdots,n-1$, and

$$\left[\left(1+\frac{\delta}{\lambda_n}\right)\mathbf{I}+\frac{c}{\lambda_n}\mathbf{D}\right]g_n(u,u_0) = \int_0^{u_0-u} g(u+y,u_0)\beta e^{-\beta y}dy + e^{-\beta(u_0-u)}. \tag{2.15}$$

Using Equation (2.14) and Equation (2.15), we obtain Equation (2.5) for $g(u,u_0)$ on $[b,\infty)$.□

It needs to be mentioned that from the proof of Lemma 2.1, we know that

$$g_k\left(0^+,u_0\right)=0,g_k\left(b^+,u_0\right)=g_k\left(b^-,u_0\right),k=2,3,\cdots,n.$$

Therefore, Equations (2.10), (2.11), (2.14) and (2.15) yield

$$\begin{aligned}&\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathbf{I}+\frac{c}{\lambda_i}\mathbf{D}\right]g\left(b^+,u_0\right)\\&=\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_i}\mathbf{D}\right]g\left(b^-,u_0\right),\\&k=1,2,\cdots,n.\end{aligned}\tag{2.16}$$

Remark 2.1 Using a similar argument to the one used in Lemma 2.1, one can get that when the innovation waiting times follow a common generalized Erlang(n) distribution, the expected discounted dividend payments $V(u,b)$ satisfies the following integro-differential equation (see Liu *et al*. [17]).

$$\begin{aligned}&\prod_{k=1}^{n}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_k}\mathbf{D}\right]V(u,b)\\&=\int_0^{\infty}V(u+y,b)\beta e^{-\beta y}dy,0<u<b\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&\prod_{k=1}^{n}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I}+\frac{c}{\lambda_k}\mathbf{D}\right]V(u,b)\\&=\int_0^{\infty}V(u+y,b)\beta e^{-\beta y}dy+B_n,u\ge b\end{aligned}\tag{2.18}$$

with

$$B_k=\sum_{i=1}^{k}\frac{\alpha}{\lambda_i+\delta}\prod_{j=i+1}^{k}\left(1+\frac{\delta}{\lambda_j}\right),k=1,2,\cdots,n.\tag{2.19}$$

In addition, the boundary conditions for $V(u,b)$ are as follows:

$$\begin{aligned}&\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_i}\mathbf{D}\right]V\left(b^-,b\right)\\&=\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathbf{I}+\frac{c}{\lambda_i}\mathbf{D}\right]V\left(b^+,b\right)-B_k,\\&k=1,2,\cdots,n,\end{aligned}\tag{2.20}$$

$$\begin{aligned}&\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathbf{I}+\frac{c-\alpha}{\lambda_i}\mathbf{D}\right]V\left(0^+,b\right)=0,\\&k=1,2,\cdots,n-1,\end{aligned}\tag{2.21}$$

with Equation (2.19).

With the preparations made above, we can now derive analytic expressions of the expected n-th moment of the accumulated discounted tax payments for the surplus process $\{R_{\gamma,b}(t),t\ge0\}$. We claim that the process $\{R_{\gamma,b}(t),t\ge0\}$ shall up-cross the initial surplus level u at least once to avoid ruin.

Now, let

$$g_\delta(u):=E_u\left[e^{-\delta\tau_u}\right]\tag{2.22}$$

denote the Laplace transform of the first upper exit time τ_u, which is the time until the risk process $\{R_b(t),t>0\}$ starting with initial capital u reaches a new record high for the first time without leading to ruin before that event. In particular, $g_0(u):=\lim_{\delta\downarrow0}g_\delta(u)$ is the probability that the process $\{R_b(t),t>0\}$ reaches a new record high before ruin. Then the closed-form expression of the quantity $g_\delta(u)$ can be calculated as follows.

