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On a Number of Colors in Cyclically Interval Edge Colorings of Simple Cycles

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ABSTRACT

A proper edge *t*-coloring of a graph *G* is a coloring of its edges with colors $1,2,\cdots,t$ such that all colors are used, and no two adjacent edges receive the same color. A cyclically interval *t*-coloring of a graph *G* is a proper edge *t*-coloring of *G* such that for each its vertex *x*, either the set of colors used on edges incident to *x* or the set of colors not used on edges incident to *x* forms an interval of integers. For an arbitrary simple cycle, all possible values of *t* are found, for which the graph has a cyclically interval *t*-coloring.

Keywords: Proper Edge Coloring; Cyclically Interval Coloring; Simple Cycle

1. Introduction

We consider undirected, simple, finite and connected graphs. For a graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. The set of edges of G incident with a vertex $x\in V(G)$ is denoted by $J_G(x)$. For any $x\in V(G)$, $d_G(x)$ denotes the degree of the vertex x in G. For a graph G, $\Delta(G)$ denotes the maximum degree of a vertex of G. A simple cycle with n edges $(n\ge 3)$ is denoted by $C(n)$. A simple path with n edges $(n\ge 1)$ is denoted by $P(n)$. The terms and concepts that we do not define can be found in [1].

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A. The set of positive integers is denoted by $\mathbb{N}$. For any subset D of the set $\mathbb{N}$, we denote by $D_{(0)}$ and $D_{(1)}$ the subsets of all even and all odd elements of D, respectively.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p,q]$. An interval D is called a h-interval if $|D|=h$.

For any real number ξ, we denote by $\lfloor\xi\rfloor$ $(\lceil\xi\rceil)$ the maximum (minimum) integer which is less (greater) than or equal to ξ.

For any positive integer k define

$$\varepsilon(k)\equiv 1+\left\lfloor\frac{k}{2}\right\rfloor-\left\lceil\frac{k}{2}\right\rceil.$$

For any nonnegative integer k define

$$\delta(k)\equiv\begin{cases}0, & \text{if } k=0\\ 1 & \text{otherwise.}\end{cases}$$

A function $\varphi: E(G)\to[1,t]$ is called a proper edge t-coloring of a graph G, if all colors are used, and no two adjacent edges receive the same color.

The minimum value of t for which there exists a proper edge t-coloring of a graph G is denoted by $\chi'(G)$ [2].

If G is a graph, and φ is its proper edge t-coloring, where $t\in\left[\chi'(G),|E(G)|\right]$, then we define

$$U(G,\varphi)\equiv\left\{e\in E(G)/1<\varphi(e)<t\right\}.$$

If $E_0\subseteq E(G)$, $t\in\left[\chi'(G),|E(G)|\right]$, and φ is a proper edge t-coloring of a graph G, then we set

$$\varphi[E_0]\equiv\left\{\varphi(e)/e\in E_0\right\}.$$

A proper edge t-coloring $\left(t\in\left[\chi'(G),|E(G)|\right]\right)$ φ of a graph G is called an interval t-coloring of G [3-5] if for any $x\in V(G)$, the set $\varphi\left[J_G(x)\right]$ is a $d_G(x)$-interval. For any $t\in\mathbb{N}$, we denote by $\mathfrak{N}_t$ the set of graphs for which there exists an interval t-coloring. Let

$$\mathfrak{N}=\bigcup_{t\ge 1}\mathfrak{N}_t.$$

For any $G\in\mathfrak{N}$, we denote by $w_{\text{int}}(G)$ and $W_{\text{int}}(G)$ the minimum and the maximum possible value of t, respectively, for which $G\in\mathfrak{N}_t$. For a graph G, let us set $\theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{N}_t\}$.

A proper edge t-coloring $\left(t \in \left[\chi'(G), |E(G)|\right]\right)$ φ of a graph G is called a cyclically interval t-coloring of G, if for any $x \in V(G)$, at least one of the following two conditions holds:

1) $\varphi\left[J_G(x)\right]$ is a $d_G(x)$-interval,
2) $[1,t] \setminus \varphi\left[J_G(x)\right]$ is a $\left(t-d_G(x)\right)$-interval.

