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New Generating Sets of the First Order Lane-Emden Differential Equations in *N*-Dimensional Radially Symmetric Polytropes

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ABSTRACT

In the present paper, two new generating sets, of homology invariant functions will be established. Moreover, by the aid of two independent homology invariant functions of each set we established the transformed first order Lane-Emden equation. The first equation for polytropic index $n \neq -1, \pm\infty$ depends on five free parameters, while the other equation is for, $n = \pm\infty$ and depends on three free parameters.

Keywords: Homology Theorem; Lane-Emden Differential Equations; Stellar Interior

1. Introduction

The reduction of the differential equations is probably the most challenging problem in dynamics and physics. A general interpretation of reducibility includes various transformations and changes the original problem not only along mathematical lines but also in a physical sense. Such transformations will be achieved using homology theorem.

Homology is a powerful tool used by mathematicians to study the properties of spaces and maps that are insensitive to small perturbations. It was first used in a topological sense by Henri Poincaré (1895) as a relation between manifolds mapped into a manifold. The homology group was further developed for computational purposes by several investigators [1-3]. Kaczynski *et al.* [4] presented the conceptual background for computational homology and indicated how homology can be used to study nonlinear dynamics.

The important consequence of the use of homology theorem, is that, if we can find two independent homology invariant functions, say *u* and *v*, then the Lane-Emden equation transformed to *u* and *v* variables is of order one. Moreover, homology invariant functions play important role in fitting up solutions at the surface of the composite stellar models [5].

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In the present paper, two new generating sets, of homology invariant functions will be established. Moreover, by the aid of two independent homology invariant functions of each set we established the transformed first order Lane-Emden equation. The first equation for polytropic index $n \neq -1, \pm\infty$ depends on five free parameters, while, the other equation is for, $n = \pm\infty$ and depends on three free parameters.

2. Lane-Emden Differential Equations

The basic equations for *N*-dimensional radially symmetric polytropes are the generalized Lane-Emden differential equations depending on the geometric index *N*, such that, *N* = 1 (slab), *N* = 2 (cylinder) and *N* = 3 (sphere), and the polytropic index *n*. These equations are given as [6]

$$\psi'' + (N-1)\psi'/\xi = \mp\psi^n; n \neq -1, \pm\infty, \tag{1}$$

$$\psi'' + (N-1)\psi'/\xi = \exp[-\psi]; n = \pm\infty, \tag{2}$$

where $r = \alpha\xi$,

$$\alpha = \begin{cases} \left[\pm(n+1)P_0/4\pi G\rho_0^2\right]^{1/2}; & n \neq -1, \pm\infty, \\ \left[K/4\pi G\rho_0\right]^{1/2}; & n = \pm\infty, \end{cases}$$

$\psi' = \mathrm{d}\psi/\mathrm{d}\xi$; $\psi'' = \mathrm{d}^2\psi/\mathrm{d}\xi^2$ and $N = 1, 2, 3, \cdots$.

The upper sign corresponds to values of polytropic index $-1<n<\infty$, the lower one to $-\infty<n<-1$. The special case $n=-1$ appears as limiting case of two polytropic sequences having $-1<n<\infty$ and $-\infty<n<-1$, respectively. Also, r is the radial distance, $\xi,\psi=\psi(\xi)$ are the Lane-Emden variables, K is the Boltzmann constant and G is the gravitational constant. The initial conditions of Equations (1) and (2) are;

$$\text{at } \xi=0,\begin{cases}\psi=1 \text{ and } \psi'=0; n\neq -1,\pm\infty,\\ \psi=0 \text{ and } \psi'=0; n=\pm\infty.\end{cases} \tag{3}$$

If these conditions are satisfied then P_0 and ρ_0 are just equal to the pressure and density at radial distance $r=0$.

3. The Homology Theorem and Homology Invariant Functions

3.1. Theorem

If $\psi(\xi)$ is a solution of the Lane-Emden Equation (1) or (2) then, $A^{2/(n-1)}\psi(A\xi)$, (A = constant) is also a solution of the of Equation (1) and $\theta(A\xi)-2\ln A$ is also a solution of the Equation (2) [6].

