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Some L_p Inequalities for B-Operators

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ABSTRACT

If $P(z)$ is a polynomial of degree at most *n* having all its zeros in $|z| \geq 1$, then it was recently claimed by Shah and Liman ([1], estimates for the family of \$B\$-operators, Operators and Matrices, (2011), 79-87) that for every $R \geq 1$, $p \geq 1$, $\|B[P \circ \rho](z)\|_p \leq \dfrac{R^n |\phi(\lambda_0, \lambda_1, \lambda_2)| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p$, where B is a $\mathcal{B}_n$-operator with parameters $\lambda_0, \lambda_1, \lambda_2$ in the sense of Rahman [2], $\rho(z) = Rz$ and $\phi(\lambda_0, \lambda_1, \lambda_2) = \lambda_0 + \lambda_1 \dfrac{n^2}{2} + \lambda_2 \dfrac{n^3(n-1)}{8}$. Unfortunately the proof of this result is not correct. In this paper, we present certain more general sharp L_p-inequalities for $\mathcal{B}_n$-operators which not only provide a correct proof of the above inequality as a special case but also extend them for $0 \leq p < 1$ as well.

Keywords: L^p-Inequalities; $\mathcal{B}_n$-Operators; Polynomials

1. Introduction and Statement of Results

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. For $P \in \mathcal{P}_n$, define

$$\|P(z)\|_0 := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P\left(e^{i\theta}\right)\right| d\theta\right\},$$

$$\|P(z)\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p\right\}^{1/p}, 0 < p < \infty$$

$$\|P(z)\|_\infty := \underset{|z|=1}{\text{Max}} |P(z)|,$$

and denote for any complex function $\rho : \mathbb{C} \to \mathbb{C}$ the composite function of P and ρ, defined by $(P \circ \rho)(z) := P(\rho(z))$ $(z \in \mathbb{C})$, as $P \circ \rho$.

A famous result known as Bernstein's inequality (for reference, see [3, p. 531], [4, p. 508] or [5] states that if $P \in \mathcal{P}_n$, then

$$\|P'(z)\|_\infty \leq n \|P(z)\|_\infty, \tag{1.1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

$$\|P(Rz)\|_\infty \leq R^n \|P(z)\|_\infty, R \geq 1, \tag{1.2}$$

(for reference, see [6, p. 442] or [3, Vol. 1, p. 137]).

Inequalities (1.1) and (1.2) can be obtained by letting $p \to \infty$ in the inequalities

$$\|P'(z)\|_p \leq n \|P(z)\|_p, p \geq 1 \tag{1.3}$$

and

$$\|P(Rz)\|_p \leq R^n \|P(z)\|_p, R > 1, p > 0, \tag{1.4}$$

respectively. Inequality (1.3) was found by Zygmund [7] whereas inequality (1.4) is a simple consequence of a result of Hardy [8] (see also [9, Th. 5.5]). Since inequality (1.3) was deduced from M. Riesz's interpolation formula [10] by means of Minkowski's inequality, it was not clear, whether the restriction on *p* was indeed essential. This question was open for a long time. Finally Arestov [11] proved that (1.3) remains true for $0 < p < 1$ as well.

If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then Inequalities (1.1) and (1.2) can be respectively replaced by

$$\|P'(z)\|_\infty \leq \frac{n}{2} \|P(z)\|_\infty, \tag{1.5}$$

and

$$\|P(Rz)\|_\infty \leq \frac{R^n + 1}{2} \|P(z)\|_\infty \quad R > 1. \tag{1.6}$$

Inequality (1.5) was conjectured by Erdös and later verified by Lax [12], whereas Inequality (1.6) is due to

Ankey and Ravilin [13].

Both the Inequalities (1.5) and (1.6) can be obtain by letting $p \to \infty$ in the inequalities

$$\|P'(z)\|_p \le n\frac{\|P(z)\|_p}{\|1+z\|_p},\ p \ge 0 \tag{1.7}$$

and for $R > 1, p > 0,$

$$\|P(Rz)\|_p \le \frac{\|R^n z+1\|_p}{\|1+z\|_p}\|P(z)\|_p. \tag{1.8}$$

Inequality (1.7) is due to De-Bruijn [14] for $p \ge 1$. Rahman and Schmeisser [15] extended it for $0 \le p < 1$ whereas the Inequality (1.8) was proved by Boas and Rahman [16] for $p \ge 1$ and later it was extended for $0 \le p < 1$ by Rahman and Schmeisser [15].

Q. I. Rahman [2] (see also Rahman and Schmeisser [4, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that carries a polynomial $P \in \mathcal{P}_n$ into

$$\begin{aligned} B[P](z) := \lambda_0 P(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!} \\ + \lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!}, \end{aligned} \tag{1.9}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$U(z) := \lambda_0 + \lambda_1 C(n,1)z + \lambda_2 C(n,2)z^2 \tag{1.10}$$

where $C(n,r) = \dfrac{n!}{r!(n-r)!}$ $0 \le r \le n,$ lie in half plane $|z| \le |z - n/2|$.

As a generalization of Inequality (1.1) and (1.5), Q. I. Rahman [2, inequality 5.2 and 5.3] proved that if $P \in \mathcal{P}_n$, and $B \in \mathcal{B}_n$ then for $|z| \ge 1,$

$$|B[P](z)| \le |\phi_n(\lambda_0,\lambda_1,\lambda_2)|\|P(z)\|_\infty, \tag{1.11}$$

and if $P \in \mathcal{P}_n$, $P(z) \ne 0$ in $|z| < 1,$ then $|z| \ge 1,$

$$|B[P](z)| \le \frac{1}{2}\{|\phi_n(\lambda_0,\lambda_1,\lambda_2)| + |\lambda_0|\}\|P(z)\|_\infty, \tag{1.12}$$

where

$$\phi_n(\lambda_0,\lambda_1,\lambda_2) = \lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}. \tag{1.13}$$

As a corresponding generalization of Inequalities (1.2) and (1.4), Rahman and Schmeisser [4, p. 538] proved that if $P \in \mathcal{P}_n$, then $|z| = 1,$

$$|B[P \circ \rho](z)| \le R^n|\phi_n(\lambda_0,\lambda_1,\lambda_2)|\|P(z)\|_\infty. \tag{1.14}$$

and if $P \in \mathcal{P}_n$, $P(z) \ne 0$ in $|z| < 1,$ then as a special case of Corollary 14.5.6 in [4, p. 539], we have

$$\begin{aligned} &|B[P \circ \rho](z)| \\ &\le \frac{1}{2}\{R^n|\phi_n(\lambda_0,\lambda_1,\lambda_2)| + |\lambda_0|\}\|P(z)\|_\infty, \end{aligned} \tag{1.15}$$

where $\rho(z) := Rz, R \ge 1$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13).

Inequality (1.15) also follows by combining the Inequalities (5.2) and (5.3) due to Rahman [2].

As an extension of Inequality (1.14) to L_p-norm, recently Shah and Liman [1, Theorem 1] proved:

Theorem A. *If* $P \in \mathcal{P}_n$, *then for every* $R \ge 1$ *and* $p \ge 1$,

$$\|B[P \circ \rho](z)\|_p \le R^n|\phi_n(\lambda_1,\lambda_2,\lambda_3)|\|P(z)\|_p, \tag{1.16}$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ *and* $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ *is defined by* (1.13).