For any $t \in \mathbb{N}$, we denote by $\mathfrak{M}_t$ the set of graphs for which there exists a cyclically interval t-coloring. Let

$$\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t .$$

For any $G \in \mathfrak{M}$, we denote by $w_{cyc}(G)$ and $W_{cyc}(G)$ the minimum and the maximum possible value of t, respectively, for which $G \in \mathfrak{M}_t$. For a graph G, let us set $\Theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{M}_t\}$.

It is clear that for any $G \in \mathfrak{N}$, an arbitrary interval t-coloring $\left(t \in \theta(G)\right)$ of a graph G is also a cyclically interval t-coloring of G. Thus, for any $t \in \mathbb{N}$, $\mathfrak{N}_t \subseteq \mathfrak{M}_t$ and $\mathfrak{N} \subseteq \mathfrak{M}$. Let us also note that for an arbitrary graph G, $\theta(G) \subseteq \Theta(G)$. It is also clear that for any $G \in \mathfrak{N}$, the following inequality is true:

$$\Delta(G) \leq \chi'(G) \leq w_{cyc}(G) \leq w_{\mathrm{int}}(G)$$

and

$$W_{\mathrm{int}}(G) \leq W_{cyc}(G) \leq |E(G)|.$$

In [5,6], for any tree G, it is proved that $G \in \mathfrak{N}$, $\theta(G)$ is an interval, and the exact values of the parameters $w_{\mathrm{int}}(G)$, $W_{\mathrm{int}}(G)$ are found. In [7,8], for any tree G, it is proved that $\Theta(G) = \theta(G)$. Some interesting results on cyclically interval t-colorings and related topics were obtained in [9-14].

In this paper, for any integer $n \geq 3$, it is proved that $C(n) \in \mathfrak{M}$, and the set $\Theta(C(n))$ is found.

2. Main Results

Remark 1. Clearly, for any integer $n \geq 3$,

$$\chi'(C(n)) = 3 - \varepsilon(n),\ \left|E(C(n))\right| = n .$$

Therefore, if $t \notin \left[3-\varepsilon(n), n\right]$, then a proper edge t-coloring of $C(n)$ does not exist, and $C(n) \notin \mathfrak{N}_t$.

Remark 2. It is not difficult to see that for any integer $k \geq 2$, $C(2k) \in \mathfrak{N}$ and $\theta(C(2k)) = [2, k+1]$.

Proposition 1. For any integer $n \geq 3$, $C(n) \in \mathfrak{M}$, $n \in \Theta(C(n))$. $\Theta(C(3)) = \{3\}$, $\Theta(C(4)) = \{2,3,4\}$.

Proof is trivial.

Theorem 1. For any integers n and t, satisfying the conditions $n \geq 5$ and $t \in \left[3-\varepsilon(n), n\right]$, $C(n) \notin \mathfrak{M}_t$ if and only if

$$t \in \left[4 + \varepsilon(n) \cdot \left(\frac{n}{2} + \varepsilon\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 2\right), n-1\right]_{(\varepsilon(n))} .$$

Proof. First let us prove, that if $n \in \mathbb{N}$, $n \geq 5$ and

$$t \in \left[4 + \varepsilon(n) \cdot \left(\frac{n}{2} + \varepsilon\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - 2\right), n-1\right]_{(\varepsilon(n))}$$

then $C(n) \notin \mathfrak{M}_t$.

Assume the contrary: there are $n_0 \in \mathbb{N}$, $n_0 \geq 5$ and

$$t_0 \in \left[4 + \varepsilon(n_0) \cdot \left(\frac{n_0}{2} + \varepsilon\left(\left\lfloor \frac{n_0}{2} \right\rfloor\right) - 2\right), n_0-1\right]_{(\varepsilon(n_0))},$$

for which a cyclically interval t_0-coloring α of the graph $C(n_0)$ exists.