When $u\ge b,g_\delta(u)=g(u,u)$. When $0<u<b$, using a simple sample path argument, we immediately have,

$$V(u,u)=g_\delta(u)\int_0^{\infty}\beta e^{-\beta y}V(u+y,u)dy,$$

or, equivalently

$$g_\delta(u)=\frac{V(u,u)}{\int_0^{\infty}\beta e^{-\beta y}V(u+y,u)dy}.\tag{2.23}$$

Let $\sigma_0=0$ and define

$$\sigma_n=\inf\left\{t>\sigma_{n-1}:R_{\gamma,b}(t)\ge\max_{0\le s<t}R_{\gamma,b}(s)\right\},\tag{2.24}$$

to be the n-th taxation time point. Thus,

$$\begin{aligned}&M_n(u,b):=E_u\left[\left(D_{\gamma,\delta}\right)^n\right]\\&:=E_u\left[\left(\frac{\gamma}{1-\gamma}\sum_{n=1}^{\infty}e^{-\delta\sigma_n}\left(R_{\gamma,b}(\sigma_n)-R_{\gamma,b}(\sigma_{n-1})\right)\mathbf{1}_{\{\sigma_n<T_{\gamma,b}\}}\right)^n\right]\end{aligned}\tag{2.25}$$

denotes the n-th moment of the accumulated discounted tax payments over the life time of the surplus process $\{R_{\gamma,b}(t),t\ge0\}$.

We will consider a recursive formula of $M_n(u,b)$ in the following theorem 2.2.

Theorem 2.2 When $0<u<b$, we have

$$M_n(u,b)=\frac{n\gamma}{1-\gamma}g_{n\delta}(u)e^{-\frac{\beta}{1-\gamma}\int_u^b(1-g_{n\delta}(t))dt}\times\left(\int_b^{\infty}\frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)}e^{-\frac{\beta}{1-\gamma}\int_b^s(1-g_{n\delta}(t))dt}ds+\int_u^b\frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)}e^{\frac{\beta}{1-\gamma}\int_s^b(1-g_{n\delta}(t))dt}ds\right),\tag{2.26}$$

and when $u \ge b$, we have

$$M_n(u,b) = \frac{n\gamma}{1-\gamma} g_{n\delta}(u) \int_u^\infty \frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)} \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_u^s (1-g_{n\delta}(t))\mathrm{d}t} \mathrm{d}s. \tag{2.27}$$

Proof Given that the after-tax excess of the surplus level over u at time τ_u is exponentially distributed with mean $(1-\gamma)/\beta$ due to the memoryless property of the exponential distribution. By a probabilistic argument, one easily shows that for $u > 0$

$$\begin{aligned} M_n(u,b) &= g_{n\delta}(u) \int_0^\infty \frac{\beta}{1-\gamma} \mathrm{e}^{-\frac{\beta}{1-\gamma}x} E\left[\left(D_{\gamma,\delta}(u+x) + \frac{\gamma}{1-\gamma}x\right)^n\right] \mathrm{d}x \\ &= g_{n\delta}(u) \int_u^\infty \frac{\beta}{1-\gamma} \mathrm{e}^{-\frac{\beta}{1-\gamma}(x-u)} E\left[\left(D_{\gamma,\delta}(x) + \frac{\gamma}{1-\gamma}(x-u)\right)^n\right] \mathrm{d}x \end{aligned} \tag{2.28}$$

Differentiating with respect to u yields

$$\begin{aligned} M_n'(u,b) &= \left(\frac{g_{n\delta}'(u)}{g_{n\delta}(u)} + \frac{\beta}{1-\gamma}(1-g_{n\delta}(u))\right) M_n(u,b) \\ &\quad - \frac{n\gamma}{1-\gamma} g_{n\delta}(u) \\ &\quad \cdot \left[\int_u^\infty \frac{\beta}{1-\gamma} \mathrm{e}^{-\frac{\beta}{1-\gamma}(x-u)} E\left[\left(D_{\gamma,\delta}(x) + \frac{\gamma}{1-\gamma}(x-u)\right)^{n-1}\right] \mathrm{d}x\right] \\ &= \left(\frac{g_{n\delta}'(u)}{g_{n\delta}(u)} + \frac{\beta}{1-\gamma}(1-g_{n\delta}(u))\right) M_n(u,b) \\ &\quad - \frac{n\gamma}{1-\gamma} \frac{g_{n\delta}(u)}{g_{(n-1)\delta}(u)} M_{n-1}(u,b). \end{aligned} \tag{2.29}$$