While seeking the analogous result of (1.15) in L_p norm, they [1, Theorem 2] have made an incomplete attempt by claiming to have proved the following result:

Theorem B. *If* $P \in \mathcal{P}_n$, *and* $P(z)$ *does not vanish for* $|z| \le 1$, *then for each* $p \ge 1$, $R \ge 1$,

$$\|B[P \circ \rho](z)\| \le \frac{R^n|\phi_n(\lambda_1,\lambda_2,\lambda_3)| + |\lambda_0|}{\|1+z\|_p}\|P(z)\|_p, \tag{1.17}$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ *and* $\phi_n(\lambda_1,\lambda_2,\lambda_3)$ *is defined by* (1.13).

Further, it has been claimed in [1] to have proved the Inequality (1.17) for self-inversive polynomials as well.

Unfortunately the proof of Inequality (1.17) and other related results including the key lemma [1, Lemma 4] given by Shah and Liman is not correct. The reason being that the authors in [1] deduce:

1) line 10 from line 7 on page 84,
2) line 19 on page 85 from Lemma 3 [1] and,
3) line 16 from line 14 on page 86,

by using the argument that if $P^\star(z) := z^n\overline{P(1/\bar{z})}$, then for $\rho(z) = Rz$, $R \ge 1$ and $|z| = 1,$

$$|B[P^\star \circ \rho](z)| = \left|B\left[(P^\star \circ \rho)^\star\right](z)\right|,$$

which is not true, in general, for every $R \ge 1$ and $|z| = 1$. To see this, let

$$P(z) = a_n z^n + \cdots + a_k z^k + \cdots + a_1 z + a_0$$

be an arbitrary polynomial of degree n, then

$$\begin{aligned} P^\star(z) &:= z^n\overline{P(1/\bar{z})} \\ &= \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_k z^{n-k} + \cdots + \bar{a}_n. \end{aligned}$$

Now with $\omega_1 := \lambda_1 n/2$ and $\omega_2 := \lambda_2 n^2/8$, we have

$$\begin{aligned} &B[P^\star \circ \rho](z) \\ &= \sum_{k=0}^{n}(\lambda_0 + \omega_1(n-k) + \omega_2(n-k)(n-k-1))\bar{a}_k z^{n-k}R^{n-k}, \end{aligned}$$

and in particular for $|z|=1$, we get

$$B\left[P^{\star}\circ\rho\right](z)=R^n z^n \cdot\sum_{k=0}^{n}\overline{\left(\lambda_0+\omega_1(n-k)+\omega_2(n-k)(n-k-1)\right)a_k\left(\frac{z}{R}\right)^k},$$

whence

$$\left|B\left[P^{\star}\circ\rho\right](z)\right| = R^n\left|\sum_{k=0}^{n}\overline{\left(\lambda_0+\omega_1(n-k)+\omega_2(n-k)(n-k-1)\right)a_k}\left(\frac{z}{R}\right)^k\right|.$$

But

$$\left|B\left[\left(P^{\star}\circ\rho\right)^{\star}\right](z)\right| = R^n\left|\sum_{k=0}^{n}\left(\lambda_0+\omega_1 k+\omega_2 k(k-1)\right)a_k\left(\frac{z}{R}\right)^k\right|,$$

so the asserted identity does not hold in general for every $R\ge 1$ and $|z|=1$ as e.g. the immediate counterexample of $P(z):=z^n$ demonstrates in view of $P^{\star}(z)=1$, $\left|B\left[P^{\star}\circ\rho\right](z)\right|=|\lambda_0|$ and

$$\left|B\left[\left(P^{\star}\circ\rho\right)^{\star}\right](z)\right|=\left|\lambda_0+\lambda_1\left(n^2/2\right)+\lambda_2 n^3(n-1)/8\right|$$

for $|z|=1$.

Authors [1] have also claimed that Inequality (1.17) and its analogue for self-inversive polynomials are sharp has remained to be verified. In fact, this claim is also wrong.

The main aim of this paper is to establish L_p-mean extensions of the inequalities (1.14) and (1.15) for $0\le p<\infty$ and present correct proofs of the results mentioned in [1]. In this direction, we first present the following result which is a compact generalization of the Inequalities (1.1), (1.2), (1.14) and (1.16) and also extend Inequality (1.17) for $0\le p<1$ as well.

Theorem 1. *If $P\in\mathcal{P}_n$ then for $\alpha\in\mathbb{C}$ with $|\alpha|\le 1$, $0\le p<\infty$ and $R>r\ge 1$,*

$$\begin{aligned}&\left\|B\left[P\circ\rho_R\right](z)-\alpha B\left[P\circ\rho_r\right](z)\right\|_p\\ &\le\left|R^n-\alpha r^n\right|\left|\phi_n\left(\lambda_0,\lambda_1,\lambda_2\right)\right|\left\|P(z)\right\|_p,\end{aligned}\tag{1.18}$$

where $B\in\mathcal{B}_n$, $\rho_t(z)=tz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is given by (1.13). The result is best possible and equality holds in (1.18) for $P(z)=z^n$.

If we choose $\alpha=0$ in (1.18), we get the following result which extends Theorem A to $0\le p<1$,

Corollary 1. *If $P\in\mathcal{P}_n$ then for $0\le p<\infty$ and $R>1$,*

$$\left\|B\left[P\circ\rho\right](z)\right\|_p\le R^n\left|\phi_n\left(\lambda_0,\lambda_1,\lambda_2\right)\right|\left\|P(z)\right\|_p,\tag{1.19}$$

where $B\in\mathcal{B}_n$, $\rho(z)=Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is given by (1.13).

Remark 1. Taking $\lambda_0=0=\lambda_2$ in (1.19) and noting that in this case all the zeros of $U(z)$ defined in (1.10) lie in $|z|\le|z-n/2|$, we get for $R>1$ and $0\le p<\infty$

$$\left\|P'(Rz)\right\|_p\le nR^{n-1}\left\|P(z)\right\|_p,$$

which includes (1.4) as a special case. Next if we choose $\lambda_1=0=\lambda_2$ in (1.19), we get inequality (1.4). Inequality (1.11) also follows from Theorem 1 by letting $p\to\infty$ in (1.18).

Theorem 1 can be sharpened if we restrict ourselves to the class of polynomials $P(z)$ which does not vanish in $|z|<1$ In this direction, we next present the following interesting compact generalization of Theorem B which yields L_p mean extension of the inequality (1.12) for $0\le p<\infty$ which among other things includes a correct proof of inequality (1.17) for $1\le p<\infty$ as a special case.

Theorem 2. *If $P\in\mathcal{P}_n$ and $P(z)$ does not vanish for $|z|<1$ then for $\alpha\in\mathbb{C}$ with $|\alpha|\le 1$, $0\le p<\infty$ and $R>r\ge 1$,*

$$\begin{aligned}&\left\|B\left[P\circ\rho_R\right](z)-\alpha B\left[P\circ\rho_r\right](z)\right\|_p\\ &\le\frac{\left\|\left(R^n-\alpha r^n\right)\phi_n\left(\lambda_0,\lambda_1,\lambda_2\right)z+(1-\alpha)\lambda_0\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p\end{aligned}\tag{1.20}$$

where $B\in\mathcal{B}_n$, $\rho_t(z)=tz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13). The result is best possible and equality holds in (1.18) for $P(z)=az^n+b$, $|a|=|b|=1$.

