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The Brunn-Minkowski Inequalities for Centroid Body

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ABSTRACT

In [1], the authors established the Brunn-Minkowski inequality for centroid body. In this paper, we give an isolate form and volume difference of it, respectively. Both of these results are strength versions of the original.

Keywords: Centroid Body; The Brunn-Minkowski Inequality

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors). Let B_n and S^{n-1} denote the unit ball and unit sphere in $\mathbb{R}^n$, respectively. If $K \in K^n$, then the support function of K, $h_K = h(K,\cdot): S^{n-1} \to \mathbb{R}$, is defined by

$$h(K,u) = \max\{u \cdot x : x \in K\}, u \in S^{n-1} \tag{1.1}$$

where $u \cdot x$ denotes the standard inner product of u and x.

For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, denoted by $V(K)$ its n-dimensional volume. The centroid body ΓK of K is the origin-symmetric convex body whose support function is given by (see [2])

$$h(\Gamma K, u) = \frac{1}{V(K)} \int_K |u \cdot x| \mathrm{d}x, \tag{1.2}$$

where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$.

Centroid body was attributed by Blaschke and Dupin (see [3,4]), it was defined and investigated by Petty [2]. More results regarding centroid body see [2-7].

For star body K and L, let $K \hat{+} L$ denote the harmonic Blaschke addition of K and L. In [1], the authors established the following Brunn-Minkowski inequality for centroid body.

Theorem A. *Let* K, L *be star bodies in* $\mathbb{R}^n$. *Then*

$$V(\Gamma(K \hat{+} L))^{\frac{1}{n}} \geq V(\Gamma K)^{\frac{1}{n}} + V(\Gamma L)^{\frac{1}{n}}, \tag{1.3}$$

the equality holds if and only if ΓK *and* ΓL *are homothetic*.

In this paper, we give two strength versions of (1.3). Our main results are the following two theorems.

Theorem 1.1. *Let* K, L *be star bodies in* $\mathbb{R}^n$ *and* $0 \leq \alpha \leq 1$.

$$\begin{aligned} & V(\Gamma(K \hat{+} L))^{\frac{1}{n}} \\ & \geq V(\Gamma(\alpha K \hat{+} (1-\alpha)))L^{\frac{1}{n}} + V(\Gamma((1-\alpha)K \hat{+} \alpha L))^{\frac{1}{n}} \\ & \geq V(\Gamma K)^{\frac{1}{n}} + V(\Gamma L)^{\frac{1}{n}}, \end{aligned}$$

the equality holds if and only if ΓK *and* ΓL *are homothetic*.

Theorem 1.2. *Let* K *and* L *be star bodies in* $\mathbb{R}^n$. *Ellipsoid* $E_1 \subset K$, *and* $E_2 \subset L$ *is a homothetic copy of* E_1. *Then*

$$\begin{aligned} & \left[V(\Gamma(K \hat{+} L)) - V(\Gamma(E_1 \hat{+} E_2))\right]^{\frac{1}{n}} \\ & \geq \left[V(\Gamma K) - V(\Gamma E_1)\right]^{\frac{1}{n}} + \left[V(\Gamma L) - V(\Gamma E_2)\right]^{\frac{1}{n}}, \end{aligned}$$

the equality holds if and only if ΓK *and* ΓL *are homothetic and*

$$(V(\Gamma K), V(\Gamma E_1)) = \mu(V(\Gamma L), V(\Gamma E_2)),$$

where μ *is a constant*.

Remark. Let $\alpha = 1$ or $\alpha = 0$ in Theorem 1.1, or let $E_1 = E_2 = \varnothing$ in Theorem 1.2, we can both get the Theorem A.

2. Notation and Preliminary Works

For a compact subset L of $\mathbb{R}^n$, with the origin in its interior, star-shaped with respect to the origin, the radial

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function $\rho(L,\cdot): S^{n-1} \to \mathbb{R}$, is defined by

$$\rho(L,u) = \max\{\lambda : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body. Let φ_o^n denote the set of star bodies in $\mathbb{R}^n$.

The mixed volume $V(K_1,\cdots,K_n)$ of the compact convex subsets $K_1,\cdots,K_n$ of $\mathbb{R}^n$ is defined by

$$V(K_1,K_2,\cdots,K_n) = \frac{1}{n!}\sum_{j=1}^{n}(-1)^{n+j}\sum_{i_1<\cdots<i_j} V\left(K_{i_1}+K_{i_2}+\cdots+K_{i_j}\right).$$

If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, then $V(K_1,K_2,\cdots,K_n)$ will be denote as $V_i(K,L)=V(K,n-i;L,i)$. If $L=B_n$, then $V_i(K,B_n)$ is called the quermassintegrals of K; it will often be written as $W_i(K)(i=0,1,\cdots,n)$.