When $0 < u < b$, we have

$$M_n(u,b) = g_{n\delta}(u) \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_u^b (1-g_{n\delta}(t))\mathrm{d}t} \cdot \left(C + \frac{n\gamma}{1-\gamma}\int_u^b \frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)} \mathrm{e}^{\frac{\beta}{1-\gamma}\int_s^b (1-g_{n\delta}(t))\mathrm{d}t} \mathrm{d}s\right). \tag{2.30}$$

When $u \ge b$, the general solution of Equation (3.20) can be expressed as

$$M_n(u,b) = g_{n\delta}(u) \left(\frac{M_n(\infty)}{g_{n\delta}(\infty)} \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_u^\infty (1-g_{n\delta}(t))\mathrm{d}t} + \frac{n\gamma}{1-\gamma}\int_u^\infty \frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)} \mathrm{e}^{\frac{\beta}{1-\gamma}\int_u^s (1-g_{n\delta}(t))\mathrm{d}t} \mathrm{d}s\right) \tag{2.31}$$

Due to the facts that $M_n(\infty) < \infty$ and $0 < g_{n\delta}(\infty) < \infty$, we have for $u \ge b$

$$M_n(u,b) = \frac{n\gamma}{1-\gamma} g_{n\delta}(u) \int_u^\infty \frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)} \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_u^s (1-g_{n\delta}(t))\mathrm{d}t} \mathrm{d}s. \tag{2.32}$$

Now, it remains to determine the unknown constant C in Equation (3.20). The continuity of $M(u,b)$ on b and Equation (3.22) lead to

$$C = \frac{n\gamma}{1-\gamma} \int_b^\infty \frac{M_{n-1}(s)}{g_{(n-1)\delta}(s)} \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_b^s (1-g_{n\delta}(t))\mathrm{d}t} \mathrm{d}s. \tag{2.33}$$

Plugging Equation (2.33) into Equation (2.30), we arrive at Equation (2.26). □

The special case $n = 1$ leads to an expression for the expected discounted total sum of tax payments over the life time of the risk process

$$M_1(u,b) = \frac{\gamma}{1-\gamma} g_\delta(u) \int_u^\infty \mathrm{e}^{-\frac{\beta}{1-\gamma}\int_u^s (1-g_\delta(t))\mathrm{d}t} \mathrm{d}s. \tag{2.34}$$

for all $u > 0$.

3. Explicit Results for Erlang(2) Innovation Waiting Times

In this section, we assume that W_i's are Erlang(2) distributed with parameters λ_1 and λ_2. We also assume that $\lambda_1 < \lambda_2$ (without loss of generality).

Example 3.1 Note that

$$\begin{aligned} &(\beta\mathbf{I} - \mathbf{D})\left(\int_0^{u_0-u} g(u+y,u_0)\beta\mathrm{e}^{-\beta y}\mathrm{d}y + \mathrm{e}^{-\beta(u_0-u)}\right) \\ &= \beta g(u,u_0). \end{aligned} \tag{3.1}$$

Applying the operator $(\beta\mathbf{I} - \mathbf{D})$ to Equations (2.4) and (2.5) gives

$$\begin{aligned} &(\beta\mathbf{I} - \mathbf{D})\prod_{k=1}^n \left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I} + \frac{c-\alpha}{\lambda_k}\mathbf{D}\right] g(u,u_0) = \beta g(u,u_0), \\ &0 \le u < b, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &(\beta\mathbf{I} - \mathbf{D})\prod_{k=1}^n \left[\left(1+\frac{\delta}{\lambda_k}\right)\mathbf{I} + \frac{c}{\lambda_k}\mathbf{D}\right] g(u,u_0) = \beta g(u,u_0), \\ &u \ge b, \end{aligned} \tag{3.3}$$

