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Numerical Solutions of the Regularized Long-Wave (RLW) Equation Using New Modification of Laplace-Decomposition Method

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ABSTRACT

In this paper the new modification of Laplace Adomian decomposition method (ADM) to obtain numerical solution of the regularized long-wave (RLW) equation is presented. The performance of the method is illustrated by solving two test examples of the problem. To see the accuracy of the method, L_2 and L_∞ error norms are calculated.

Keywords: Adomian Decomposition Method; Regularized Long-Wave (RLW); Laplace Transform

1. Introduction

The regularized long wave (RLW) equation which can be shown in the form

$$u_t + u_x + \epsilon u u_x - \mu u_{xxt} = 0 \tag{1}$$

where ϵ, μ are positive parameters, is an important nonlinear wave equation. This equation plays a major role in the study of nonlinear dispersive waves. The RLW equation particularly describes the behavior of the undular bore [1-3], it has also been derived from the study of water waves and ion acoustic plasma waves.

The RLW equation has been solved analytically only for restricted set of boundary and initial conditions. Therefore, the numerical solution of this equation has been the subject of many papers [4-7]. Recently a great deal of interest has been focused on application of Adomian decomposition method (ADM) to solve a wide variety of nonlinear problems [8,9]. In this paper, we will apply the new modification of Laplace ADM to the RLW Equation (1). The soliton solution of RLW equation has the form

$$u(x,t) = 3c\operatorname{sech}^2\left(p\left(x - vt - x_0\right)\right) \tag{2}$$

where p is an arbitrary constant and $p = \frac{1}{2}\left(\frac{c}{\mu(1+c)}\right)^{\frac{1}{2}}$, $v = c+1$, $\varepsilon = 1$, and $c > 0$ is a constant [10]. In this work, a new modification of Laplace ADM is used to solve the RLW equation with the initial condition

$$u(x,0) = f(x) \tag{3}$$

where $f(x)$ is a localized disturbance inside the considered interval.

2. Description of Method

We begin by consider Equation (1) in an operator form

$$L_t u + u_x - \mu R(u) + \epsilon N(u) = 0 \tag{4}$$

where $\left(L_t = \frac{\partial}{\partial t}\right)$ is a linear operator and R its remainder of the linear operator. The nonlinear term is represented by $N(u)$. Thus we get

$$L_t u = -u_x + \mu R(u) - \epsilon N(u) \tag{5}$$

We represent solution as an infinite series given below,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{6}$$

The nonlinear term Nu can be decomposed into infinite series of polynomial given by:

$$Nu = uu_x = \sum_{n=0}^{\infty} A_n \tag{7}$$

where A_n are Adomian polynomials [11] of $u_0, u_1, \cdots, u_n$ and it can be calculated by formula given below:

$$\begin{gathered} A_0 = u_{0x}u_0 \\ A_1 = u_{0x}u_1 + u_{1x}u_0 \\ A_2 = u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0 \\ \vdots \end{gathered}$$

and so on. The rest of the polynomials can be constructed in a similar manner.

By applying the Laplace transform to both sides of Equation (5), we obtain

$$\mathcal{L}\{L_t(u)\} = -\mathcal{L}\{u_x\} + \mu\mathcal{L}\{Ru\} - \epsilon\mathcal{L}\{Nu\} \tag{8}$$

Thus

$$\mathcal{L}\{u(x,t)\} = \frac{1}{s}u(x,0) - \frac{1}{s}\left(\mathcal{L}\{u_x\} + \mu\mathcal{L}\{Ru\} - \epsilon\mathcal{L}\{Nu\}\right) \tag{9}$$

In the new modification of ADM [12], Wazwaz replaced the initial condition $u(x,0)$ by a series of infinite components *i.e.*,

$$u(x,0) = \sum_{n=0}^{\infty} u_{0,n}(x) \tag{10}$$

and the new recursive relationship can be expressed in the form

$$\begin{aligned} U_0(x,s) &= \frac{1}{s}u_{0,0}(x,0), \\ U_{n+1}(x,s) &= \frac{1}{s}u_{0,n+1}(x,0) \\ &\quad - \frac{1}{s}\left(\mathcal{L}\{u_x\} + \mu\mathcal{L}\{Ru\} - \epsilon\mathcal{L}\{Nu\}\right), n \ge 0 \end{aligned} \tag{11}$$

where

$$\mathcal{L}\{u(x,t)\} = U(x,s) = \sum_{n=0}^{\infty} U_n(x,s) \tag{12}$$

Now, by applying inverse Laplace transformation we get:

$$\begin{aligned} u_0(x,t) &= \mathcal{L}^{-1}\{U_0(x,s)\} \\ u_{n+1}(x,t) &= \mathcal{L}^{-1}\{U_{n+1}(x,s)\}, n \ge 0 \end{aligned} \tag{13}$$

Using (13) the series solution follows immediately.