If we take $\alpha=0$ in (1.20), we get the following result which is the generalization of Theorem B for $p\ge 1$ but also extends it for $0\le p<\infty$

Corollary 2. *If $P\in\mathcal{P}_n$ and $P(z)$ does not vanish for $|z|<1$ then for $0\le p<\infty$ and $R>1$,*

$$\left\|B\left[P\circ\rho\right](z)\right\|_p\le\frac{\left\|R^n\phi_n\left(\lambda_0,\lambda_1,\lambda_2\right)z+\lambda_0\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p,\tag{1.21}$$

$B\in\mathcal{B}_n$, $\rho(z)=Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13).

By triangle inequality, the following result is an immediately follows from Corollary 2.

Corollary 3. *If $P\in\mathcal{P}_n$ and $P(z)$ does not vanish for $|z|<1$ then for $0\le p<\infty$ and $R>1$,*

$$\left\|B\left[P\circ\rho\right](z)\right\|_p\le\frac{R^n\left|\phi_n\left(\lambda_0,\lambda_1,\lambda_2\right)\right|+\left|\lambda_0\right|}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p\tag{1.22}$$

$B\in\mathcal{B}_n$, $\rho(z)=Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13).

Remark 2. Corollary 3 establishes a correct proof of a result due to Shah and Liman [1, Theorem 3] for $p\ge 1$ and also extends it for $0\le p<1$ as well.

Remark 3. If we choose $\lambda_0 = 0 = \lambda_2$ in (1.21), we get for $R > 1$ and $0 \le p < \infty$,

$$\left\|P'(Rz)\right\|_p \le \frac{nR^{n-1}}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p$$

which, in particular, yields Inequality (1.7). Next if we take $\lambda_1 = 0 = \lambda_2$ in (1.21), we get Inequality (1.8). Inequality (1.12) can be obtained from corollary 2 by letting $p \to \infty$ in (1.20).

By using triangle inequality, the following result immediately follows from Theorem 2.

Corollary 4. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$ $0 \le p < \infty$ and $R > r \ge 1$,*

$$\begin{aligned}&\left\|B[P\circ\rho_R](z) - \alpha B[P\circ\rho_r](z)\right\|_p\\ &\le \frac{\left[\left|\left(R^n - \alpha r^n\right)\phi_n(\lambda_0,\lambda_1,\lambda_2)\right| + \left|(1-\alpha)\lambda_0\right|\right]}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p\end{aligned} \quad (1.23)$$

$B \in \mathcal{B}_n$, $\rho_t(t) = tz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13).

A polynomial $P \in \mathcal{P}_n$ is said be self-inversive if $P(z) \equiv \nu P^\star(z)$ where $|\nu| = 1$ and $P^\star(z)$ is the conjugate polynomial of $P(z)$, that is, $P^\star(z) := z^n\overline{P(1/\overline{z})}$.

Finally in this paper, we establish the following result for self-inversive polynomials , which includes a correct proof of an another result of Shah and Liman [1, Theorem 2] as a special case.

Theorem 3. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$ $0 \le p < \infty$ and $R > r \ge 1$,*

$$\begin{aligned}&\left\|B[P\circ\rho_R](z) - \alpha B[P\circ\rho_r](z)\right\|_p\\ &\le \frac{\left\|\left(R^n - \alpha r^n\right)\phi_n(\lambda_0,\lambda_1,\lambda_2)z + (1-\alpha)\lambda_0\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p,\end{aligned} \quad (1.24)$$

where $B \in \mathcal{B}_n$, $\rho_t(t) = tz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is given by (1.13). The result is sharp and an extremal polynomial is $P(z) = c\left(az^n + \overline{a}\right)$, $ac \ne 0$.

For $\alpha = 0$, we get the following result.

Corollary 5. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $0 \le p < \infty$ and $R > 1$,*

$$\begin{aligned}&\left\|B[P\circ\rho](z)\right\|_p\\ &\le \frac{\left\|R^n\phi_n(\lambda_0,\lambda_1,\lambda_2)z + \lambda_0\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p,\end{aligned} \quad (1.25)$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is given by (1.13).

The following result is an immediate consequence of Corollary 5.

Corollary 6 *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $0 \le p < \infty$ and $R > 1$,*

$$\begin{aligned}&\left\|B[P\circ\rho](z)\right\|_p\\ &\le \frac{\left[\left|R^n\phi_n(\lambda_0,\lambda_1,\lambda_2)\right| + \left|\lambda_0\right|\right]}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p,\end{aligned} \quad (1.26)$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is given by (1.13).

Remark 4. Corollary 6 establishes a correct proof of a result due to Shah and Liman [1, Theorem 3] for $p \ge 1$ and also extends it for $0 \le p < 1$ as well.

Remark 5. A variety of interesting results can be easily deduced from Theorem 3 in the same way as we have deduced from Theorem 2. Here we mention a few of these. Taking $\lambda_0 = 0 = \lambda_2$ = in (1.25), we get for $R > 1$ and $0 \le p < \infty$,

$$\left\|P'(Rz)\right\|_p \le \frac{nR^{n-1}}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p,$$

which, in particular, yields a result due to Dewan and Govil [17] and A. Aziz [18] for polynomials $P \in \mathcal{P}_n^\star$. Next if we choose $\lambda_1 = 0 = \lambda_2$ in (1.25), we get for $R < 1$; $0 \le p < \infty$

$$\left\|P(Rz)\right\|_p \le \frac{\left\|R^n z + 1\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p.$$

The above inequality is a special case of a result proved by Aziz and Rather [19].

Lastly letting $p \to \infty$ in (1.25), it follows that if $P(z)$, is a self-inversive polynomial then

$$\begin{aligned}&\left\|B[P\circ\rho](z)\right\|_\infty\\ &\le \frac{1}{2}\left\{R^n\left|\phi_n(\lambda_0,\lambda_1,\lambda_2)\right| + \left|\lambda_0\right|\right\}\left\|P(z)\right\|_\infty,\end{aligned} \quad (1.27)$$

where $B \in \mathcal{B}_n$, $\rho(z) = Rz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13). The result is sharp.

Inequality (1.27) is a special case of a result due to Rahman and Schmeisser [4, Cor. 14.5.6].

2. Lemma

For the proof of above theorems we need the following Lemmas:

The following lemma follows from Corollary 18.3 of [20, p. 86].

Lemma 1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le 1$, then all the zeros of $B[P](z)$ also lie in $|z| \le 1$.*

Lemma 2. *If $P \in \mathcal{P}_n$ and $P(z)$ have all its zeros in $|z| \le 1$ then for every $R \ge r \ge 1$, and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq 1$, we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leq 1$. Now for $0 \leq \theta < 2\pi$, $R \geq r \geq 1$, we have

$$\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| = \left\{\frac{R^2 + r_j^2 - 2Rr_j\cos(\theta - \theta_j)}{r + r_j^2 - 2rr_j\cos(\theta - \theta_j)}\right\}^{1/2}$$
$$\geq \left\{\frac{R + r_j}{r + r_j}\right\} \geq \left\{\frac{R+1}{r+1}\right\}, \text{for } j = 1, 2, \cdots, n.$$

Hence

$$\left|\frac{P(R\,e^{i\theta})}{P(re^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{R\,e^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|$$
$$\geq \prod_{j=1}^{n}\left(\frac{R+1}{r+1}\right) = \left(\frac{R+1}{r+1}\right)^n,$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$ and $R \geq r \geq 1$,

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|,$$

which completes the proof of Lemma 2.