Let us construct a graph H_{00} removing from the graph $C(n_0)$ the subset $U\left(C(n_0), \alpha\right)$ of its edges. Let us construct a graph H_0 removing from the graph H_{00} all its isolated vertices.

Case A. H_0 is a connected graph.

Let us denote by F the simple path with pendant edges e' and e'' which is isomorphic to the graph $P\left(n_0 - \left|E(H_0)\right| + 2\right)$.

*Case A.*1. n_0 is odd.

Clearly, $t_0 \in \left[4, n_0 - 1\right]_{(0)}$. It means that t_0 is an even number, satisfying the inequality $4 \leq t_0 \leq n_0 - 1$.

Case A.1.1. $\left|E(H_0)\right|$ is odd.

Clearly, $\left|E(H_0)\right| \geq 3$. Since α is a cyclically interval t_0-coloring of $C(n_0)$, we conclude from the definition of H_0, that for a graph F, there exists an interval $(t_0 - 1)$-coloring β_1 with $\beta_1(e') = \beta_1(e'')$. Consequently, the number $n_0 - \left|E(H_0)\right| + 2$ is odd, what contradicts the same parity of n_0 and $\left|E(H_0)\right|$.

Case A.1.2. $\left|E(H_0)\right|$ is even.

Clearly, $\left|E(H_0)\right| \geq 2$. Since α is a cyclically interval t_0-coloring of $C(n_0)$, we conclude from the definition of H_0, that for a graph F, there exists an interval t_0-coloring β_2 with $\beta_2(e') = 1$ and $\beta_2(e'') = t_0$. Consequently, the number $n_0 - \left|E(H_0)\right| + 2$ is even, what contradicts the different parity of n_0 and $\left|E(H_0)\right|$.

Case A.2. n_0 is even.

Clearly, $t_0 \in \left[\frac{n_0}{2} + 2 + \varepsilon\left(\frac{n_0}{2}\right), n_0 - 1\right]_{(1)}$. It means that t_0 is an odd number, satisfying the inequality

$$\frac{n_0}{2} + 2 + \varepsilon\left(\frac{n_0}{2}\right) \leq t_0 \leq n_0 - 1 .$$

Case A.2.1. $\left|E(H_0)\right|$ is odd.

Clearly, $\left|E(H_0)\right| \geq 3$. Since α is a cyclically interval t_0-coloring of $C(n_0)$, we can conclude from the definition of H_0, that for a graph F, there exists an interval (t_0-1)-coloring β_3 with $\beta_3(e')=\beta_3(e'')$. Consequently,

$$n_0 > n_0 - |E(H_0)| + 2 = |E(F)|$$
$$\geq 2t_0 - 3 \geq n_0 + 1 + 2\cdot\varepsilon\left(\frac{n_0}{2}\right) > n_0,$$

which is impossible.

*Case A.*2.2. $|E(H_0)|$ is even.

Clearly, $|E(H_0)| \geq 2$. Since α is a cyclically interval t_0-coloring of $C(n_0)$, we can conclude from the definition of H_0, that for a graph F, there exists an interval t_0-coloring β_4 with $\beta_4(e')=1$ and $\beta_4(e'')=t_0$. Since t_0 is odd, the number $n_0 - |E(H_0)| + 2$ is also odd, but it is impossible because of the same parity of n_0 and $|E(H_0)|$.

Case B. H_0 is a graph with m connected components, $m \geq 2$.

Assume that:

1) $H_1,\cdots,H_m$ are connected components of H_0 numbered in succession at bypassing of the graph $C(n_0)$ in some fixed direction,

2) $v_1,\cdots,v_{n_0}$ are vertices of $C(n_0)$ numbered in succession at bypassing mentioned in 1),

3) $e_1,\cdots,e_{n_0}$ are edges of $C(n_0)$ numbered in succession at bypassing mentioned in 1),

4) $v_1 \in V(H_1)$, $v_2 \in V(H_1)$, $v_{n_0} \notin V(H_1)$, $e_1 = (v_1, v_2)$.