3. Numerical Examples and Results

In this section, the new modification of Laplace ADM will be demonstrated on illustrative examples and we compare the approximate solution obtained for our RLW equation with known exact solutions. We define u_m to be m-term approximate solution, *i.e.*

$$u_m = \sum_{i=0}^{m} u_i(x,t)$$

u_e the exact solution and e_m the absolute error between the exact solution and the approximate solution

$$e_m = |u_e - u_m|$$

In order to show how good the numerical solutions are in comparison with the exact ones, we will use the L_2 and L_∞ error norms defined by

$$L_2 = u_e - u_{m2} = \left[\Delta x \sum_{i=1}^{m} |u_{e,i} - u_{m,i}|^2\right]^{\frac{1}{2}}$$

$$L_\infty = \|u_e - u_m\|_\infty = \max_i |u_{e,i} - u_{m,i}|$$

Example (1)

We consider Equation (1) with the initial condition

$$u(x,0) = 3c\operatorname{sech}^2\left(p(x - x_0)\right)$$

The exact solution of this problem is given by Equation (2). This solution corresponds to the motion of a single solitary wave with amplitude 3*c* and width *p*, initially centered at x_0, where $v = 1 + c$ is the wave velocity. We use the new modification of Laplace ADM to solve this equation, all computations are done for the parameters $\varepsilon = 1$, $\mu = 1$ and $x_0 = 0$.

We consider $c = 0.1$, as in [13], so the initial condition $u(x,0)$ can expressed as a series of infinite components *i.e.*

$$\begin{aligned} u(x,0) = 0.3 - 0.00681818x^2 + 0.000103306x^4 \\ -1.33045x^6 + 1.5754x^8 - 1.77355x^{10} + O[x]^{12} \end{aligned}$$

Using recursive relation (11) yield the components

$$u_0(x,t) = \mathcal{L}^{-1}\{U_0(x,s)\} = \mathcal{L}^{-1}\left\{\frac{0.3}{s}\right\} = 0.3$$

$$\begin{aligned} u_1(x,t) &= \mathcal{L}^{-1}\{U_1(x,s)\} = \mathcal{L}^{-1}\left\{\frac{-0.00681818}{s}x^2\right\} \\ &= -0.00681818x^2 \end{aligned}$$

$$\begin{aligned} u_2(x,t) &= \mathcal{L}^{-1}\{U_2(x,s)\} \\ &= 0.0177273tx + 0.000103306x^4 \end{aligned}$$

$$\begin{aligned} u_3(x,t) &= \mathcal{L}^{-1}\{U_3(x,s)\} \\ &= -0.0115227t^2 - 0.000630165tx^3 \\ &\quad -1.33045(10)^{-6}x^6 \end{aligned}$$

And so on, in this manner the rest of components of the decomposition series were obtained. The results are given in **Table 1**. The error norms for (c = 0.1) are recorded in **Table 2** for different value of *m*.

Also in **Figure 1** we show the exact solution and numerical solution with new modification of Laplace ADM for t = 0.1 and t = 0.5. **Figure 2** shows the exact solution and numerical solution with new modification for t = 0.5 at the interval $5 \le x \le 5$.

Example (2)

In the second test problem [14], a smaller solitary wave of amplitude 0.109 (c = 0.05), has been modeled.