Lemma 3. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *has no zero in* $|z| < 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \leq 1$, $R > r \geq 1$ *and* $|z| = 1$,

$$\begin{aligned}&|B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)| \\ &\leq \left|B\left[P^\star \circ \rho_R\right](z) - \alpha B\left[P^\star \circ \rho_r\right](z)\right|,\end{aligned} \tag{2.1}$$

where $P^\star(z) := z^n\overline{P(1/\overline{z})}$ and $\rho_t(z) = tz$.

Proof. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number λ with $|\lambda| > 1$, the polynomial $f(z) = P(z) - \lambda P^\star(z)$, where $P^\star(z) := z^n\overline{P(1/\overline{z})}$ has all zeros in $|z| \leq 1$. Applying Lemma 2 to the polynomial $f(z)$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left|f\left(Re^{i\theta}\right)\right| \geq \left(\frac{R+1}{r+1}\right)^n \left|f\left(re^{i\theta}\right)\right|. \tag{2.2}$$

Since $f\left(Re^{i\theta}\right) \neq 0$ for every $R > r \geq 1$, $0 \leq \theta < 2\pi$ and $R + 1 > r + 1$, it follows from (2.2) that

$$\left|f\left(Re^{i\theta}\right)\right| > \left(\frac{R+1}{r+1}\right)^n \left|f\left(Re^{i\theta}\right)\right| \geq \left|f\left(re^{i\theta}\right)\right|,$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. This gives

$$|f(rz)| < |f(Rz)| \text{ for } |z| = 1, \text{ and } R > r \geq 1.$$

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq 1/R < 1$, we conclude that the polynomial

$$\begin{aligned}T(z) &= f(Rz) - \alpha f(rz) \\ &= \{P(Rz) - \alpha P(rz)\} - \lambda\{P^\star(Rz) - \alpha P^\star(rz)\}\end{aligned}$$

has all its zeros in $|z| < 1$ for every real or complex α with $|\alpha| \geq 1$ and $R > r \geq 1$.

Applying Lemma 1 to polynomial $T(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned}B[T](z) &= B[f \circ \rho_R](z) - \alpha B[f \circ \rho_r](z) \\ &= \{B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)\} \\ &\quad -\lambda\left\{B\left[P^\star \circ \rho_R\right](z) - \alpha B\left[P^\star \circ \rho_r\right](z)\right\}\end{aligned}$$

lie in $|z| < 1$ where $\rho_t(z) = tz$. This implies

$$\begin{aligned}&|B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)| \\ &\leq \left|B\left[P^\star \circ \rho_R\right](z) - \alpha B\left[P^\star \circ \rho_r\right](z)\right|\end{aligned} \tag{2.3}$$

for $|z| \geq 1$ and $R > r \geq 1$. If Inequality (2.3) is not true, then there exits a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned}&|B[P \circ \rho_R](z_0) - \alpha B[P \circ \rho_r](z_0)| \\ &\leq \left|B\left[P^\star \circ \rho_R\right](z_0) - \alpha B\left[P^\star \circ \rho_r\right](z_0)\right|\end{aligned} \tag{2.4}$$

But all the zeros of $P^\star(Rz)$ lie in $|z| < 1/R < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $P^\star(Rz) - \alpha P^\star(rz)$ lie in $|z| < 1$ Hence, by Lemma 1, we have

$$B\left[P^\star \circ \rho_R\right](z_0) - \alpha B\left[P^\star \circ \rho_r\right](z_0) \neq 0.$$

We take

$$\lambda = \frac{B[P \circ \rho_R](z_0) - \alpha B[P \circ \rho_r](z_0)}{B\left[P^\star \circ \rho_R\right](z_0) - \alpha B\left[P^\star \circ \rho_r\right](z_0)},$$

then λ is well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $B[T](z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $B[T](z)$ lie in $|z| < 1$. Thus (2.3) holds true for $|\alpha| \leq 1$ and $R > r \geq 1$.

Next we describe a result of Arestov [11].

For $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:

1) $P(z)$ has all its zeros in $\{z\in\mathbb{C}:|z|\le 1\}$.

2) $P(z)$ has all its zeros in $\{z\in\mathbb{C}:|z|\ge 1\}$.

The result of Arestov [11] may now be stated as follows.

Lemma 4. [11, *Theorem* 4] *Let* $\phi(x)=\psi(\log x)$ *where* ψ *is a convex non decreasing function on* $\mathbb{R}$. *Then for all* $P\in\mathcal{P}_n$ *and each admissible operator* Λ_δ,

$$\int_0^{2\pi}\phi\left(\left|\Lambda_\delta P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|\right)\mathrm{d}\theta \le \int_0^{2\pi}\phi\left(C(\delta,n)\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|\right)\mathrm{d}\theta,$$

where $C(\delta,n)=\max\left(|\delta_0|,|\delta_n|\right)$.

In particular, Lemma 4 applies with $\phi:x\to x^p$ for every $p\in(0,\infty)$. Therefore, we have

$$\left\{\int_0^{2\pi}\left(\left|\Lambda_\delta P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p\right)\mathrm{d}\theta\right\}^{1/p} \le C(\delta,n)\left\{\int_0^{2\pi}\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p\mathrm{d}\theta\right\}^{1/p}. \tag{2.5}$$

We use (2.5) to prove the following interesting result.

Lemma 5. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z|<1$, *then for every* $p>0$, $R>r\ge 1$ *and for* σ *real*, $0\le\sigma<2\pi$,

$$\begin{aligned}&\int_0^{2\pi}\Big|\left\{B[P\circ\rho_R]\left(\mathrm{e}^{\mathrm{i}\theta}\right)-\alpha B[P\circ\rho_r]\left(\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\sigma}\right.\\&\left.+\left\{B\left[P^\star\circ\rho_R\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}\right\}\Big|\mathrm{d}\theta\\&\le\left|\left(R^n-\alpha r^n\right)\phi(\lambda_0,\lambda_1,\lambda_2)\mathrm{e}^{\mathrm{i}\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p\\&\int_0^{2\pi}\left|P\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p\mathrm{d}\theta,\end{aligned} \tag{2.6}$$

where $B\in\mathcal{B}_n$, $\rho_t(z):=tz$, $B\left[P^\star\circ\rho_t\right]^\star(z):=\left(B\left[P^\star\circ\rho_t\right](z)\right)^\star$ and $\phi(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13).

Proof. Since $P\in\mathcal{P}_n$ and $P^\star(z):=z^n\overline{P(1/\bar{z})}$, by Lemma 3, we have for $|z|\ge 1$,

$$\begin{aligned}&\left|B[P\circ\rho_R](z)-\alpha B[P\circ\rho_r](z)\right|\\&\le\left|B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)\right|,\end{aligned} \tag{2.7}$$

