

Chiral Dirac Equation Derived from Quaternionic Maxwell's Systems

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ABSTRACT

In the present article, we propose a simple equality involving the Dirac operator and the Maxwell operators under chiral approach. This equality establishes a direct connection between solutions of the two systems and moreover, we show that it is valid when the natural relation between the frequency of the electromagnetic wave and the energy of the Dirac particle is fulfilled if the electric field \mathbf{E} is parallel to the magnetic field \mathbf{H} . Our analysis is based on the quaternionic form of the Dirac equation and on the quaternionic form of the Maxwell equations. In both cases these reformulations are completely equivalent to the traditional form of the Dirac and Maxwell systems. This theory is a new quantum mechanics (QM) interpretation. The below research proves that the QM represents the electrodynamics of the curvilinear closed chiral waves. It is entirely according to the modern interpretation and explains the particularities and the results of the quantum field theory. Also this work may help to clarify the controversial relation between Maxwell and Dirac equations while presenting an original way to derive the Dirac equation from the chiral electrodynamics, leading, perhaps, to novel conception in interactions between matter and electromagnetic fields. This approach may give a reinterpretation of Majorana equation, neutrino mass, violation of Heisenberg's measurement-disturbance relationship and mass generation in systems like graphene devices.

Keywords: Quaternion; Dirac Equation; Maxwell System

1. Introduction

In mathematical physics, the relation between the two most important first order systems of partial differential equations, (Dirac equation and Maxwell's equations), is among those topics which attract attention because of their general significance and solutions of particular problems concerning physical models. The connection between solutions of massless Dirac and Maxwell equations was well established [1]. It is shown that the massless Dirac equation is invariant under three different representations of Poincaré algebra. Two coupled Dirac equations with masses m and $-m$ has also possesses this symmetries. The Maxwell equations can be represented in a Dirac like form in different ways (e.g., [1-10]). The Beltrami equations are another approach [11]. Also, solutions of Maxwell's system can be related to solutions of the Dirac equation through some nonlinear equations (e.g., [12]).

However, in [13] we find that the author shows that in the formulation of [12], based on spinor form there is no physically meaningful way to transform Maxwell's and Dirac's equations into each other. However, this statement is valid for standard Maxwell fields, but not for

parallel electromagnetic fields that will be discussed in this paper.

Maxwell's equations are formulated in a number of different representations: a) As a single four-component spinor equation whose transformation properties are almost identical with those of the Dirac equation; b) As a pair of uncoupled two-component spinor equations, in two different representations. One of these is similar to the Weyl equation for the neutrino field and the other to the two-component spinor form of the Dirac equation; c) As a single equation in which the field variables are 2×2 matrices.

Nevertheless, in spite of these significant efforts there remain some important conceptual questions. For example, what is the meaning of this close relation between the Maxwell system and the Dirac equation and how this relation is connected with the wave-particle dualism. In the present article we propose a simple equality involving the Dirac quaternionic operator and the Maxwell quaternionic operators under chiral approach. Dirac derived the linear relativistic wave equation for fermions by "taking the square root" of the Klein-Gordon equation, which is quadratic in time and space derivatives. In this paper,

we find a chiral electromagnetic wave equation of fourth order which can be linearized to the Dirac way to obtain two linear Beltrami equations.

We propose to find an equivalence between the Dirac equation and the Beltrami equations in quaternionic coordinates. This equality establishes a direct connection between solutions of the two systems and moreover, we show that it is valid when a quite natural relation between the frequency of the electromagnetic wave and the energy of the Dirac particle is fulfilled. This condition is satisfied when \mathbf{E} is parallel to \mathbf{H} . Our analysis is based on the quaternionic form of the Dirac equation [8] and on the quaternionic form of the Maxwell equations [7,8] (see also [9]). In both cases our quaternionic reformulations are completely equivalent to the traditional form of the Dirac and Maxwell systems. Chiral approach means that our Universe is observable area of basic space-time where temporal coordinate is positive and all particles bear positive masses (energies). The mirror Universe is an area where temporal coordinate is negative and all particles bear negative masses. Also, from viewpoint of our world observer, the mirror Universe is a world with reverse flow of time, where particles travel from future into past in respect to us. The two worlds are separated by a membrane—an area of space-time inhabited by light-like particles that travel along light-like right or left-handed (isotropic-chiral) spirals. On the scales of elementary particles such space can be attributed to particles that possess spirality (e.g. Chiral photons). The membrane prevents mixing of positive and negative-mass particles and thus their total annihilation. The link between both universes is the chiral factor defined by β_c (see Section 3).