Define functions

$$\zeta : [1,m] \to [1, n_0 - 1],$$
$$\eta : [1,m] \to [1, n_0 - 1],$$
$$y : [1,2m] \to \{0,1\}$$

as follows. For any $i \in [1,m]$, set:

$$\zeta(i) \equiv \min\{k / e_k \in E(H_i)\},$$
$$\eta(i) \equiv \max\{k / e_k \in E(H_i)\}.$$

For any $j \in [1,2m]$, set

$$y(j) \equiv \begin{cases} \delta\left(\alpha\left(e_{\zeta\left(\frac{j+1}{2}\right)}\right)-1\right), & \text{if } j \text{ is odd} \\ \delta\left(\alpha\left(e_{\eta\left(\frac{j}{2}\right)}\right)-1\right), & \text{if } j \text{ is even.} \end{cases}$$

Now let us define subgraphs $H'_1,\cdots,H'_m$ of the graph $C(n_0)$.

For any $i \in [1, m-1]$, let H'_i be the subgraph of $C(n_0)$ induced by the subset

$$\{v_{\eta(i)}, v_{\eta(i)+1}, \cdots, v_{\zeta(i+1)}, v_{\zeta(i+1)+1}\}$$

of its vertices. Let H'_m be the subgraph of $C(n_0)$ induced by the subset

$$\{v_{\eta(m)}, v_{\eta(m)+1}, \cdots, v_{n_0}, v_1, v_2\}$$

of its vertices.

Let

$$M_1 \equiv \{i \in [1,m] / 1 \in \alpha[E(H'_i)]\},$$
$$M_2 \equiv \{i \in [1,m] / t_0 \in \alpha[E(H'_i)]\}.$$

For any $j \in [1,2m]$, we define a point π_j of the 2-dimensional rectangle coordinate system by the following way: $\pi_j \equiv (j, y(j))$.

Let us define a graph $\tilde{H}$. Set

$$V(\tilde{H}) \equiv \{\pi_1, \cdots, \pi_{2m}\},$$
$$E(\tilde{H}) \equiv \{(\pi_{2m}, \pi_1)\} \cup \{(\pi_j, \pi_{j+1}) / j \in [1, 2m-1]\}.$$

Clearly, $\tilde{H} \cong C(2m)$.

Let

$$E_1(\tilde{H}) \equiv \{(\pi_{2q-1}, \pi_{2q}) / q \in [1,m]\},$$
$$E_2(\tilde{H}) \equiv E(\tilde{H}) \setminus E_1(\tilde{H}).$$

An edge (π', π'') of the graph $\tilde{H}$ is called horizontal if the points π' and π'' have the same ordinate.

Let us denote by $E_-(\tilde{H})$ the set of all horizontal edges of the graph $\tilde{H}$. Set $E_|(\tilde{H}) \equiv E(\tilde{H}) \setminus E_-(\tilde{H})$. It is easy to note that the numbers $|E_-(\tilde{H})|$ and $|E_|(\tilde{H})|$ are both even.

Now let us define a function $\psi : E(\tilde{H}) \to [1, n_0 - 1]$ by the following way. For an arbitrary $e \in E(\tilde{H})$ set:

$$\psi(e) \equiv \begin{cases} |E(H_q)|, & \text{if } e = (\pi_{2q-1}, \pi_{2q}), \text{where } q \in [1,m] \\ |E(H'_q)|, & \text{if } e = (\pi_{2q}, \pi_{2q+1}), \text{where } q \in [1, m-1] \\ |E(H'_m)|, & \text{if } e = (\pi_{2m}, \pi_1). \end{cases}$$

Clearly,

$$\sum_{e \in E(\tilde{H})} \psi(e) = n_0 + 2m.$$

*Case B.*1. n_0 is odd.

Clearly, $t_0 \in [4, n_0 - 1]_{(0)}$. It means that t_0 is an even number, satisfying the inequality $4 \leq t_0 \leq n_0 - 1$. It is not difficult to see that in this case, for an arbitrary $e \in E_-(\tilde{H})$, $\psi(e)$ is odd, and, moreover, for an arbitrary $e \in E_|(\tilde{H})$, $\psi(e)$ is even. Since $|E_-(\tilde{H})|$ is even, we conclude that the odd number

$$n_0 + 2m = \sum_{e\in E_-(\tilde{H})} \psi(e) + \sum_{e\in E_|(\tilde{H})} \psi(e)$$

is represented as a sum of two even numbers, which is impossible.