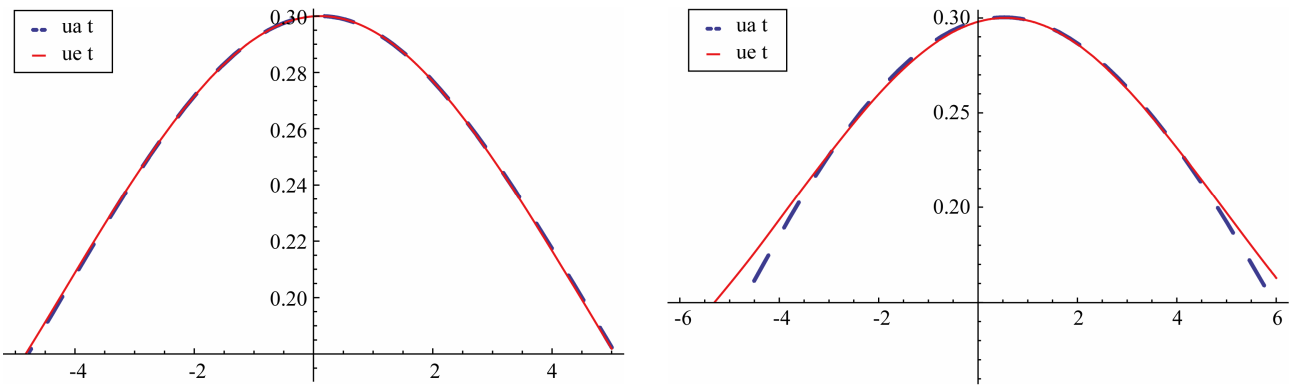
The initial condition $u(x,0)$ can expressed as a series of infinite components *i.e.*

Table 1. Absolute errors for Example (1) with $c = 0.1$ and $m = 10$.

x/t	0.01	0.02	0.03	0.04	0.05
0.1	4.60152×10^{-7}	4.29123×10^{-7}	9.14459×10^{-8}	1.09975×10^{-6}	2.59382×10^{-6}
0.2	1.15346×10^{-6}	1.8112×10^{-6}	1.97626×10^{-6}	1.65183×10^{-6}	8.41272×10^{-7}
0.3	1.82435×10^{-6}	3.1446×10^{-6}	3.96518×10^{-6}	4.29071×10^{-6}	4.12595×10^{-6}
0.4	2.4624×10^{-6}	4.40862×10^{-6}	5.84456×10^{-6}	6.77623×10^{-6}	7.20981×10^{-6}
0.5	3.05691×10^{-6}	5.58214×10^{-6}	7.58301×10^{-6}	9.06702×10^{-6}	1.00418×10^{-5}

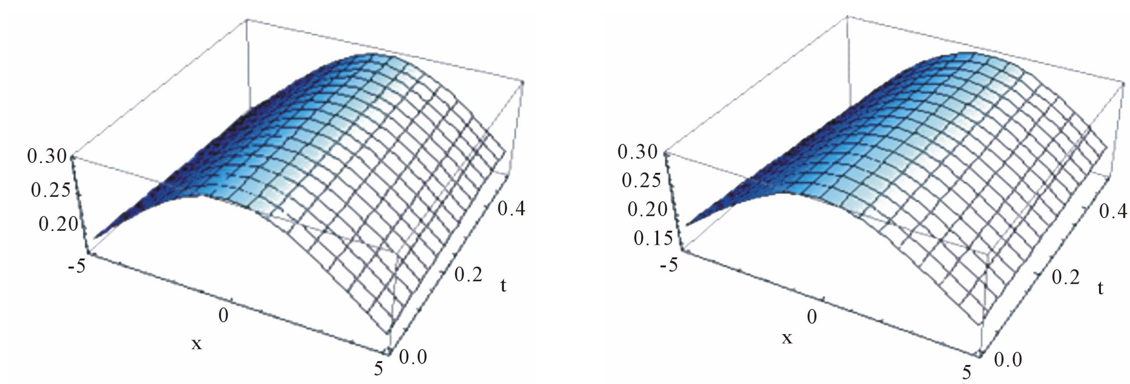
Table 2. L_2 and L_∞ errors for Example (1) with m = 4, 6 and 10.