The aim of this work is to link the Dirac equation in the Weyl representation with Maxwell's equations in the chiral formulation of Born Fedorov and show that only when the \mathbf{E} field and \mathbf{H} field are spatially parallel, we have that this field distribution can generate mass and the Dirac equation can be obtained from the chiral electrodynamics. These fields are circulatory and stationary. Under this condition we have no radiation and the vector Poynting $\mathbf{E} \times \mathbf{H}$ is zero.

2. The Dirac Equation in Quaternionic Form

The algebra of complex quaternions is denoted by $H(\mathbb{C})$. Each complex quaternion q is of the form $q = \sum_{k=0}^3 q_k e_k$ where $\{a_k\} \subset \mathbb{C}$, e_0 is the unit and $\{e_k | k=1,2,3\}$ are the quaternionic imaginary units:

$$e_0^2 = e_0 = -e_k^2; \quad e_0 e_k = e_k e_0 = e_k, \quad k = 1, 2, 3;$$

$$e_1 e_2 = -e_2 e_1 = e_3; \quad e_2 e_3 = -e_3 e_2 = e_1; \quad e_3 e_1 = -e_1 e_3 = e_2$$

The complex imaginary unit i commutes with e_k , $k = 0, 1, 2, 3, 4$. We further introduce a complex conju-

gate operation $*$ which takes $i^* = -i$ but leaves e_k , $k = 0, \dots, 3$ unchanged, as well as a quaternionic conjugation operation $-$, which leaves i unchanged but takes

$$e_k \rightarrow \{\bar{e}_k = -e_k | k = 1, 2, 3\}$$

We will use the vector representation of complex quaternions: $q = Sc(q) + Vec(q)$, where $Sc(q) = q_0$ and

$$Vec(q) = \mathbf{q} = \sum_{k=1}^3 q_k e_k.$$

That is each complex quaternion is a sum of its scalar part and its vector part. Complex vectors we identify with complex quaternions whose scalar part is equal to zero. In vector terms, the multiplication of two arbitrary complex quaternions q and b can be written as follows:

$$qb = q_0 b_0 - \mathbf{q} \cdot \mathbf{b} + [\mathbf{q} \times \mathbf{b}] + q_0 \mathbf{b} + b_0 \mathbf{q},$$

where

$$\mathbf{q} \cdot \mathbf{b} := \sum_{k=1}^3 q_k b_k \in \mathbb{C}, \quad [\mathbf{q} \times \mathbf{b}] := \begin{vmatrix} e_1 & e_2 & e_3 \\ q_1 & q_2 & q_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \in \mathbb{C}^3$$

We shall consider continuously differentiable $H(\mathbb{C})$ -valued functions depending on three real variables $x = (x_1, x_2, x_3)$. On this set the well known (see, e.g., [1,2,5,8,9]) Moisil-Theodoresco operator is defined by the expression

$$D := \sum_{k=1}^3 e_k \partial_k, \quad \bar{D} := -\sum_{k=1}^3 e_k \partial_k \quad \text{where } \partial_k = \frac{\partial}{\partial x_k}.$$

The action of the operator D on an $H(\mathbb{C})$ -valued function f can be written in a vector form:

$$Df = -divf + rotf. \tag{1}$$

In a good number of physical applications the operators $D_\alpha = D + M^\alpha$ and $D_{-\alpha} = D - M^\alpha$ are needed, where α is a complex quaternion and M^α denotes the operator of multiplication by α from the right-hand side: $M^\alpha f = f\alpha$. Here we will be interested in two special cases when α is a scalar, that is $\alpha = \alpha_0$ or when α is a vector $\alpha = \boldsymbol{\alpha}$. The first case corresponds to the Maxwell equations and the second to the Dirac equation (see [4,8]).