Also, since

$$\begin{aligned}&P^\star(Rz)-\alpha P^\star(rz)\\&=R^nz^n\overline{P(1/R\bar{z})}-\alpha r^nz^n\overline{P(1/r\bar{z})},\end{aligned}$$

$$\begin{aligned}&B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)\\&=\lambda_0\left\{R^nz^n\overline{P(1/R\bar{z})}-\alpha r^nz^n\overline{P(1/r\bar{z})}\right\}\\&+\lambda_1\left(\frac{nz}{2}\right)\Big\{\left(nR^nz^{n-1}\overline{P(1/R\bar{z})}-R^{n-1}z^{n-2}\overline{P'(1/R\bar{z})}\right)\\&-\alpha\left(nr^nz^{n-1}\overline{P(1/r\bar{z})}-r^{n-1}z^{n-2}\overline{P'(1/r\bar{z})}\right)\Big\}\\&+\frac{\lambda_2}{2!}\left(\frac{nz}{2}\right)^2\Big\{\Big(n(n-1)R^nz^{n-2}\overline{P(1/R\bar{z})}\\&-2(n-1)R^{n-1}z^{n-3}\overline{P'(1/R\bar{z})}+R^{n-2}z^{n-4}\overline{P''(1/R\bar{z})}\Big)\\&-\alpha\Big(n(n-1)r^nz^{n-2}\overline{P(1/r\bar{z})}-2(n-1)r^{n-1}z^{n-3}\overline{P'(1/r\bar{z})}\\&+r^{n-2}z^{n-4}\overline{P''(1/r\bar{z})}\Big)\Big\}\end{aligned}$$

and therefore,

$$\begin{aligned}&B\left[P^\star\circ\rho_R\right]^\star(z)-\alpha B\left[P^\star\circ\rho_r\right]^\star(z)\\&=\left(B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)\right)^\star\\&=\left(\bar{\lambda}_0+\bar{\lambda}_1\frac{n^2}{2}+\bar{\lambda}_2\frac{n^3(n-1)}{8}\right)\left\{R^nP(z/R)-\bar{\alpha}r^nP(z/r)\right\}\\&-\left(\bar{\lambda}_1\frac{n}{2}+\bar{\lambda}_2\frac{n^2(n-1)}{4}\right)\left\{R^{n-1}zP'(z/R)-\bar{\alpha}r^{n-1}zP'(z/r)\right\}\\&+\bar{\lambda}_2\frac{n^2}{8}\left\{R^{n-2}z^2P''(z/R)-\bar{\alpha}r^{n-2}z^2P''(z/r)\right\}.\end{aligned} \tag{2.8}$$

Also, for $|z|=1$,

$$\begin{aligned}&\left|B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)\right|\\&=\left|B\left[P^\star\circ\rho_R\right]^\star(z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(z)\right|.\end{aligned}$$

Using this in (2.7), we get for $|z|=1$,

$$\begin{aligned}&\left|B[P\circ\rho_R](z)-\alpha B[P\circ\rho_r](z)\right|\\&\le\left|B\left[P^\star\circ\rho_R\right]^\star(z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(z)\right|.\end{aligned}$$

As in the proof of Lemma 3, the polynomial $P^\star\circ\rho_R(z)-\alpha P^\star\circ\rho_r(z)$, has all its zeros in $|z|<1$ and by Lemma 1, $B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)$, also has all its zero in $|z|<1$, therefore, $B\left[P^\star\circ\rho_R\right]^\star(z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(z)$ has all its zeros in $|z|\ge 1$. Hence by the maximum modulus principle, for $|z|=1$,

$$\begin{aligned}&\left|B[P\circ\rho_R](z)-\alpha B[P\circ\rho_r](z)\right|\\&<\left|B\left[P^\star\circ\rho_R\right]^\star(z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(z)\right|.\end{aligned} \tag{2.9}$$

A direct application of Rouche's theorem shows that with $P(z) = a_n z^n + \cdots + a_0$,

$$\begin{aligned}&\Lambda_\delta P(z)\\&= \{B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)\} \mathrm{e}^{\mathrm{i}\sigma} + B[P^\star \circ \rho_R]^\star(z)\\&-\bar{\alpha} B[P^\star \circ \rho_r]^\star(z)\\&= \left\{(R^n - \alpha r^n)\left(\lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}\right)\mathrm{e}^{\mathrm{i}\sigma} + (1-\bar{\alpha})\bar{\lambda}_0\right\}\\&\cdot a_n z^n + \cdots\\&+ \left\{(R^n - \bar{\alpha} r^n)\left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8}\right) + \mathrm{e}^{\mathrm{i}\sigma}(1-\alpha)\lambda_0\right\}\\&\cdot a_0,\end{aligned}$$

has all its zeros in $|z| \geq 1$ for every real σ, $0 \leq \sigma \leq 2\pi$. Therefore, Λ_δ is an admissible operator. Applying (2.5) of Lemma 4, the desired result follows immediately for each $p > 0$.

From Lemma 5, we deduce the following more general result.

Lemma 6. *If* $P \in \mathcal{P}_n$, *then for every* $p > 0$, $R > r \geq 1$ *and* σ *real* $0 \leq \sigma \leq 2\pi$,

$$\begin{aligned}&\int_0^{2\pi} \Big| \{B[P \circ \rho_R](\mathrm{e}^{\mathrm{i}\theta}) - \alpha B[P \circ \rho_r](\mathrm{e}^{\mathrm{i}\theta})\} \mathrm{e}^{\mathrm{i}\sigma}\\&+ \{B[P^\star \circ \rho_R]^\star(\mathrm{e}^{\mathrm{i}\theta}) - \bar{\alpha} B[P^\star \circ \rho_r]^\star(\mathrm{e}^{\mathrm{i}\theta})\}\Big|^p \mathrm{d}\theta\\&\leq \left|(R^n - \alpha r^n)\phi(\lambda_0, \lambda_1, \lambda_2)\mathrm{e}^{\mathrm{i}\sigma} + (1-\bar{\alpha})\bar{\lambda}_0\right|^p\\&\cdot \int_0^{2\pi} \left|P(\mathrm{e}^{\mathrm{i}\theta})\right|^p \mathrm{d}\theta,\end{aligned} \tag{2.10}$$

Proof. Let $P \in \mathcal{P}_n$ and let $z_1, z_2, \cdots, z_n$ be the zeros of $P(z)$. If $|z_j| \geq 1$ for all $j = 1, 2, \cdots, n$, then the result follows by Lemma 5. Henceforth, we assume that $P(z)$ has at least one zero in $|z| < 1$ so that we can write

$$\begin{aligned}P(z) &= P_1(z) P_2(z)\\&= a \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (z - z_j),\\&0 \leq k \leq n-1, a \neq 0,\end{aligned}$$

where the zeros $z_1, z_2, \cdots, z_k$ of $P_1(z)$ lie in $|z| \geq 1$ and the zeros $z_{k+1}, z_{k+2}, \cdots, z_n$ of $P_2(z)$ lie in $|z| < 1$. First we suppose that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $P_1(z)$ lie in $|z| > 1$. Since all the zeros of $(n-k)$ th degree polynomial $P_2(z)$ lie in $|z| < 1$, all the zeroes of its conjugate polynomial $P_2^\star(z) = z^{n-k}\overline{P_2(1/\bar{z})}$ lie in $|z| > 1$ and $|P_2^\star(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$\begin{aligned}f(z) &= P_1(z) P_2^\star(z)\\&= a \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (1 - z\bar{z}_j),\end{aligned}$$

then all the zeroes of $f(z)$ lie in $|z| > 1$, and for $|z| = 1$,

$$\begin{aligned}|f(z)| &= |P_1(z)||P_2^\star(z)|\\&= |P_1(z)||P_2(z)| = |P(z)|.\end{aligned} \tag{2.11}$$