The characteristic equation for Equation (3.2) is

$$(\beta - r)\prod_{k=1}^2 \left[\left(1+\frac{\delta}{\lambda_k}\right) + \frac{c-\alpha}{\lambda_k} r\right] = \beta \tag{3.4}$$

without loss of generality, we assume that $\lambda_1 < \lambda_2$. We know that Equation (3.4) has three real roots, say r_1, r_2 and r_3 which satisfies

$$\beta > r_1 > 0 > r_2 > -\frac{\lambda_1+\delta}{c-\alpha} > -\frac{\lambda_2+\delta}{c-\alpha} > r_3 > -\frac{\lambda_1+\delta}{c-\alpha} - \frac{\lambda_2+\delta}{c-\alpha}.$$

With c replace $c-\alpha$ in Equation (3.4), we get the characteristic equation of Equation (3.3), whose roots are denoted by r_4, r_5 and r_6 with

$$\beta > r_4 > 0 > r_5 > -\frac{\lambda_1+\delta}{c} > -\frac{\lambda_2+\delta}{c} > r_6 > -\frac{\lambda_1+\delta}{c} - \frac{\lambda_2+\delta}{c}$$

Thus, we have

$$g(u,u_0) = c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}, 0 \le u < b, \tag{3.5}$$

and

$$g(u,u_0) = c_4 e^{r_4 u} + c_5 e^{r_5 u} + c_6 e^{r_6 u}, u \ge b, \tag{3.6}$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants. To determine the arbitrary constants, we insert Equations (3.5) and (3.6) into Equation (2.3) and obtain

$$c_1 + c_2 + c_3 = 0, \tag{3.7}$$

and

$$\begin{aligned} &e^{r_1 b} c_1 + e^{r_2 b} c_2 + e^{r_3 b} c_3 - e^{r_4 b} c_4 \\ &-e^{r_5 b} c_5 - e^{r_6 b} c_6 = 0. \end{aligned} \tag{3.8}$$

Apply Equation (2.10) together with Equations (2.3) and (3.5) when $k = 1$, we get

$$r_1 c_1 + r_2 c_2 + r_3 c_3 = 0. \tag{3.9}$$

Insert Equation (3.5) into Equation (2.4), we have

$$\frac{\beta e^{r_1 u_0}}{\beta - r_1} c_1 + \frac{\beta e^{r_2 u_0}}{\beta - r_2} c_2 + \frac{\beta e^{r_3 u_0}}{\beta - r_3} c_3 = 1. \tag{3.10}$$

In addition, plugging Equations (3.5) and (3.6) into Equation (2.16) yields

$$\begin{aligned} &(c-\alpha) r_1 e^{r_1 b} c_1 + (c-\alpha) r_2 e^{r_2 b} c_2 + (c-\alpha) r_3 e^{r_3 b} c_3 \\ &-c r_4 e^{r_4 b} c_4 - c r_5 e^{r_5 b} c_5 - c r_6 e^{r_6 b} c_6 = 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{\beta e^{r_1 b}}{\beta - r_1} c_1 + \frac{\beta e^{r_2 b}}{\beta - r_2} c_2 + \frac{\beta e^{r_3 b}}{\beta - r_3} c_3 - \frac{\beta e^{r_4 b}}{\beta - r_4} c_4 \\ &-\frac{\beta e^{r_5 b}}{\beta - r_5} c_5 - \frac{\beta e^{r_6 b}}{\beta - r_6} c_6 = 0. \end{aligned} \tag{3.12}$$