Following [8,14,15], but by considering the chiral representation, the Dirac equation in its covariant form

$$\left(\hbar \left(\frac{\gamma_{0c}}{c} \partial_t + \sum_{k=1}^3 \gamma_{kc} \partial_k \right) + imc \right) Q(t, x) = 0.$$

For a wave function with a given energy we have $Q(t, x) = q(x) e^{\frac{iEt}{\hbar}}$, where q satisfies the time harmonic Dirac equation

$$\left(\frac{iE}{\hbar} \gamma_{0c} + \sum_{k=1}^3 \gamma_{kc} \partial_k + \frac{imc}{\hbar} \right) q(x) = 0. \tag{2}$$

Denote

$$D_c - \frac{imc}{\hbar} := \frac{iE}{c\hbar} \gamma_{0c} + \sum_{k=1}^3 \gamma_{kc} \partial_k .$$

where the gamma matrices are given in the chiral representation. Also, $q(x)$ is written in terms of the two component complex spinor $q(x) = (q_R, q_L)^T$

$$\gamma_{0c} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{kc} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

With 0 and 1 are the 2×2 zero and unit matrix and σ_k the Pauli matrices.

Equation (2) can be written as a two component Weyl equation

$$\begin{pmatrix} -\frac{imc}{\hbar} & \frac{i\varepsilon}{c\hbar} + \sum_{k=1}^3 \sigma_{kc} \partial_k \\ \frac{i\varepsilon}{c\hbar} - \sum_{k=1}^3 \sigma_{kc} \partial_k & -\frac{imc}{\hbar} \end{pmatrix} \begin{pmatrix} q_R \\ q_L \end{pmatrix} = 0$$

Let us introduce an auxiliary notation

$\tilde{f} := f(t, x_1, x_2, -x_3)$. The transformation which allows us to rewrite the Dirac Equation (2) in a quaternionic form we denote as \mathcal{A} defined as a function

$Q: \mathbb{R}^3 \rightarrow \mathbb{C}^4$ which is transformed into a function

$F: \mathbb{R}^3 \rightarrow H(\mathbb{C})$ by the rule $F = \mathcal{A}[Q]$, [7].

The inverse transformation \mathcal{A}^{-1} is $Q = \mathcal{A}^{-1}[F]$. The introduced transformations relates the components of a \mathbb{C}^4 -valued function Q with the components of an $H(\mathbb{C})$ -valued function F :

Following [7,8] an important equality is obtained

$$D_{\tilde{\alpha}_c} = D + \alpha_c = -\mathcal{A} \gamma_{1c} \gamma_{2c} \gamma_{3c} D_c \mathcal{A}^{-1},$$

where

$$\alpha_c := \frac{1}{\hbar} \left(i \frac{E}{c} e_3 - mce_2 \right) \quad (3)$$

The difference between this α_c and α of [8,14,15] is that here the Dirac Equation (2) is in the Weyl or chiral representation. This equality shows that instead of Equation (2) we can consider the equivalent quaternionic equation

$$(D + \alpha_c) f = D_{\alpha_c} f = 0$$

This last equation can be written as the two-component field $f = (f_R, f_L)^T$

$$\begin{pmatrix} -\frac{1}{\hbar} mce_2 & D - \frac{1}{\hbar} i \frac{E}{c} e_3 \\ \frac{1}{\hbar} D + \frac{1}{\hbar} i \frac{E}{c} e_3 & -\frac{1}{\hbar} mce_2 \end{pmatrix} \begin{pmatrix} f_R \\ f_L \end{pmatrix} = 0 \quad (4)$$

and the relation between solutions of (2) and (4) is established by means of the invertible transformation $\mathcal{A}: f = \mathcal{A}q$.

Next we develop the Maxwell equations in quaternionic representation.