n	4		6		10	
x	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	7.08453×10^{-7}	5.4329×10^{-6}	2.30062×10^{-7}	2.44677×10^{-7}	1.88979×10^{-6}	2.59382×10^{-6}
0.2	1.71987×10^{-6}	1.8958×10^{-5}	9.54522×10^{-7}	5.43345×10^{-6}	9.42236×10^{-7}	8.41272×10^{-7}
0.3	2.7237×10^{-6}	3.2340×10^{-5}	1.70526×10^{-6}	1.05194×10^{-5}	9.18072×10^{-7}	4.12595×10^{-6}
0.4	3.71444×10^{-6}	4.5525×10^{-5}	2.43975×10^{-6}	1.54669×10^{-5}	1.73297×10^{-6}	7.20981×10^{-6}
0.5	4.6877×10^{-6}	5.8456×10^{-5}	3.15018×10^{-6}	2.02399×10^{-5}	2.61274×10^{-6}	1.00418×10^{-5}



(a) (b)

Figure 1. The exact solution and numerical solution with new modification of Laplace ADM for Example (1), for $t = 0.1$ and (b) for $t = 0.5$.



(a) (b)

Figure 2. (a) The exact solution for $t = 0.5$ and $-5 \le x \le 5$; (b) Numerical solution with new modification for $t = 0.5$ and $-5 \le x \le 5$.

$$u(x,0)=0.15-0.00178571x^2 +0.0000141723x^4-9.5607\times(10)^{-8}x^6 +5.93001\times(10)^{-10}x^8+O[x]^{12}$$

Using recursive relation (11) yield the components

$$u_0(x,t)=\mathcal{L}^{-1}\{U_0(x,s)\} =\mathcal{L}^{-1}\left\{\frac{0.15}{s}\right\}=0.15$$

$$u_1(x,t)=\mathcal{L}^{-1}\{U_1(x,s)\}=\mathcal{L}^{-1}\left\{\frac{-0.00178571}{s}x^2\right\} =-0.00178571x^2$$

$$u_2(x,t)=\mathcal{L}^{-1}\{U_2(x,s)\} =0.00410714tx+0.0000141723x^4$$

$$u_3(x,t)=\mathcal{L}^{-1}\{U_3(x,s)\} =-0.00236161t^2-0.0000715703tx^3 -9.5607\times(10)^{-8}x^6$$

And so on, in this manner the rest of components of the decomposition series were obtained.

The results are given in **Table 3** and the error norms for (c = 0.05) are recorded in **Table 4** for different value of m. Also in **Figure 3** we show the exact solution and numerical solution with new modification of Laplace ADM for t = 0.1 and t = 0.5. **Figure 4** show the exact solution and numerical solution with new modification for t = 0.5 at the interval $10 \le x \le 10$.

4. Conclusion

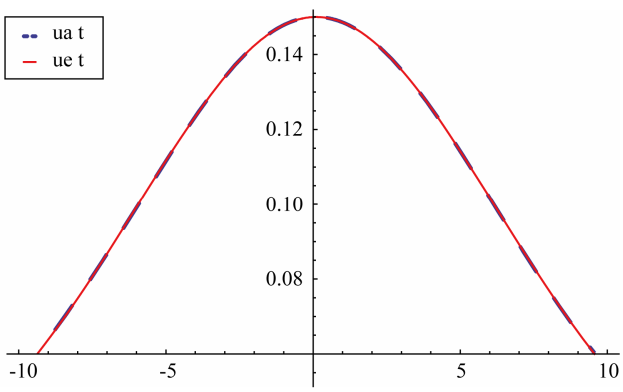
In this paper, we use the new modification of Laplace

Table 3. Absolute errors for Example (2) with *c* = 0.05, *k* = 0.109 and *m* = 10.

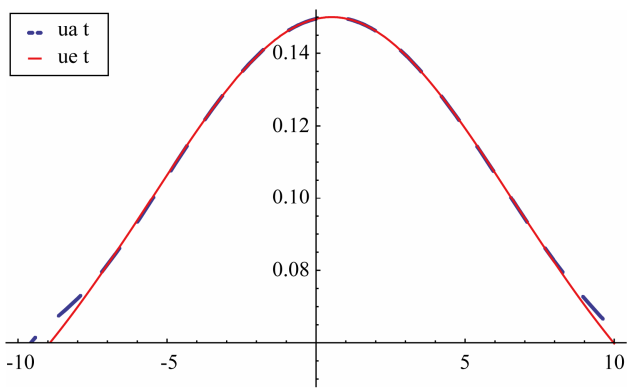
x/t	0.01	0.02	0.03	0.04	0.05
0.1	5.68488×10^{-9}	2.48779×10^{-9}	9.54769×10^{-9}	3.03715×10^{-8}	5.99272×10^{-8}
0.2	1.56382×10^{-8}	2.23134×10^{-8}	2.01032×10^{-8}	9.09157×10^{-9}	1.06314×10^{-8}
0.3	2.52853×10^{-8}	4.14523×10^{-8}	4.86129×10^{-8}	4.6885×10^{-8}	3.63928×10^{-8}
0.4	3.44902×10^{-8}	5.96351×10^{-8}	7.55814×10^{-8}	8.2482×10^{-8}	8.04953×10^{-8}
0.5	4.31131×10^{-8}	7.65863×10^{-8}	1.00602×10^{-7}	1.15349×10^{-7}	1.21020×10^{-7}