Some calculations give

$$\begin{aligned}
c_1 &= \frac{(r_3 - r_2)/\beta}{\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0}}, \\
c_2 &= \frac{(r_1 - r_3)/\beta}{\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0}}, \\
c_3 &= \frac{(r_2 - r_1)/\beta}{\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0}}, \\
c_4 &= \frac{(r_1 - r_2) e^{r_3 b} \theta_1(r_3, r_5, r_6) + (r_3 - r_1) e^{r_2 b} \theta_1(r_2, r_5, r_6) + (r_2 - r_3) e^{r_1 b} \theta_1(r_1, r_5, r_6)}{-c\beta e^{r_4 b} \left(\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0} \right) \left(\frac{r_6 - r_5}{\beta - r_4} + \frac{r_4 - r_6}{\beta - r_5} + \frac{r_5 - r_4}{\beta - r_6} \right)}, \\
c_5 &= \frac{(r_2 - r_1) e^{r_3 b} \theta_1(r_3, r_4, r_6) + (r_1 - r_3) e^{r_2 b} \theta_1(r_2, r_4, r_6) + (r_3 - r_2) e^{r_1 b} \theta_1(r_1, r_4, r_6)}{-c\beta e^{r_5 b} \left(\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0} \right) \left(\frac{r_6 - r_5}{\beta - r_4} + \frac{r_4 - r_6}{\beta - r_5} + \frac{r_5 - r_4}{\beta - r_6} \right)}, \\
c_6 &= \frac{(r_1 - r_2) e^{r_3 b} \theta_1(r_3, r_4, r_5) + (r_3 - r_1) e^{r_2 b} \theta_1(r_2, r_4, r_5) + (r_2 - r_3) e^{r_1 b} \theta_1(r_1, r_4, r_5)}{-c\beta e^{r_6 b} \left(\frac{r_3 - r_2}{\beta - r_1} e^{r_1 u_0} + \frac{r_1 - r_3}{\beta - r_2} e^{r_2 u_0} + \frac{r_2 - r_1}{\beta - r_3} e^{r_3 u_0} \right) \left(\frac{r_6 - r_5}{\beta - r_4} + \frac{r_4 - r_6}{\beta - r_5} + \frac{r_5 - r_4}{\beta - r_6} \right)},
\end{aligned} \tag{3.13}$$

with

$$\theta_1\left(r_i,r_j,r_k\right)=\frac{c\left(r_k-r_j\right)}{\beta-r_i}+\frac{(c-\alpha)r_i-cr_k}{\beta-r_j}+\frac{cr_j-(c-\alpha)r_i}{\beta-r_k} \quad (3.14)$$

Remark 3.1 Now, we give the explicit results for $g_\delta(u)$. By Equations (3.6) and (3.13), we have for $u\ge b$

$$g_\delta(u)=g(u,u)=\frac{l_4(b)\mathrm{e}^{r_4u}+l_5(b)\mathrm{e}^{r_5u}+l_6(b)\mathrm{e}^{r_6u}}{\frac{r_3-r_2}{\beta-r_1}\mathrm{e}^{r_1u}+\frac{r_1-r_3}{\beta-r_2}\mathrm{e}^{r_2u}+\frac{r_2-r_1}{\beta-r_3}\mathrm{e}^{r_3u}}, \quad (3.15)$$

with

$$l_4(b)=\frac{(r_1-r_2)\mathrm{e}^{r_3b}\theta_1(r_3,r_5,r_6)+(r_3-r_1)\mathrm{e}^{r_2b}\theta_1(r_2,r_5,r_6)+(r_2-r_3)\mathrm{e}^{r_1b}\theta_1(r_1,r_5,r_6)}{-c\beta\mathrm{e}^{r_4b}\left(\frac{r_6-r_5}{\beta-r_4}+\frac{r_4-r_6}{\beta-r_5}+\frac{r_5-r_4}{\beta-r_6}\right)},$$

$$l_5(b)=\frac{(r_2-r_1)\mathrm{e}^{r_3b}\theta_1(r_3,r_4,r_6)+(r_1-r_3)\mathrm{e}^{r_2b}\theta_1(r_2,r_4,r_6)+(r_3-r_2)\mathrm{e}^{r_1b}\theta_1(r_1,r_4,r_6)}{-c\beta\mathrm{e}^{r_5b}\left(\frac{r_6-r_5}{\beta-r_4}+\frac{r_4-r_6}{\beta-r_5}+\frac{r_5-r_4}{\beta-r_6}\right)}, \quad (3.16)$$