3. The Maxwell Equations in Chiral Quaternionic Form

Chiral materials, in which the mirror reflection symmetry is broken, are ubiquitous in nature. Even the quantum vacuum of the standard model of particle physics is chiral [16], so that the behavior of the left-handed and right-handed elementary particles (quarks and leptons) are essentially different. One of the consequences is the chiral anomaly—the anomalous nonconservation of a chiral current, as first described by Adler [17] and by Ginzburg and Landau [18]. The chiral anomaly provides one explanation for the baryogenesis in the early Universe and the huge excess of matter over antimatter in the present Universe. Here, we consider a chiral vacuum. A Chiral Vacuum will be defined as a vacuum for which the constitutive matrices represented by $[\beta_c]$ are not zero, but for which there are no real charge densities or current densities. Such an assumption, which if applicable to the vacuum, would imply that the chiral vacuum, and there for the universe itself, may not have a center of symmetry. The concept of 'spontaneous symmetry breakdown' has proved to be extraordinarily fruitful in many areas of physics and I consider it worthwhile to try to incorporate it into chiral electromagnetism. The modern formulation of this concept appears to originate with the work of V. Ginzburg and L. Landau [18] and A. Zee [19].

First we consider the chiral factor $\beta_c = \hbar/2mc$ as a scalar parameter and then we extend the analysis to a chiral matrix. We will consider the Maxwell equations for a sourceless anisotropic chiral homogeneous medium with $\partial_t \rightarrow \partial_t (1 + \beta_c \nabla \times)$ [20-22]. β_c is a chiral parameter so the Maxwell's equations without charges are expressed as:

$$rot \mathbf{H} = \frac{1}{c} \partial_t (1 + \beta_c \nabla \times) \mathbf{E}, \quad div \mathbf{E} = 0$$

$$rot \mathbf{E} = -\frac{1}{c} \partial_t (1 + \beta_c \nabla \times) \mathbf{H}, \quad div \mathbf{H} = 0$$

So the time-harmonic Maxwell equations are

$$rot \mathbf{H} = -\frac{i\omega}{c} (1 + \beta_c \nabla \times) \mathbf{E}, \quad (5)$$

$$rot \mathbf{E} = \frac{i\omega}{c} (1 + \beta_c \nabla \times) \mathbf{H}, \quad (6)$$

$$div \mathbf{E} = 0, \quad (7)$$

$$div \mathbf{H} = 0. \quad (8)$$

Here, $rot = \nabla \times$, β_c is the chiral scalar parameter, ω is the frequency. Application of rot to (5) and (6) allows us to separate the equations for \mathbf{E} and \mathbf{H} and to obtain

in this way the wave equation for chiral medium.

$$\text{rot}(\text{rot}\mathbf{E}) + \frac{1}{c^2} \partial_t^2 \mathbf{E} + \frac{2\beta_c}{c^2} \partial_t^2 \text{rot}\mathbf{E} + \frac{\beta_c^2}{c^2} \partial_t^2 \text{rot}(\text{rot}\mathbf{E}) = 0 \quad (9)$$

this chiral generalization represents an equation of fourth order. When

$$\left(1 + \frac{\beta_c^2}{c^2} \frac{\partial^2}{\partial t^2}\right) = 0 \quad (10)$$

we have an important linearization which transforms a fourth order equation to a first order one.

$$\mathbf{E} + 2\beta_c (\text{rot}\mathbf{E}) = 0, \quad \mathbf{E} = i\mathbf{H} \quad (11)$$

This result corresponds to a self dual electromagnetic fields represented by standing waves possessing zero Poynting vector [23]. In this case, we can link the Dirac Equation (4) with the Beltrami Equation (11). Obviously system (11) can be written in the form of a quaternionic equation if we define

$\mathbf{E} \sim \mathbf{E}e^{ie_3\epsilon t/\hbar}$, $\omega \rightarrow \omega e_3$, $\beta_c \rightarrow \beta_c e_2$, so Equation (10) is transformed to

$$\left(1 - \frac{\beta_c^2 e_2^2}{c^2} \omega^2 e_3^2\right) = 0 \quad (12)$$

$$(D + 1/2 \beta_c e_2)\mathbf{E} = D_{1/2\beta_c e_2} \mathbf{E} = 0 \quad (13)$$

where

$$D := \sum_{k=1}^3 e_k \partial_k$$

A similar equation can be obtained for \mathbf{H} . Taking into account (1) and \mathbf{E} and \mathbf{H} simultaneously, we introduce the following quaternionic operator

$$M_c = \frac{\beta_c}{c} \frac{\partial}{\partial t} D + \frac{1}{c} \frac{\partial}{\partial t} - iD$$