The mixed quermassintegrals $W_i(K,L)(i=0,1,\cdots,n-1)$ of $K,L\in K^n$, are defined by [8]

$$(n-i)W_i(K,L) = \lim_{\varepsilon\to 0^+}\frac{W_i(K+\varepsilon L)-W_i(K)}{\varepsilon}, \tag{2.2}$$

Since $W_i(\lambda K)=\lambda^{n-i}W_i(K)$, it follows that $W_i(K,K)=W_i(K)$, for all i. Since the quermassintegrals W_{n-1} is Minkowski linear, it follows that $W_{n-1}(K,L)=W_{n-1}(L)$ for all K.

Aleksandrov [9] and Fenchel and Jessen [10] have shown that for $K\in\mathcal{K}^n$ and $i=0,1,\cdots,n-1$, there exists a regular Borel measure $S_i(K,\cdot)$ on S^{n-1}, such that the mixed quermassintegrals $W_i(K,L)$ has the following integral representation:

$$W_i(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)\,\mathrm{d}S_i(K,u), \tag{2.3}$$

for all $L\in\mathcal{K}^n$. The measure $S_{n-1}(K,\cdot)$ is independent of the body K and is just ordinary Lebesgue measure, S on S^{n-1}. The surface area measure $S_0(K,\cdot)$ will frequently be written simply as $S(K,\cdot)$.

Suppose $K,L\in\varphi_o^n$, λ and μ are nonnegative real numbers and not both zero. To define the harmonic Blaschke addition, $\lambda K \hat{+} \mu L$, first define $\xi>0$ by [6]

$$\xi^{1/(n+1)} = \frac{1}{n}\int_{S^{n-1}}\Big[\lambda V(K)^{-1}\rho(K,u)^{n+1} + \mu V(L)^{-1}\rho(L,u)^{n+1}\Big]^{n/(n+1)}\mathrm{d}u. \tag{2.4}$$

The body $\lambda K \hat{+} \mu L \in \varphi_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(\lambda K \hat{+} \mu L,\cdot)^{n+1} = \lambda V(K)^{-1}\rho(K,\cdot)^{n+1} + \mu V(L)^{-1}\rho(L,\cdot)^{n+1}. \tag{2.5}$$

3. Inequalities for Centroid Body

In this section, we will establish the inequality more general than Theorem 1.1 as follows.

Theorem 3.1. *Let* $K,L\in\varphi_o^n$, $0\le i<n-1$ *and* $0\le\alpha\le 1$. *Then*

$$\begin{aligned} W_i\left(\Gamma(K \hat{+} L)\right)^{\frac{1}{n-i}} &\ge W_i\left(\Gamma(\alpha K \hat{+} (1-\alpha)L)\right)^{\frac{1}{n-i}} + W_i\left(\Gamma((1-\alpha)K \hat{+} \alpha L)\right)^{\frac{1}{n-i}} \\ &\ge W_i(\Gamma K)^{\frac{1}{n-i}} + W_i(\Gamma L)^{\frac{1}{n-i}}, \end{aligned}$$

with equality holds if and only if ΓK *and* ΓL *are homothetic*.

To prove Theorem 3.1, the following preliminary results will be needed:

Lemma 3.2. ([8]). *Let* $K,L\in\mathcal{K}^n$ *and* $0\le i<n-1$. *Then*

$$W_i(K,L)^{n-i} \ge W_i(K)^{n-i-1}W_i(L), \tag{3.1}$$

with equality if and only if K and L are homothetic.

Lemma 3.3. ([11]). *Let* $K,L\in\mathcal{K}^n$, $0\le i<n-1$. *Then*

$$W_i(K+L)^{1/(n-i)} \ge W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{3.2}$$

with equality if and only if K and L are homothetic.

Proof of Theorem 3.1.

By (2.4), (2.5) and the polar coordinate formula for volume, we can get $\xi=V(K \hat{+} L)$. Hence from (2.5), we obtain

$$\frac{\rho(\lambda K \hat{+} \mu L,\cdot)^{n+1}}{V(\lambda K \hat{+} \mu L)} = \frac{\lambda\rho(K,\cdot)^{n+1}}{V(K)} + \frac{\mu\rho(L,\cdot)^{n+1}}{V(L)}. \tag{3.3}$$

Using polar coordinates, (1.2) can be written as an integral over S^{n-1}

$$h(\Gamma K,u) = \frac{1}{V(K)}\int_{S^{n-1}}|u\cdot v|\,\rho_K(v)^{n+1}\,\mathrm{d}v. \tag{3.4}$$

Then from (3.3) and (3.4), we have

$$h\left(\Gamma(\lambda K \hat{+} \mu L),u\right) = \lambda h(\Gamma K,u) + \mu h(\Gamma L,u). \tag{3.5}$$