Case B.2. n_0 is even.

Clearly,

$$t_0 \in \left[\frac{n_0}{2} + 2 + \varepsilon\left(\frac{n_0}{2}\right), n_0 - 1\right]_{(1)}.$$

It means that t_0 is an odd number, satisfying the inequality

$$\frac{n_0}{2} + 2 + \varepsilon\left(\frac{n_0}{2}\right) \le t_0 \le n_0 - 1.$$

It is not difficult to see that in this case, for an arbitrary $e \in E_2(\tilde{H}) \cup E_-(\tilde{H})$, $\psi(e)$ is odd, and, moreover, for an arbitrary $e \in E_1(\tilde{H}) \cap E_|(\tilde{H})$, $\psi(e)$ is even.

Case B.2.1. $\left|E_2(\tilde{H}) \cap E_|(\tilde{H})\right| \ge 2$.

In this case, evidently, there are different integers i' and i'' in the set $[1,m]$, for which there exist interval t_0-colorings β' and β'' of the graphs $H'_{i'}$ and $H'_{i''}$, respectively. Consequently,

$$\begin{aligned} n_0 &= \left|E(C(n_0))\right| \ge \left|E(H'_{i'}) \cup E(H'_{i''})\right| \\ &= \left|E(H'_{i'})\right| + \left|E(H'_{i''})\right| - \left|E(H'_{i'}) \cap E(H'_{i''})\right| \\ &\ge \left|E(H'_{i'})\right| + \left|E(H'_{i''})\right| - 2 \ge 2t_0 - 2 \\ &\ge n_0 + 2 + 2\varepsilon\left(\frac{n_0}{2}\right) > n_0, \end{aligned}$$

which is impossible.

Case B.2.2. $\left|E_2(\tilde{H}) \cap E_|(\tilde{H})\right| = 1$.

Without loss of generality assume that

$$E_2(\tilde{H}) \cap E_|(\tilde{H}) = \{e^0\}.$$

Since $\left|E_-(\tilde{H})\right|$ is even, we conclude that the even number

$$\begin{aligned} &n_0 + 2m \\ &= \sum_{e\in E_-(\tilde{H})} \psi(e) + \sum_{e\in E_|(\tilde{H})} \psi(e) \\ &= \sum_{e\in E_2(\tilde{H})\cap E_|(\tilde{H})} \psi(e) + \sum_{e\in E_1(\tilde{H})\cap E_|(\tilde{H})} \psi(e) + \sum_{e\in E_-(\tilde{H})} \psi(e) \\ &= \psi(e^0) + \sum_{e\in E_1(\tilde{H})\cap E_|(\tilde{H})} \psi(e) + \sum_{e\in E_-(\tilde{H})} \psi(e) \end{aligned}$$

is represented as a sum of one odd and two even numbers, which is impossible.

Case B.2.3. $\left|E_2(\tilde{H}) \cap E_|(\tilde{H})\right| = 0$.

Clearly, for any $i \in [1,m]$, the set $\alpha\left[E(H'_i)\right]$ contains exactly one of the colors 1 and t_0.

Case B.2.3.*a*). $M_1 \ne \varnothing$, $M_2 = \varnothing$.

It is not difficult to see that in this case there is $i_1 \in M_1$, for which the set $\alpha\left[E(H'_{i_1})\right]$ contains the color $t_0 - 1$. It means that there exists an interval $(t_0 - 1)$-coloring of the graph H'_{i_1} which colors pendant edges of H'_{i_1} by the color 1. Consequently,

$$n_0 > \left|E(H'_{i_1})\right| \ge 2t_0 - 3 \ge n_0 + 1 + 2\varepsilon\left(\frac{n_0}{2}\right) > n_0,$$

which is impossible.