If we consider the purely vectorial biquaternionic function $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and the complex conjugate $\mathbf{F}^* = \mathbf{E}^* - i\mathbf{H}^*$ we have

$$M_c (\mathbf{E} + i\mathbf{H}) = 0, M_c^* (\mathbf{E}^* - i\mathbf{H}^*) = 0 \quad (14)$$

The first equation is equivalent to the Maxwell system (5)-(8). We can rewrite this system in matrix form as

$$D\mathbf{F} = \frac{i\omega e_3}{c} (1 + \beta_c D)\mathbf{F}, \quad (15)$$

Equation (15) can be diagonalized and transformed in the following way [5,6] (see also [8,14]). Following [5] we define

$$\mathbf{F} \rightarrow F = (0, F_1, F_2, F_3)^T = (F_0, F_1, F_2, F_3)^T$$

$$F' = S^* F$$

$$F' = (F'_R, F'_L)^T$$

$$F'_R = \begin{pmatrix} iF_1 \\ -iF_2 + F_3 \end{pmatrix}; \quad F'_L = \begin{pmatrix} -iF_1 \\ iF_2 + F_3 \end{pmatrix} \quad (16)$$

Where the unitary matrix S ($S^* = S^{-1}$) is given by

$$S = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad (17)$$

Thus we are transformed the quaternionic Maxwell Equation (15) in the form of two component equations

$$\begin{pmatrix} -\frac{1}{\hbar} mce_2 & D - \frac{1}{\beta_c} e_1 \\ \bar{D} + \frac{1}{\beta_c} e_1 & -\frac{1}{\hbar} mce_2 \end{pmatrix} \begin{pmatrix} F'_R \\ F'_L \end{pmatrix} = 0 \quad (18)$$

$-\kappa^2 = (\omega e_3)^2 = c^2 / (\beta_c e_2)^2$ is the square wave number. Applying the operator D and D^* ($\bar{D} = -D^*$) to the functions F'_R and F'_L respectively one can see that F'_R satisfies the equation

$$(D - \kappa) F'_R = \frac{mc}{\hbar} F'_L, \quad (19)$$

and F'_L satisfies the equation

$$(D^* + \kappa) F'_L = \frac{mc}{\hbar} F'_R. \quad (20)$$

Solutions of (19) and (20) are called the Beltrami fields (see, e.g., [11]). Thus, if we compare (19) and (20) with Equation (4), we infer that for parallel fields, the coupled system (19-20) is equivalent to Equation (4). R and L , subscripts are associated with circularly polarized photons (right or left-handed spirals or chiral photons).

4. The Chiral Electromagnetic Dirac Equation

In the preceding sections it was shown that the Dirac Equation (2) is equivalent to the quaternionic equation

$$D_{\alpha_c} f = 0 \quad \text{with} \quad \alpha_c := \frac{1}{\hbar} \left(i \frac{\epsilon}{c} e_3 - mce_2 \right)$$

and the Maxwell Equations (5)-(8) are equivalent to the pair of quaternionic Equations (19) and (20) when the electric field \mathbf{E} is parallel to the magnetic field \mathbf{H} , ($\mathbf{E} = i\mathbf{H}$). If f is a solution of (4), then $\mathbf{E} = i\mathbf{H}$ are solutions of Equations (19) and (20) with $\beta_c^2 \omega^2 = c^2$ and the coupled Equations (19) and (20) are equivalent to the chiral Dirac Equation (4).

Now we will show a simple relation between these objects. Observing Equation (10) or (12) we have

$$\frac{1}{4\beta_c^2} = \kappa^2. \quad (21)$$

Note that from Equation (3)

$$\alpha_c^2 = \alpha_c \cdot \alpha_c = \frac{1}{\hbar^2} \left(\frac{E^2}{c^2} - m^2 c^2 \right). \quad (22)$$

Thus, when \mathbf{E} is parallel to \mathbf{H} , (21) is equal to (22) we have the important result

$$\frac{1}{4\beta_c^2} = \kappa^2 = \frac{1}{\hbar^2} \left(\frac{\mathcal{E}^2}{c^2} - m^2 c^2 \right) \quad (23)$$

Thus, relation between the Dirac operator and the Maxwell operators is valid if the condition (23) is fulfilled. This happens if and only if \mathbf{E} is parallel to \mathbf{H} , that is $\mathbf{E} = i\mathbf{H}$ and the vector Poynting is null.