Thus, if one solution $\psi=\psi(\xi)$ of the Lane-Emden equation is known, we can derive a whole homologous family $\{\psi(\xi)\}$ of solutions. In particular, if ψ is just the Lane-Emden function defined by the initial conditions of Equation (3), then its homologous family $\{\psi(\xi)\}$ defines a whole set of solutions that are all finite at the origin $\xi=0$. Solutions that are finite at the origin are called E-solutions and denoted by ψ_E. The Lane-Emden function defined by the initial conditions from Equation (3) is just a particular member of the set $\{\psi_E(\xi)\}$ of E-solutions. All *E*-solutions can be found from the Lane-Emden function through the homology transformations

$$\psi(\xi)\to A^{2/(n-1)}\psi(A\xi),(n\neq\pm1,\pm\infty), \tag{4.1}$$

$$\psi(\xi)\to\psi(A\xi)-2\ln A,(n=\pm\infty). \tag{4.2}$$

It should also be noted that, any solution $\psi_E=\psi_E(\xi)$ that is finite at the origin $\xi=0$ is an *E*-solution, and its derivative is zero $\left(\mathrm{d}\psi_E/\mathrm{d}\xi\right)_{\xi=0}=0$ The general solution of the second order Lane-Emden equation must characterized by two integration constants. According to the homology theorem one of the two constants must be "trivial" in the sense that it defines merely the scale factor A of the homology transformation, and we should be able throughout the introduction of two independent homology invariant functions to transform the second order Lane-Emden equation into a first order differential equation [7].

3.2. Homology Invariant Functions

In what follows the definition and the basic properties of the homology invariant functions are

1) A function Q (say) is said to *homology invariant* if it is invariant to the homologous transformations:

$$\psi^*(\xi)=A^{\omega}\psi(A\xi) \text{ or } \psi^*\left(\frac{\xi}{A}\right)=A^{\omega}\psi(\xi);\quad \omega=2/(n-1); n\neq-1,\pm\infty, \tag{5.1}$$

or

$$\psi^*(\xi)=\psi(A\xi)-2\ln A \quad \text{or } \psi^*\left(\frac{\xi}{A}\right)=\psi(\xi)-2\ln A; \quad n=\pm\infty \tag{5.2}$$

So, to prove that, Q is homology invariant function, we have to prove that

$$Q^*(\xi)=Q(A\xi) \text{ or } Q^*\left(\frac{\xi}{A}\right)=Q(\xi) \tag{6}$$

2) The homology transformation for the derivatives are:

$$\left.\frac{\mathrm{d}^k\theta^*(\xi)}{\mathrm{d}\xi^k}\right|_{\xi=\frac{\xi}{A}}=A^{\omega+k}\frac{\mathrm{d}^k\theta(\xi)}{\mathrm{d}\xi^k}; n\neq-1,\pm\infty, \tag{7.1}$$

$$\left.\frac{\mathrm{d}^k\theta^*(\xi)}{\mathrm{d}\xi^k}\right|_{\xi=\frac{\xi}{A}}=A^{k}\frac{\mathrm{d}^k\theta(\xi)}{\mathrm{d}\xi^k}; n=\pm\infty, \tag{7.2}$$

4. New Generating Sets of Homology Invariant Functions

In this section, two new generating sets, (one for $n\neq-1$, $\pm\infty$, and other for $n=\pm\infty$) of homology invariant functions will be established,

1) $n\neq-1,\pm\infty$

$$U(\xi)=\xi^{\ell_1}\psi^{\ell_2}(\xi)/\psi'(\xi) \text{ and } V(\xi)=\xi^{k_1}\psi^{k_2}(\xi)\psi'(\xi), \tag{8}$$

where ℓ_1 and k_1 are real numbers, while k_2 and ℓ_2 are given in terms of ℓ_1 and k_1 and the polytropic index n $\left(\omega=2/(n-1)\right)$ from

$$k_2=(k_1-\omega-1)/\omega \text{ and } \ell_2=(\ell_1+\omega+1)/\omega. \tag{9}$$

The two functions $U(\xi)$ and $V(\xi)$ are homology invariant functions.