For $K,L\in\varphi_o^n$ and $0\le\alpha\le 1$. Let

$$F = \Gamma\left(\alpha K \hat{+} (1-\alpha)L\right),$$

$$G = \Gamma\left((1-\alpha)K \hat{+} \alpha L\right),$$

By (2.3) and (3.5), we have

$$\begin{aligned} W_i\left(\Gamma\left(K \hat{+} L\right)\right) &= \frac{1}{n}\int_{S^{n-1}} h\left(\Gamma\left(K \hat{+} L\right),u\right)\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right) = \frac{1}{n}\int_{S^{n-1}}\left[h\left(\Gamma K,u\right)+h\left(\Gamma L,u\right)\right]\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right) \\ &= \frac{1}{n}\int_{S^{n-1}}\left[\alpha h\left(\Gamma K,u\right)+\left(1-\alpha\right)h\left(\Gamma L,u\right)+\left(1-\alpha\right)h\left(\Gamma K,u\right)+\alpha h\left(\Gamma L,u\right)\right]\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right) \\ &= \frac{1}{n}\int_{S^{n-1}}\left[h\left(\Gamma\left(\alpha K \hat{+}\left(1-\alpha\right)L\right),u\right)+h\left(\Gamma\left(\left(1-\alpha\right)K \hat{+}\alpha L\right),u\right)\right]\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right) \\ &= \frac{1}{n}\int_{S^{n-1}}\left[h\left(F,u\right)+h\left(G,u\right)\right]\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right) = \frac{1}{n}\int_{S^{n-1}} h\left(F+G,u\right)\mathrm{d}S_i\left(\Gamma\left(K \hat{+} L\right),u\right). \end{aligned}$$

That is

$$W_i\left(\Gamma\left(K \hat{+} L\right)\right) = W_i\left(\Gamma\left(K \hat{+} L\right),F+G\right). \tag{3.6}$$

By Lemma 3.2, we get

$$W_i\left(\Gamma\left(K \hat{+} L\right)\right) \ge W_i\left(K \hat{+} L\right)^{(n-i-1)/(n-i)} W_i\left(F+G\right)^{1/(n-i)},$$

which implies that,

$$W_i\left(\Gamma\left(K \hat{+} L\right)\right) \ge W_i\left(F+G\right), \tag{3.7}$$

with equality holds if and only if $\Gamma\left(K \hat{+} L\right)$ and $F+G$ are homothetic.

The Brunn-Minkowski inequality (3.2) can now be used to conclude that

$$W_i\left(F+G\right)^{1/(n-i)} \ge W_i\left(F\right)^{1/(n-i)} + W_i\left(G\right)^{1/(n-i)}, \tag{3.8}$$

with equality holds if and only if F and G are homothetic.

By (3.7) and (3.8), we get the first inequality of Theorem 3.1. By the equality conditions of (3.7) and (3.8), the first equality of Theorem 3.1 holds if and only if ΓK and ΓL are homothetic.

By (3.5) and Lemma 3.3, we get

$$\begin{aligned} W_i\left(F\right)^{1/(n-i)} &= W_i\left(\Gamma\left(\alpha K \hat{+}\left(1-\alpha\right)\right)\right)^{1/(n-i)} \\ &= W_i\left(\alpha\Gamma K+\left(1-\alpha\right)\Gamma L\right)^{1/(n-i)} \\ &\ge \alpha W_i\left(\Gamma K\right)^{1/(n-i)} + \left(1-\alpha\right)W_i\left(\Gamma L\right)^{1/(n-i)}, \end{aligned}$$

Similarly,

$$W_i\left(G\right)^{1/(n-i)} \ge \left(1-\alpha\right)W_i\left(\Gamma K\right)^{1/(n-i)} + \alpha W_i\left(\Gamma L\right)^{1/(n-i)}.$$

Hence,

$$\begin{aligned} &W_i\left(F\right)^{1/(n-i)} + W_i\left(G\right)^{1/(n-i)} \\ &\ge W_i\left(\Gamma K\right)^{1/(n-i)} + W_i\left(\Gamma L\right)^{1/(n-i)}, \end{aligned}$$

with equality holds if and only if ΓK and ΓL are homothetic. This completes the proof.

Let $i = 0$ in Theorem 3.1, we obtain an isolate form of Brunn-Minkowski inequality for centroid body.

Corollary 3.4. *Let K, L be star bodies in $\mathbb{R}^n$ and $0 \le \alpha \le 1$.*

$$\begin{aligned} &V\left(\Gamma\left(K \hat{+} L\right)\right)^{\frac{1}{n}} \\ &\ge V\left(\Gamma\left(\alpha K \hat{+}\left(1-\alpha\right)L\right)\right)^{\frac{1}{n}} + V\left(\Gamma\left(\left(1-\alpha\right)K \hat{+}\alpha L\right)\right)^{\frac{1}{n}} \\ &\ge V\left(\Gamma K\right)^{\frac{1}{n}} + V\left(\Gamma L\right)^{\frac{1}{n}}, \end{aligned}$$

the equality holds if and only if ΓK and ΓL are homothetic.