Case B.2.3.*b*). $M_1 = \varnothing$, $M_2 \ne \varnothing$.

It is not difficult to see that in this case there is $i_2 \in M_2$, for which the set $\alpha\left[E(H'_{i_2})\right]$ contains the color 2. It means that there exists an interval $(t_0 - 1)$-coloring of the graph H'_{i_2} which colors pendant edges of H'_{i_2} by the color 1. Consequently,

$$n_0 > \left|E(H'_{i_2})\right| \ge 2t_0 - 3 \ge n_0 + 1 + 2\varepsilon\left(\frac{n_0}{2}\right) > n_0,$$

which is impossible.

Case B.2.3.*c*). $M_1 \ne \varnothing$, $M_2 \ne \varnothing$.

Let us choose $i_3 \in M_1$ and $i_4 \in M_2$ satisfying the conditions

$$\left|\alpha\left[E(H'_{i_3})\right]\right| = \max_{i\in M_1}\left|\alpha\left[E(H'_i)\right]\right|,$$

$$\left|\alpha\left[E(H'_{i_4})\right]\right| = \max_{i\in M_2}\left|\alpha\left[E(H'_i)\right]\right|.$$

Let $j^{(3)}$ be the maximum color of the set

$$\alpha\left[E(H'_{i_3})\right].$$

Let $j^{(4)}$ be the minimum color of the set

$$\alpha\left[E(H'_{i_4})\right].$$

Clearly, $j^{(3)} \ge j^{(4)} - 1$.

It is not difficult to see that there exists an interval $j^{(3)}$-coloring of the graph H'_{i_3} which colors pendant edges of H'_{i_3} by the color 1. Hence,

$$\left|E(H'_{i_3})\right| \ge 2j^{(3)} - 1.$$

It is not difficult to see that there exists an interval $\left(t_0 - j^{(4)} + 1\right)$-coloring of the graph H'_{i_4} which colors pendant edges of H'_{i_4} by the color 1. Hence,

$$\left|E(H'_{i_4})\right| \ge 2\cdot\left(t_0 - j^{(4)} + 1\right) - 1 = 2t_0 - 2j^{(4)} + 1.$$

Consequently, we obtain that

$$\begin{aligned} n_0 &> \left|E(H'_{i_3}) \cup E(H'_{i_4})\right| = \left|E(H'_{i_3})\right| + \left|E(H'_{i_4})\right| \\ &\ge 2t_0 + 2\left(j^{(3)} - j^{(4)}\right) \ge 2t_0 - 2 \ge n_0 + 2 + 2\varepsilon\left(\frac{n_0}{2}\right) > n_0, \end{aligned}$$

which is impossible.

Thus, we have proved that if $n \in \mathbb{N}$, $n \geq 5$ and

$$t \in \left[4+\varepsilon(n)\cdot\left(\frac{n}{2}+\varepsilon\left(\left\lfloor\frac{n}{2}\right\rfloor\right)-2\right), n-1\right]_{(\varepsilon(n))},$$

then $C(n) \notin \mathfrak{M}_t$.

Now let us prove that if

$$n \in \mathbb{N},\ n \geq 5,\ t \in [3-\varepsilon(n), n],\ C(n) \notin \mathfrak{M}_t,$$

then

$$t \in \left[4+\varepsilon(n)\cdot\left(\frac{n}{2}+\varepsilon\left(\left\lfloor\frac{n}{2}\right\rfloor\right)-2\right), n-1\right]_{(\varepsilon(n))}.$$

Assume the contrary. It means that there are $n_0 \in \mathbb{N}$, $n_0 \geq 5$, and $t_0 \in [3-\varepsilon(n_0), n_0]$, which satisfy the conditions $C(n_0) \notin \mathfrak{M}_{t_0}$ and

$$t_0 \notin \left[4+\varepsilon(n_0)\cdot\left(\frac{n_0}{2}+\varepsilon\left(\left\lfloor\frac{n_0}{2}\right\rfloor\right)-2\right), n_0-1\right]_{(\varepsilon(n_0))}.$$

Case 1. n_0 is odd.