Therefore, it follows by Rouche's Theorem that the polynomial $g(z) = P(z) + \beta f(z)$ has all its zeros in $|z| > 1$ for every β, with $|\beta| > 1$ so that all the zeros of $T(z) = g(\tau z)$ lie in $|z| \geq 1$ for some $\tau > 1$. Applying (2.9) and (2.8) to the polynomial $T(z)$, we get for $R > 1$ and $|z| < 1$,

$$\begin{aligned}&|B[T \circ \rho_R](z) - \alpha B[T \circ \rho_r](z)|\\&< \left|B[T^\star \circ \rho_R]^\star(z) - \bar{\alpha} B[T^\star \circ \rho_r]^\star(z)\right|\\&= \Bigg|\left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8}\right)\\&\cdot \{R^n T(z/R) - \bar{\alpha} r^n T(z/r)\}\\&- \left(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4}\right)\\&\cdot \{R^{n-1} z T'(z/R) - \bar{\alpha} r^{n-1} z T'(z/r)\}\\&+ \bar{\lambda}_2 \frac{n^2}{8} \{R^{n-2} z^2 T''(z/R) - \bar{\alpha} r^{n-2} z^2 T''(z/r)\}\Bigg|,\end{aligned}$$

that is,

$$\begin{aligned}&|B[T \circ \rho_R](z) - \alpha B[T \circ \rho_r](z)|\\&< \Bigg|\left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8}\right)\\&\{R^n g(\tau z/R) - \bar{\alpha} r^n g(\tau z/r)\}\\&- \left(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4}\right)\\&\times \{R^{n-1} \tau z g'(\tau z/R) - \bar{\alpha} r^{n-1} \tau z g'(\tau z/r)\}\\&+ \bar{\lambda}_2 \frac{n^2}{8} \{R^{n-2} z^2 \tau^2 g''(\tau z/R) - \bar{\alpha} r^{n-2} z^2 \tau^2 g''(\tau z/r)\}\Bigg|,\end{aligned} \tag{2.12}$$

for $|z| < 1$. If $z = \mathrm{e}^{\mathrm{i}\theta}/\tau, 0 \leq \theta < 2\pi$, then $|z| = (1/\tau) < 1$ as $\tau > 1$ and we get

$$|B[g \circ \rho_R](\mathrm{e}^{\mathrm{i}\theta}/\tau) - \alpha B[g \circ \rho_r](\mathrm{e}^{\mathrm{i}\theta}/\tau)|$$

$$< \Bigg| \left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8} \right)$$
$$\cdot \left\{ R^n g\left(\mathrm{e}^{\mathrm{i}\theta}/R\right) - \bar{\alpha} r^n g\left(\mathrm{e}^{\mathrm{i}\theta}/r\right) \right\}$$
$$- \left(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4} \right)$$
$$\times \left\{ R^{n-1} \mathrm{e}^{\mathrm{i}\theta} g'\left(\mathrm{e}^{\mathrm{i}\theta}/R\right) - \bar{\alpha} r^{n-1} \mathrm{e}^{\mathrm{i}\theta} g'\left(\mathrm{e}^{\mathrm{i}\theta}/r\right) \right\}$$
$$+ \bar{\lambda}_2 \frac{n^2}{8} \left\{ R^{n-2} \mathrm{e}^{\mathrm{i}\theta} g''\left(\mathrm{e}^{\mathrm{i}\theta}/R\right) - \bar{\alpha} r^{n-2} \mathrm{e}^{\mathrm{i}\theta} g''\left(\mathrm{e}^{\mathrm{i}\theta}/r\right) \right\} \Bigg|,$$

Equivalently, for $|z| = 1$,

$$\left| B[g \circ \rho_R](z) - \alpha B[g \circ \rho_r](z) \right|$$
$$< \left| B\left[g^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[g^\star \circ \rho_r\right]^\star(z) \right|,$$

where $\rho_t(z) = tz$.

Since $g(z)$ has all its zeros in $|z| > 1$, it follows that $g^\star(z)$ has its zeros in $|z| < 1$ and hence (proceeding similarly as in proof of Lemma 3) the polynomial $g^\star \circ \rho_R(z) - \alpha g^\star \circ \rho_r(z)$ also has all its zeros in $|z| < 1$. By Lemma 1, $B\left[g^\star \circ \rho_R\right](z) - \alpha B\left[g^\star \circ \rho_r\right](z)$ has all zeros in $|z| < 1$ and thus $B\left[g^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[g^\star \circ \rho_r\right]^\star(z)$ does not vanish in $|z| < 1$.

An application of Rouche's theorem shows that the polynomial

$$L(z) = \left\{ B[g \circ \rho_R](z) - \alpha B[g \circ \rho_r](z) \right\} \mathrm{e}^{\mathrm{i}\sigma} + B\left[g^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[g^\star \circ \rho_r\right]^\star(z) \tag{2.13}$$

has all zeros in $|z| > 1$. Writing in $g(z) := P(z) + \beta f(z)$ and noting that B is a linear operator, it follows that the polynomial

$$\begin{aligned} L(z) = {} & \left\{ B[g \circ \rho_R](z) - \alpha B[g \circ \rho_r](z) \right\} \mathrm{e}^{\mathrm{i}\sigma} \\ & + \left\{ B\left[g^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[g^\star \circ \rho_r\right]^\star(z) \right\} \\ & + \beta \Big[\left\{ B[f \circ \rho_R](z) - \alpha B[f \circ \rho_R](z) \right\} \mathrm{e}^{\mathrm{i}\sigma}, \\ & \quad + \left\{ B\left[f^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star(z) \right\} \Big] \end{aligned} \tag{2.14}$$

has all its zeros in $|z| > 1$ for every β with $|\beta| > 1$.

We claim

$$\begin{aligned} & \left| \left\{ B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[P^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[P^\star \circ \rho_r\right]^\star(z) \right\} \right| \\ & \le \left| \left\{ B[f \circ \rho_R](z) - \alpha B[f \circ \rho_R](z) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[f^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star(z) \right\} \right|, \end{aligned} \tag{2.15}$$

for $|z| \le 1$. If Inequality (2.15) is not true, then there exists a point $z = z_0$ with $|z_0| \le 1$ such that

$$\begin{aligned} & \left| \left\{ B[P \circ \rho_R](z_0) - \alpha B[P \circ \rho_r](z_0) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[P^\star \circ \rho_R\right]^\star(z_0) - \bar{\alpha} B\left[P^\star \circ \rho_r\right]^\star(z_0) \right\} \right| \\ & > \left| \left\{ B[f \circ \rho_R](z_0) - \alpha B[f \circ \rho_R](z_0) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[f^\star \circ \rho_R\right]^\star(z_0) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star(z_0) \right\} \right|, \end{aligned}$$

Since $f(z)$ has all its zeros in $|z| > 1$, proceeding similarly as in the proof of (2.13), it follows that $\left\{ B[f \circ \rho_R](z) - \alpha B[f \circ \rho_R](z) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[f^\star \circ \rho_R\right]^\star(z) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star(z) \right\} \ne 0$ for $|z| \le 1$ We take

$$\beta = \frac{\left[\left\{ B[P \circ \rho_R](z_0) - \alpha B[P \circ \rho_r](z_0) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[P^\star \circ \rho_R\right]^\star(z_0) - \bar{\alpha} B\left[P^\star \circ \rho_r\right]^\star(z_0) \right\} \right]}{\left[\left\{ B[f \circ \rho_R](z_0) - \alpha B[f \circ \rho_R](z_0) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[f^\star \circ \rho_R\right]^\star(z_0) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star(z_0) \right\} \right]}$$

so that β is a well-defined real or complex number with $|\beta| > 1$ and with this choice of β, from (2.14), we get $L(z_0) = 0$. This clearly is a contradiction to the fact that $L(z)$ has all its zeros in $|z| > 1$. Thus (2.15) holds, which in particular gives for each $p > 0$ and σ real,

$$\begin{aligned} & \int_0^{2\pi} \left| \left\{ B[P \circ \rho_R]\left(\mathrm{e}^{\mathrm{i}\theta}\right) - \alpha B[P \circ \rho_r]\left(\mathrm{e}^{\mathrm{i}\theta}\right) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[P^\star \circ \rho_R\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right) - \bar{\alpha} B\left[P^\star \circ \rho_r\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right) \right\} \right|^p \mathrm{d}\theta \\ & \le \int_0^{2\pi} \left| \left\{ B[f \circ \rho_R]\left(\mathrm{e}^{\mathrm{i}\theta}\right) - \alpha B[f \circ \rho_R]\left(\mathrm{e}^{\mathrm{i}\theta}\right) \right\} \mathrm{e}^{\mathrm{i}\sigma} + \left\{ B\left[f^\star \circ \rho_R\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right) - \bar{\alpha} B\left[f^\star \circ \rho_r\right]^\star\left(\mathrm{e}^{\mathrm{i}\theta}\right) \right\} \right|^p \mathrm{d}\theta. \end{aligned}$$