Proof. Since

$$U^*(\xi/A)=\frac{(\xi/A)^{\ell_1}\left[\psi^*(\xi/A)\right]^{\ell_2}}{\left.\mathrm{d}\psi^*/\mathrm{d}\xi\right|_{\xi=\xi/A}},$$

$$V^*(\xi/A)=(\xi/A)^{k_1}\left[\psi*(\xi/A)\right]^{k_2}\left.\mathrm{d}\psi^*/\mathrm{d}\xi\right|_{\xi=\xi/A}$$

Applying the rules of Equations (5.1) and (7.1) we get

$$U^*(\xi/A) = \left(\xi^{\ell_1}\psi^{\ell_2}(\xi)/\psi'(\xi)\right)\left(A^{-\ell_1}A^{\omega\ell_2}/A^{\omega+1}\right)$$

and

$$V^*(\xi/A) = \xi^{k_1}\psi^{k_2}(\xi)\psi'(\xi)\left(A^{-k_1}A^{\omega k_2}A^{\omega+1}\right)$$

Using the values of ℓ_2 and k_2 from Equation (9) we get

$$\left(A^{-\ell_1}A^{\omega\ell_2}/A^{\omega+1}\right) = 1 \text{ and } \left(A^{-k_1}A^{\omega k_2}A^{\omega+1}\right) = 1.$$

So

$$U^*(\xi/A) = \xi^{\ell_1}\psi^{\ell_2}(\xi)/\psi'(\xi) = U(\xi)$$

and

$$V^*(\xi/A) = \xi^{k_1}\psi^{k_2}(\xi)\psi'(\xi) = V(\xi).$$

That is, the two functions $U(\xi)$ and $V(\xi)$ are homology invariant functions. □

2) $n = \pm\infty$ $U(\xi) = \xi^k \exp\left[-k\psi(\xi)/2\right]$ and

$$V(\xi) = \xi^{m_1}\exp\left[-(m_1-m_2)\psi(\xi)/2\right]\left[\psi'(\xi)\right]^{m_2}, \quad (10)$$

where k, m_1 and m_2 are real numbers.

The two functions $U(\xi)$ and $V(\xi)$ are homology invariant functions.

Proof. Since

$$U^*(\xi/A) = (\xi/A)^k \exp\left[-k\psi^*(\xi/A)/2\right],$$

$$V^*(\xi/A) = (\xi/A)^{m_1}\exp\left[-(m_1-m_2)\psi^*(\xi/A)/2\right] \times\left[\mathrm{d}\psi^*/\mathrm{d}\xi\big|_{\xi=\xi/A}\right]^{m_2}.$$

Applying the rules of Equations (5.2) and (7.2) we get

$$U^*(\xi/A) = (\xi/A)^k \exp\left[-k\left(\psi(\xi)-2\ln A\right)/2\right],$$

$$V^*(\xi/A) = (\xi/A)^{m_1}\exp\left[-(m_1-m_2)\left(\psi(\xi)-2\ln A\right)/2\right] \times\left[A\,\mathrm{d}\psi(\xi)/\mathrm{d}\xi\big|_{\xi=\xi/A}\right]^{m_2}.$$

So

$$U^*(\xi/A) = \xi^k\exp\left[-k\psi(\xi)/2\right] = U(\xi),$$

$$V^*(\xi/A) = \xi^{m_1}\exp\left[-(m_1-m_2)\psi(\xi)/2\right]\left[\psi'(\xi)\right]^{m_2} = V(\xi).$$

That is the two functions $U(\xi)$ and $V(\xi)$ are homology invariant functions. □

5. Reduction to the First Order-Differential Equation

Now, since the two functions $U(\xi)$ and $V(\xi)$ are homology invariant functions with respect to the transformations of the homology theorem, then we can reduce with the aid of these functions the second order Lane-Emden equation to one of the first order. This will be of the subject of the present section.

1) $n \neq -1, \pm\infty$

Since $\omega(\ell_2 + k_2) = \ell_1 + k_1$, then we get from Equation (8) that

$$\psi = \left(UV/\xi^{\ell_1+k_1}\right)^{\omega/\ell_1+k_1} \quad (11)$$

Also from Equation (8) we have $\psi' = \xi^{\ell_1}\psi^{\ell_2}/U$, then by using Equation (11) we get

$$\psi' = U^a V^b/\xi^{\omega+1}, \quad (12)$$

where

$$a = \omega - k_1 + 1/\ell_1 + k_1,\ b = \ell_1 + \omega + 1/\ell_1 + k_1.$$

We have also from the original Lane-Emden equation

$$\psi'' = \mp\psi^n - (N-1)\psi'/\xi \quad (13)$$

Differentiating Equations (8) logarithmically and then using Equations (11) and (12) we obtain