In general, if in (23) we formally use the de Broglie equality $p = \mp \hbar \kappa = \mp \hbar / 2\beta_c$, we again obtain the fundamental Einstein relation

$$E^2 = p^2 c^2 + m^2 c^4. \quad (24)$$

From $p\beta_c = \mp \hbar/2$ we obtain naturally the value of the electron spin $-\hbar/2$ and the positron spin $+\hbar/2$.

We can note that $\beta_c = \hbar/2mc$ and $\hbar\omega = 2mc^2$ so the original energy photon is transformed in the pair particle-antiparticle ($\pm mc^2$).

If we consider p and β_c as arbitrary observable operators \hat{p} and $\hat{\beta}_c$, a new Heisenberg uncertainty principle (HUP) is obtained

$$\Delta\hat{p}\Delta\hat{\beta}_c \geq \frac{\hbar}{2} = \left\langle \left\langle F'_{R,L} \left[\hat{p}, \hat{\beta}_c \right] F'_{R,L} \right\rangle \right\rangle \quad (25)$$

While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system often referred to as "Heisenberg's uncertainly principle" HUP, recently it has been shown experimentally violation of Heisenberg's measurements-disturbance relationship by weak measurements, MDR [24,25].

$$\varepsilon(\hat{p})\eta(\hat{\beta}_c) \geq \frac{\hbar}{2} = \left\langle \left\langle F'_{R,L} \left[\hat{p}, \hat{\beta}_c \right] F'_{R,L} \right\rangle \right\rangle$$

It is formally incorrect because the disturbance $\eta(\hat{\beta}_c) \rightarrow 0$ in some eigenstate $|F'_{R,L}\rangle$.

Equation (25) is appropriate to obtain the corrected MDR

$$\varepsilon(\hat{p})\eta(\hat{\beta}_c) + \varepsilon(\hat{p})\Delta\hat{\beta}_c + \eta(\hat{\beta}_c)\Delta\hat{p} \geq \frac{\hbar}{2}$$

The authors [24,25] have designed an apparatus to measure the polarization of a single photon. They needed to measure how much that apparatus disturbed that photon. To do this, they needed to measure the photon before the apparatus but how that measurement would also disturb the photon. Our approach is related precisely to the polarization of photons (see Equations (19) and (20)). Our theory may be important on study of uncertainly relations, specifically in the setting of quantum informa-

tion and quantum cryptography.

Also, with this theory we can explore the neutrino physics. The importance of neutrino electromagnetic properties was first mentioned by Pauli in 1930. Systematic theoretical studies of neutrino electromagnetic properties started after it was shown that in the extended Standard Model with right-handed neutrinos the magnetic moment of a massive neutrino is, in general, non vanishing and that its value is determined by the neutrino mass [26].

As in known, uncharged fermions having non-zero mass and spin 1/2 may be subject to one of the two equations: the Dirac equation, similar to that for charged particles such electron (Equation (18)), and the Majorana equation in which the inversion (on our case of β_c) takes helicity particles into antiparticles (in our theory if we take $(F'_R)^c = F'_L$).

Neutrino electromagnetic properties are important because they are directly connected to fundamentals of particle physics. For example, neutrino electromagnetic properties can be used to distinguish Dirac and Majorana neutrinos and also as probes of new physics that might exist beyond the Standard Model. Equation (18) is well suited to study the electromagnetic properties of neutrinos. Our result Equation (18) is appropriate to study Dirac and Majorana particles. As $\beta_c = \hbar/2mc$ our theory lends itself to study mass generation in graphene devices [22]. Also, classically the Heisenberg expression can be written in terms of time and energy and can be useful in problems of shear waves [27].

5. Conclusions

The main result of this paper is that the Dirac equation can be derived from the Maxwell's equation under chiral quaternionic approach. Equations (15)-(20) are our main results which are compared with Equation (4). With this theory we can study electromagnetic interactions in neutrinos which can be considered as Majorana fermions, favoured by simplicity because they have only two degrees of freedom.

Also, our approach is related precisely to the polarization of photons so this theory may be important on study of uncertainly relations, specifically in the setting of quantum information and quantum cryptography.

Also, our theory lends itself to study mass generation in graphene devices.

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