Table 4. L_2 and L_∞ errors for Example (2) with *m* = 4, 6 and 10.

n	4		6		10	
x	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	4.34078×10^{-6}	1.3424×10^{-6}	1.37871×10^{-6}	4.32171×10^{-7}	4.61577×10^{-7}	1.48914×10^{-7}
0.2	4.97100×10^{-6}	1.70414×10^{-6}	1.59318×10^{-6}	5.56933×10^{-7}	5.4413×10^{-7}	1.98079×10^{-7}
0.3	5.60954×10^{-6}	2.0710×10^{-6}	1.80913×10^{-6}	6.83084×10^{-7}	6.26085×10^{-7}	2.47459×10^{-7}
0.4	6.26191×10^{-6}	2.4466×10^{-6}	2.0284×10^{-6}	8.11842×10^{-7}	7.08138×10^{7}	2.97527×10^{-7}
0.5	6.93387×10^{-6}	2.8344×10^{-6}	2.25293×10^{-6}	9.44397×10^{-7}	7.91008×10^{-7}	3.48733×10^{-7}

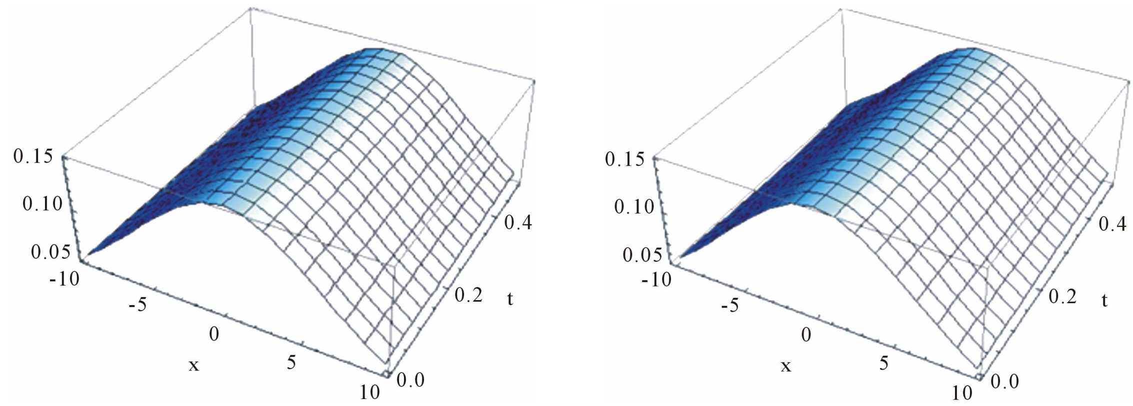


(a)



(b)

Figure 3. The exact solution and numerical solution with new modification of Laplace ADM for Example (1), (a) for *t* = 0.1 and (b) for *t* = 0.5.



(a) (b)

Figure 4. (a) The exact solution for $t = 0.5$ and $-10 \le x \le 10$; (b) Numerical solution with new modification for $t = 0.5$ and $-10 \le x \le 10$.

ADM to solve the RLW equation. The decomposition series solutions are converge very rapidly in real physical problems. The numerical results we obtained justify the advantage of this methodology, even in the few terms approximation is accurate. The method is tested on the problem of single solitary motion and high accuracy was achieved with the L_2 and L_∞ error norms. The new Laplace ADM presented here is for the RLW equation, but it can be implemented to a large number of physically important nonlinear wave problems.

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