$$\begin{aligned}&(1/U)\,\mathrm{d}U/\mathrm{d}\xi \\ &= \ell_1/\xi + \ell_2\psi'/\psi \pm \psi^n/\psi' + (N-1)/\xi \\ &= (\ell_1 + N - 1)/\xi + \ell_2\left(U^aV^b/\xi^{\omega+1}\right)\times\left(\xi^{\omega}/U^cV^c\right) \\ &\quad \pm\left(U^{nc}V^{nc}/\xi^{n\omega}\right)\times\left(\xi^{\omega+1}/U^aV^b\right),\end{aligned}$$

where $c = \omega/\ell_1 + k_1$. Then

$$(1/U)\,\mathrm{d}U/\mathrm{d}\xi = \left\{\ell_1 + N - 1 + \ell_2U^fV^g \pm U^hV^q\right\}/\xi, \quad (14)$$

where

$$f = 1 - k_1/\ell_1 + k_1,\ g = 1 + \ell_1/\ell_1 + k_1,\ h = 1 + k_1/\ell_1 + k_1,$$

and $q = 1 - \ell_1/\ell_1 + k_1$.

Similarly we get

$$(1/V)\,\mathrm{d}V/\mathrm{d}\xi = \left\{k_1 - N + 1 + k_2U^fV^g \mp U^hV^q\right\}/\xi. \quad (15)$$

The required differential equation between U and V is obtained by dividing Equations (14) and (15) and we get for, $N = 1, 2, \cdots, n \neq \pm1, \pm\infty$

$$(U/V)\,\mathrm{d}V/\mathrm{d}U = \beta/\lambda, \quad (16)$$

where

$$\begin{aligned}\beta &= 2(k_1 - N + 1) \\ &\quad + \left[(n-1)(k_1+1) - 2\right]U^fV^g \mp 2U^hV^q, \\ \lambda &= 2(\ell_1 + N - 1) \\ &\quad + \left[(n-1)(\ell_1+1) + 2\right]U^fV^g \pm 2U^hV^q.\end{aligned}$$

2) $n = \pm\infty$

Form Equation (10) we get

$$\psi = \ln\left(\xi^2U^{k/2}\right), \quad (17)$$

$$V = \xi^{m_2}U^{k(m_2-m_1)/4}\left[\psi'\right]^{m_2}. \quad (18)$$

From Equations (18) and (10) we have

$$\psi' = V^{1/m_2} U^{k(m_2-m_1)/4m_2} \Big/ \xi. \quad (19)$$

We have also from the original Lane-Emden equation

$$\psi'' = -(N-1)\psi'/\xi + \exp[-\psi]. \quad (20)$$

From this equation and Equation (18) we get

$$\psi'' = \left(U^{-k/2} - (N-1)V^{1/m_2} U^{k(m_1-m_2)/4m_2}\right) \Big/ \xi^2. \quad (21)$$

From Equations (10) and (18)

$$(U'/U) = k\left(1 - \left(V^{1/m_2} U^{k(m_1-m_2)/4m_2}\right) \Big/ 2\right) \Big/ \xi. \quad (22)$$

Similarly we get

$$\begin{aligned}(V'/V) = &-m_2(N-2)/\xi + k^2(m_2-m_1)/4\xi \\ &\times\left(1 - V^{1/m_2} U^{k(m_1-m_2)/4m_2}\right) \\ &+ m_2\left(V^{-1/m_2} U^{k(m_1+m_2)/4m_2}\right) \Big/ \xi.\end{aligned} \quad (23)$$

The required differential equation between U and V is obtained by dividing Equations (22) and (23) and we get for, $N = 1, 2, \cdots, n = \pm\infty$

$$\begin{aligned}&(U/V)(\mathrm{d}V/\mathrm{d}U) \\ &= -k^2(-2+\eta)(m_1-m_2)/2 + 4(N-2)m_2 \\ &\quad -4m_2 V^{-1/m_2} U^{-k(m_1+m_2)/4m_2} \Big/ \left(2k(-2+\eta)\right),\end{aligned}$$

where $\eta = V^{1/m_2} U^{k(m_1/m_2-1)/4}$.

6. Conclusion

In concluding the present paper, we stress that, two new generating sets, of homology invariant functions was established. Moreover, by the aid of two independent homology invariant functions of each set we established the transformed first order Lane-Emden equation. The first equation for polytropic index $n \neq -1, \pm\infty$ depends on five free parameters, while, the other equation is for, $n = \pm\infty$ and depends on three free parameters.

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