$$l_6(b)=\frac{(r_1-r_2)\mathrm{e}^{r_3b}\theta_1(r_3,r_4,r_5)+(r_3-r_1)\mathrm{e}^{r_2b}\theta_1(r_2,r_4,r_5)+(r_2-r_3)\mathrm{e}^{r_1b}\theta_1(r_1,r_4,r_5)}{-c\beta\mathrm{e}^{r_6b}\left(\frac{r_6-r_5}{\beta-r_4}+\frac{r_4-r_6}{\beta-r_5}+\frac{r_5-r_4}{\beta-r_6}\right)}.$$

For $0<u<b$, using the explicit expressions of $V(b,b)$ in Liu *et al*. [17], we obtain

$$g_\delta(u)=\frac{V(u,u)}{\int_0^\infty\beta\mathrm{e}^{-\beta y}V(u+y,u)\mathrm{d}y}=\frac{\left(\frac{(cr_5+\delta)r_6}{\beta-r_6}-\frac{(cr_6+\delta)r_5}{\beta-r_5}\right)\left((r_3-r_2)\mathrm{e}^{r_1u}+(r_1-r_3)\mathrm{e}^{r_2u}+(r_2-r_1)\mathrm{e}^{r_3u}\right)}{l_1(r_3-r_2)\mathrm{e}^{r_1u}+l_2(r_1-r_3)\mathrm{e}^{r_2u}+l_3(r_2-r_1)\mathrm{e}^{r_3u}} \quad (3.17)$$

with

$$l_i=\frac{\beta\theta_3(i,5)}{\beta-r_6}-\frac{\beta\theta_3(i,6)}{\beta-r_5}+\theta_2(i,5,6),i=1,2,3, \quad (3.18)$$

where

$$\theta_2\left(r_i,r_j,r_k\right)=\frac{c\left(r_k-r_j\right)r_i}{\beta-r_i}+\frac{\left((c-\alpha)r_i-cr_k\right)r_j}{\beta-r_j}+\frac{\left(cr_j-(c-\alpha)r_i\right)r_k}{\beta-r_k},$$
$$1\le i<j<k\le 6,$$

and

$$\theta_3\left(r_i,r_j\right)=\frac{\left(cr_j+\delta\right)r_i}{\beta-r_i}-\frac{\left((c-\alpha)r_i+\delta\right)r_j}{\beta-r_j},1\le i<j\le 6.$$

We point out that when the innovation times are exponentially distributed, one can follow the same steps to get the explicit expressions of $g_\delta(u)$, which coincide with the results in Albrecher *et al.* (2008).

Example 3.2 (The expected discounted tax payments.) Following from Equation (2.34) of Theorem 2.2 and Remark 3.1, we have for $0<u<b$,