Lemma 4 and (2.7) applied to f, gives for each $p>0$,

$$\begin{aligned}&\int_0^{2\pi}\left|\left\{B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right\}e^{i\sigma}+\left\{B\left[P^\star\circ\rho_R\right]^\star(e^{i\theta})-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(e^{i\theta})\right\}\right|^p d\theta\\&\le\left|\left(R^n-\alpha r^n\right)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p\times\int_0^{2\pi}\left|f(e^{i\theta})\right|^p d\theta\\&=\left|\left(R^n-\alpha r^n\right)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p\times\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta.\end{aligned}\tag{2.16}$$

Now if $P_1(z)$ has a zero on $|z|=1$, then applying (2.16) to the polynomial $\tilde{P}(z)=P_1(\mu z)P_2(z)$ where $0<\mu<1$, we get for each $p>0$, $R>r\ge1$ and σ real,

$$\begin{aligned}&\int_0^{2\pi}\left|\left\{B\left[\tilde{P}\circ\rho_R\right](e^{i\theta})-\alpha B\left[\tilde{P}\circ\rho_r\right](e^{i\theta})\right\}e^{i\sigma}+\left\{B\left[\tilde{P}^\star\circ\rho_R\right]^\star(e^{i\theta})-\alpha B\left[\tilde{P}^\star\circ\rho_r\right]^\star(e^{i\theta})\right\}\right|^p d\theta\\&\le\left|\left(R^n-\alpha r^n\right)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p\int_0^{2\pi}\left|\tilde{P}(e^{i\theta})\right|^p d\theta.\end{aligned}\tag{2.17}$$

Letting $\mu\to1$ in (2.17) and using continuity, the desired result follows immediately and this proves Lemma 6.

Lemma 7. *If* $P\in\mathcal{P}_n$, *then for every* $p>0$, $R>r\ge1$ and $0\le\sigma<2\pi$,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|\left\{B[P\circ\rho_R](z)-\alpha B[P\circ\rho_r](z)\right\}+e^{i\sigma}\left\{B\left[P^\star\circ\rho_R\right](z)-\alpha B\left[P^\star\circ\rho_r\right](z)\right\}\right|^p d\theta d\sigma\\&\le\int_0^{2\pi}\left|\left(R^n-\alpha r^n\right)\phi(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p d\sigma\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,\end{aligned}\tag{2.18}$$

where $B\in\mathcal{B}_n$, $\rho_t(z)=tz$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (1.13). The result is best possible and $P(z)=bz^n$ is an extremal polynomial for any $b\ne0$.

Proof. By Lemma 6, for each $p>0$, $0\le\alpha<2\pi$ and $R>r\ge1$, the Inequality (2.6) holds. Since $B\left[P^\star\circ\rho_R\right]^\star(z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(z)$ is the conjugate polynomial of $B\left[P^\star\circ\rho_R\right](z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right](z)$,

$$\left|B\left[P^\star\circ\rho_R\right](z)-\bar{\alpha}B\left[P^\star\circ\rho_r\right](z)\right|=\left|B\left[P^\star\circ\rho_R\right]^\star(e^{i\theta})-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(e^{i\theta})\right|,$$

and therefore for each $p>0$, $R>r\ge1$ and $0\le\alpha<2\pi$, we have

$$\begin{aligned}&\int_0^{2\pi}\left|\left\{B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right\}+e^{i\sigma}\left\{B\left[P^\star\circ\rho_R\right](e^{i\theta})-\alpha B\left[P^\star\circ\rho_r\right](e^{i\theta})\right\}\right|^p d\sigma\\&=\int_0^{2\pi}\left|\left|B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right|e^{i\sigma}+\left|B\left[P^\star\circ\rho_R\right](e^{i\theta})-\alpha B\left[P^\star\circ\rho_r\right](e^{i\theta})\right|\right|^p d\sigma\\&=\int_0^{2\pi}\left|\left|B[P\circ\rho_R](z)-\alpha B[P\circ\rho_r](e^{i\theta})\right|+e^{i\sigma}\left|B\left[P^\star\circ\rho_R\right](e^{i\theta})-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(e^{i\theta})\right|\right|^p d\sigma.\end{aligned}\tag{2.19}$$

Integrating (2.19) both sides with respect to θ from 0 to 2π and using (2.6), we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|\left\{B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right\}e^{i\sigma}+\left\{B\left[P^\star\circ\rho_R\right](e^{i\theta})-\alpha B\left[P^\star\circ\rho_r\right](e^{i\theta})\right\}\right|^p d\sigma d\theta\\&=\int_0^{2\pi}\int_0^{2\pi}\left|\left|B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right|e^{i\sigma}+\left|B\left[P^\star\circ\rho_R\right]^\star(e^{i\theta})-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(e^{i\theta})\right|\right|^p d\sigma d\theta,\\&=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(B[P\circ\rho_R](e^{i\theta})-\alpha B[P\circ\rho_r](e^{i\theta})\right)e^{i\sigma}+\left(B\left[P^\star\circ\rho_R\right]^\star(e^{i\theta})-\bar{\alpha}B\left[P^\star\circ\rho_r\right]^\star(e^{i\theta})\right)\right|^p d\theta\right\}d\sigma\\&\le\left|\left(R^n-\alpha r^n\right)\phi_n(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p d\sigma\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\\&=\int_0^{2\pi}\left|\left(R^n-\alpha r^n\right)\phi_n(\lambda_0,\lambda_1,\lambda_2)e^{i\sigma}+(1-\bar{\alpha})\bar{\lambda}_0\right|^p d\sigma\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,\end{aligned}$$

which establishes Inequality (2.18).