Now, we establish the volume difference of Brunn-Minkowski inequality for centroid body.

Theorem 3.5. *Let K and L be star bodies in $\mathbb{R}^n$. Ellipsoid $E_1 \subset K$, and $E_2 \subset L$ is a homothetic copy of E_1. Then*

$$\begin{aligned} &\left[V\left(\Gamma\left(K \hat{+} L\right)\right)-V\left(\Gamma\left(E_1 \hat{+} E_2\right)\right)\right]^{\frac{1}{n}} \\ &\ge \left[V\left(\Gamma K\right)-V\left(\Gamma E_1\right)\right]^{\frac{1}{n}} + \left[V\left(\Gamma L\right)-V\left(\Gamma E_2\right)\right]^{\frac{1}{n}}, \end{aligned}$$

the equality holds if and only if ΓK and ΓL are homothetic and

$$\left(V\left(\Gamma K\right),V\left(\Gamma E_1\right)\right) = \mu\left(V\left(\Gamma L\right),V\left(\Gamma E_2\right)\right),$$

where μ is a constant.

To prove Theorem 3.5, we need the following two lemmas:

Lemma 3.6. (Bellman's inequality) ([12], p. 38). *Suppose that $a = \{a_1, a_2, \cdots, a_n\}$ and $b = \{b_1, b_2, \cdots, b_n\}$ are two n-tuples of positive real numbers, and $p > 1$ such that*

$$a_1^p - \sum_{i=2}^n a_i^p > 0 \text{ and } b_1^p - \sum_{i=2}^n b_i^p > 0.$$

Then

$$\begin{aligned} &\left(\left(a_1+b_1\right)^p - \sum_{i=2}^n\left(a_i+b_i\right)^p\right)^{\frac{1}{p}} \\ &\ge \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}}, \end{aligned}$$

with equality if and only if $a = \nu b$, where ν is a constant.

Lemma 3.7. (Busemann-Petty centroid inequality) ([4], p. 359). *Let* $K \in \mathcal{K}^n$. *Then*

$$V(\Gamma K) \geq \left(\frac{2k_{n-1}}{(n+1)k_n}\right)^n V(K),$$

with equality if and only if K *is a centered ellipsoid.*

Proof of Theorem 1.2. Applying inequality (1.3), we have

$$V\left(\Gamma\left(K \hat{+} L\right)\right)^{\frac{1}{n}} \geq V(\Gamma K)^{\frac{1}{n}} + V(\Gamma L)^{\frac{1}{n}}, \qquad (3.9)$$

the equality holds if and only if ΓK and ΓL are homothetic.

$$V\left(\Gamma\left(E_1 \hat{+} E_2\right)\right)^{\frac{1}{n}} = V(\Gamma E_1)^{\frac{1}{n}} + V(\Gamma E_2)^{\frac{1}{n}}, \qquad (3.10)$$

From (3.9) and (3.10), we obtain that

$$V\left(\Gamma\left(K \hat{+} L\right)\right) - V\left(\Gamma\left(E_1 \hat{+} E_2\right)\right) \geq \left[V(\Gamma K)^{\frac{1}{n}} + V(\Gamma L)^{\frac{1}{n}}\right]^n - \left[V(\Gamma E_1)^{\frac{1}{n}} + V(\Gamma E_2)^{\frac{1}{n}}\right]^n. \qquad (3.11)$$

Since $E_1 \subset K$ and $E_2 \subset L$, by Lemma 3.7, we get

$$V(\Gamma K) \geq \left(\frac{2k_{n-1}}{(n+1)k_n}\right)^n V(K) > \left(\frac{2k_{n-1}}{(n+1)k_n}\right)^n V(E_1) = V(\Gamma E_1),$$

and

$$V(\Gamma L) > V(\Gamma E_2),$$

By (3.11) and Bellman's inequality, we get

$$\left[V\left(\Gamma\left(K \hat{+} L\right)\right) - V\left(\Gamma\left(E_1 \hat{+} E_2\right)\right)\right]^{\frac{1}{n}} \geq \left[V(\Gamma K) - V(\Gamma E_1)\right]^{\frac{1}{n}} + \left[V(\Gamma L) - V(\Gamma E_2)\right]^{\frac{1}{n}}. \qquad (3.12)$$

By the equality conditions of (3.9) and the Bellman's inequality, the equality of (3.12) holds if and only if ΓK and ΓL are homothetic and

$$\left(V(\Gamma K), V(\Gamma D_1)\right) = \mu\left(V(\Gamma L), V(\Gamma D_2)\right),$$

where μ is a constant. This completes the proof.

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