$$
\begin{aligned}
M_1(u,b)={}&\frac{\gamma}{1-\gamma}\frac{\left(\frac{(cr_5+\delta)r_6}{\beta-r_6}-\frac{(cr_6+\delta)r_5}{\beta-r_5}\right)\left((r_3-r_2)\mathrm{e}^{r_1u}+(r_1-r_3)\mathrm{e}^{r_2u}+(r_2-r_1)\mathrm{e}^{r_3u}\right)}{l_1(r_3-r_2)\mathrm{e}^{r_1u}+l_2(r_1-r_3)\mathrm{e}^{r_2u}+l_3(r_2-r_1)\mathrm{e}^{r_3u}}\\
&\times\int_u^b\exp\left\{-\frac{\beta}{1-\gamma}\int_u^s\left(1-\frac{\left(\frac{(cr_5+\delta)r_6}{\beta-r_6}-\frac{(cr_6+\delta)r_5}{\beta-r_5}\right)\left((r_3-r_2)\mathrm{e}^{r_1t}+(r_1-r_3)\mathrm{e}^{r_2t}+(r_2-r_1)\mathrm{e}^{r_3t}\right)}{l_1(r_3-r_2)\mathrm{e}^{r_1t}+l_2(r_1-r_3)\mathrm{e}^{r_2t}+l_3(r_2-r)_1\mathrm{e}^{r_3t}}\right)\mathrm{d}t\right\}\mathrm{d}s\\
&+\frac{\gamma}{1-\gamma}\frac{\left(\frac{(cr_5+\delta)r_6}{\beta-r_6}-\frac{(cr_6+\delta)r_5}{\beta-r_5}\right)\left((r_3-r_2)\mathrm{e}^{r_1u}+(r_1-r_3)\mathrm{e}^{r_2u}+(r_2-r_1)\mathrm{e}^{r_3u}\right)}{l_1(r_3-r_2)\mathrm{e}^{r_1u}+l_2(r_1-r_3)\mathrm{e}^{r_2u}+l_3(r_2-r_1)\mathrm{e}^{r_3u}}\\
&\times\int_b^\infty\exp\left\{-\frac{\beta}{1-\gamma}\int_u^b\left(1-\frac{\left(\frac{(cr_5+\delta)r_6}{\beta-r_6}-\frac{(cr_6+\delta)r_5}{\beta-r_5}\right)\left((r_3-r_2)\mathrm{e}^{r_1t}+(r_1-r_3)\mathrm{e}^{r_2t}+(r_2-r_1)\mathrm{e}^{r_3t}\right)}{l_1(r_3-r_2)\mathrm{e}^{r_1t}+l_2(r_1-r_3)\mathrm{e}^{r_2t}+l_3(r_2-r_1)\mathrm{e}^{r_3t}}\right)\mathrm{d}t\right\}\mathrm{d}s\\
&\times\exp\left\{-\frac{\beta}{1-\gamma}\int_b^\infty\left(1-\frac{l_4(b)\mathrm{e}^{r_4t}+l_5(b)\mathrm{e}^{r_5t}+l_6(b)\mathrm{e}^{r_6t}}{\frac{r_3-r_2}{\beta-r_1}\mathrm{e}^{r_1t}+\frac{r_1-r_3}{\beta-r_2}\mathrm{e}^{r_2t}+\frac{r_2-r_1}{\beta-r_3}\mathrm{e}^{r_3t}}\right)\mathrm{d}t\right\}.
\end{aligned}
\tag{3.19}
$$

And, for $u\ge b$, we have

$$
M_1(u,b)=\frac{\gamma}{1-\gamma}\frac{l_4(b)\mathrm{e}^{r_4u}+l_5(b)\mathrm{e}^{r_5u}+l_6(b)\mathrm{e}^{r_6u}}{\frac{r_3-r_2}{\beta-r_1}\mathrm{e}^{r_1u}+\frac{r_1-r_3}{\beta-r_2}\mathrm{e}^{r_2u}+\frac{r_2-r_1}{\beta-r_3}\mathrm{e}^{r_3u}}\int_u^\infty\exp\left\{\frac{-\beta}{1-\gamma}\int_u^s\left(1-\frac{l_4(b)\mathrm{e}^{r_4t}+l_5(b)\mathrm{e}^{r_5t}+l_6(b)\mathrm{e}^{r_6t}}{\frac{r_3-r_2}{\beta-r_1}\mathrm{e}^{r_1t}+\frac{r_1-r_3}{\beta-r_2}\mathrm{e}^{r_2t}+\frac{r_2-r_1}{\beta-r_3}\mathrm{e}^{r_3t}}\right)\mathrm{d}t\right\}\mathrm{d}s. \tag{3.20}
$$

Then we can get that when W_i's are Erlang (2) distributed with parameters λ_1 and λ_2, the expresses of $g_\delta(u)$ can be given by Equations (3.15) and (3.17) and the expected discounted tax payments can be given by Equation (3.20).

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