In this case $t_0 \in [3, n_0]$ and $t_0 \notin [4, n_0-1]_{(0)}$, and, therefore, $t_0 \in [3, n_0]_{(1)}$. It means that there exists $m_0 \in \mathbb{N}$, for which

$$2 \leq m_0 = \frac{t_0+1}{2} \leq \frac{n_0+1}{2}.$$

Let us note that the equality $m_0 = \dfrac{n_0+1}{2}$ implies $t_0 = n_0$, which is incompatible with the condition $C(n_0) \notin \mathfrak{M}_{t_0}$. Hence, $n_0 - 2m_0 \geq 1$.

Now, to see a contradiction, it is enough to note that the existence of an interval t_0-coloring of a graph $P(2m_0-1)$ with the existence of an interval 2-coloring of a graph $P(n_0-2m_0+1)$ provides the existence of a cyclically interval t_0-coloring of the graph $C(n_0)$.

Case 2. n_0 is even.

In this case $t_0 \in [2, n_0]$ and

$$t_0 \notin \left[\frac{n_0}{2}+2+\varepsilon\left(\frac{n_0}{2}\right), n_0-1\right]_{(1)},$$

and, therefore,

$$t_0 \in \left[2, \frac{n_0}{2}+1\right] \cup \left(\left[\frac{n_0}{2}+3-\varepsilon\left(\frac{n_0}{2}\right), n_0\right]_{(0)}\right).$$

It follows from Remark 2 that

$$t_0 \in \left[\frac{n_0}{2}+3-\varepsilon\left(\frac{n_0}{2}\right), n_0\right]_{(0)}.$$

Clearly, there exists $m_0 \in \mathbb{N}$,

$$m_0 \leq \frac{1}{2}\left(\frac{n_0}{2}+\varepsilon\left(\frac{n_0}{2}\right)-1\right),$$

for which

$$t_0 = \frac{n_0}{2}+1-\varepsilon\left(\frac{n_0}{2}\right)+2m_0.$$

Let us note that the equality

$$m_0 = \frac{1}{2}\left(\frac{n_0}{2}+\varepsilon\left(\frac{n_0}{2}\right)-1\right)$$

implies $t_0 = n_0$, which is incompatible with the condition $C(n_0) \notin \mathfrak{M}_{t_0}$. Hence,

$$\frac{n_0}{2}+\varepsilon\left(\frac{n_0}{2}\right)-1-2m_0$$

is an even number, satisfying the inequality

$$\frac{n_0}{2}+\varepsilon\left(\frac{n_0}{2}\right)-1-2m_0 \geq 2.$$

Now, to see a contradiction, it is enough to note that the existence of an interval t_0-coloring of a graph

$$P\left(\frac{n_0}{2}+1-\varepsilon\left(\frac{n_0}{2}\right)+2m_0\right)$$

with the existence of an interval 2-coloring of a graph

$$P\left(\frac{n_0}{2}+\varepsilon\left(\frac{n_0}{2}\right)-1-2m_0\right)$$

provides the existence of a cyclically interval t_0-coloring of the graph $C(n_0)$.

Thus, we have proved, that if $n \in \mathbb{N}$, $n \geq 5$,

$$t \in [3-\varepsilon(n), n],\ C(n) \notin \mathfrak{M}_t,$$

then

$$t \in \left[4+\varepsilon(n)\cdot\left(\frac{n}{2}+\varepsilon\left(\left\lfloor\frac{n}{2}\right\rfloor\right)-2\right), n-1\right]_{(\varepsilon(n))}.$$

Theorem 1 is proved.

It means that we also have

Theorem 2. For an arbitrary integer $n \geq 5$,

$$\Theta(C(n)) = \begin{cases} [3, n]_{(1)}, & \text{if } n \text{ is odd} \\ \left[2, \frac{n}{2}+1\right] \cup \left(\left[\frac{n}{2}+3-\varepsilon\left(\frac{n}{2}\right), n\right]_{(0)}\right), & \text{if } n \text{ is even.} \end{cases}$$

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