3. Proof of Theorems

Proof of Theorem. By hypothesis $P \in \mathcal{P}_n$, we can write

$$P(z) = P_1(z)P_2(z) = a\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(z-z_j),\ k \geq 1,\ a \neq 0,$$

where the zeros $z_1, z_2, \cdots, z_k$ of $P_1(z)$ lie in $|z| \leq 1$ and the zeros $z_{k+1}, z_{k+2}, \cdots, z_n$ of $P_2(z)$ lie in $|z| > 1$. First, we suppose that all the zeros of $P_1(z)$ lie in $|z| < 1$. Since all the zeros of $P_2(z)$ lie in $|z| > 1$, the polynomial $P_2^{\star}(z) = z^{n-k}\overline{P_2(1/\bar{z})}$ has all its zeroes in $|z| < 1$ and $|P_2^{\star}(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$M(z) = P_1(z)P_2^{\star}(z) = a\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\bar{z}_j),$$

then all the zeros of $M(z)$ lie in $|z| < 1$, and for $|z| = 1$,

$$|M(z)| = |P_1(z)||P_2^{\star}(z)| = |P_1(z)||P_2(z)| = |P(z)|. \quad (3.1)$$

Observe that $P(z)/M(z) \to 1\Big/\prod_{j=k+1}^{n}(-\bar{z}_j)$ when $z \to \infty$, so it is regular even at ∞ and thus from (3.1) and by the maximum modulus principle, it follows that

$$|P(z)| \leq |M(z)| \text{ for } |z| \geq 1.$$

Since $M(z) \neq 0$ for $|z| \geq 1$, a direct application of Rouche's theorem shows that the polynomial $H(z) = P(z) + \lambda M(z)$ has all its zeros in $|z| < 1$ for every λ with $|\lambda| > 1$. Applying Lemma 2 to the polynomial $H(z)$ and noting that the zeros of $H(Rz)$ lie in $|z| < 1/R < 1$, we deduce (as in Lemma 3) that for every real or complex α with $|\alpha| \leq 1$, all the zeros of polynomial

$$G(z) = H(Rz) - \alpha H(rz) = \{P(Rz) - \alpha P(rz)\} - \lambda\{M(Rz) - \alpha M(rz)\}$$

lie in $|z| < 1$. Applying Lemma 1 to $G(z)$ and noting that B is a linear operator, it follows that all the zeroes of

$$B[G](z) = \{B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)\} - \lambda\{B[M \circ \rho_R](z) - \alpha B[M \circ \rho_r](z)\},$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies for $|z| > 1$,

$$|B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)| \leq |B[M \circ \rho_R](z) - \alpha B[M \circ \rho_r](z)|,$$

which, in particular, gives for each $p > 0$, $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi}\left|B[P \circ \rho_R](e^{i\theta}) - \alpha B[P \circ \rho_r](e^{i\theta})\right|^p d\theta \leq \int_0^{2\pi}\left|B[M \circ \rho_R](e^{i\theta}) - \alpha B[M \circ \rho_r](e^{i\theta})\right|^p d\theta. \quad (3.2)$$

Again,(as in case of $H(z)$) $M(Rz) - \alpha M(rz)$ has all its zeros in $|z| < 1$, thus by Lemma 1, $B[P \circ \rho_R](z) - \alpha B[P \circ \rho_r](z)$ also has all its zeros in $|z| < 1$. Therefore, if $E(z) = e_n z^n + \cdots + e_1 z + e_0$ has all its zeros in $|z| < 1$, then the operator Λ_δ defined by

$$\Lambda_\delta E(z) = B[E \circ \rho_R](z) - \alpha B[E \circ \rho_r](z) = (R^n - \alpha r^n)\left(\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right)e_n z^n + \cdots + (1-\alpha)\lambda_0 e_0, \quad (3.3)$$

is admissible. Since $M(z) = b_n z^n + \cdots + b_0$, has all its zeros in $|z| < 1$, in view of (3.3) it follows by (2.5) of Lemma 4 that for each $p > 0$,

$$\int_0^{2\pi}\left|B[M \circ \rho_R](e^{i\theta}) - \alpha B[M \circ \rho_r](e^{i\theta})\right|^p d\theta \leq |R^n - \alpha r^n|^p |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \int_0^{2\pi}\left|M(e^{i\theta})\right|^p d\theta, \quad (3.4)$$

Combining Inequalities (3.3), (3.4) and noting that $|M(e^{i\theta})| = |P(e^{i\theta})|$, we obtain for each $p > 0$ and $R > 1$,

$$\int_0^{2\pi}\left|B[P \circ \rho_R](e^{i\theta}) - \alpha B[P \circ \rho_r](e^{i\theta})\right|^p d\theta \leq |R^n - \alpha r^n|^p |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta, \quad (3.5)$$

In case $P_1(z)$ has a zero on $|z| = 1$, then Inequality (3.5) follows by continuity. This proves Theorem 1 for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$.

Proof of Theorem 2. By hypothesis $P(z)$ does not vanish in $|z| < 1$, $\rho_t(z) = tz$ and $R > r \geq 1$, therefore, for $0 \leq \theta < 2\pi$, (2.1) holds. Also, for each $p > 0$ and σ real, (2.18) holds.

Now it can be easily verified that for every real number σ and $s \geq 1$,

$$|s + e^{i\sigma}| \geq |1 + e^{i\sigma}|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi}|s + e^{i\sigma}|^p d\sigma \geq \int_0^{2\pi}|1 + e^{i\sigma}|^p d\sigma. \quad (3.6)$$

If $B[P \circ \rho_R](e^{i\theta}) - \alpha B[P \circ \rho_r](e^{i\theta}) \neq 0$, we take

$$s = \frac{B[P^{\star} \circ \rho_R](e^{i\theta}) - \alpha B[P^{\star} \circ \rho_r](e^{i\theta})}{B[P \circ \rho_R](e^{i\theta}) - \alpha B[P \circ \rho_r](e^{i\theta})},$$

then by (2.1), $s \geq 1$ and we get with the help of (3.6),

$$\int_0^{2\pi}\left|\left\{B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right\}+e^{i\sigma}\left\{B\left[P^\star\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P^\star\circ\rho_r\right]\left(e^{i\theta}\right)\right\}\right|^p d\sigma$$

$$=\left|B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right|^p\int_0^{2\pi}\left|1+e^{i\sigma}\frac{B\left[P^\star\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P^\star\circ\rho_r\right]\left(e^{i\theta}\right)}{B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)}\right|^p d\sigma$$

$$=\left|B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right|^p\int_0^{2\pi}\left|1+e^{i\sigma}\left|\frac{B\left[P^\star\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P^\star\circ\rho_r\right]\left(e^{i\theta}\right)}{B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)}\right|\right|^p d\sigma$$

$$\geq\left|B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right|^p\int_0^{2\pi}\left|1+e^{i\sigma}\right|^p d\sigma.$$

For $B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)=0$, this inequality is trivially true. Using this in (2.18), we conclude that for each $p>0$,

$$\int_0^{2\pi}\left|B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right|^p d\theta\int_0^{2\pi}\left|1+e^{i\sigma}\right|^p d\sigma$$
$$\leq\int_0^{2\pi}\left|\left(R^n-\alpha r^n\right)\phi\left(\lambda_0,\lambda_1,\lambda_2\right)e^{i\sigma}+\left(1-\alpha\right)\lambda_0\right|^p d\sigma$$
$$\cdot\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta,$$

from which Theorem 2 follows for $p>0$. To establish this result for $p=0$, we simply let $p\to 0+$.

Proof of Theorem 3. Since $P(z)$ is a self-inversive polynomial, then we have for some ν, with $|\nu|=1$ $P(z)=\nu P^\star(z)$ for all $z\in\mathbb{C}$, where $P^\star(z)$ is the conjugate polynomial $P(z)$. This gives, for $0\leq\theta<2\pi$

$$\left|B\left[P\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P\circ\rho_r\right]\left(e^{i\theta}\right)\right|$$
$$=\left|B\left[P^\star\circ\rho_R\right]\left(e^{i\theta}\right)-\alpha B\left[P^\star\circ\rho_r\right]\left(e^{i\theta}\right)\right|.$$

Using this in place of (2.1) and proceeding similarly as in the proof of Theorem 2, we get the desired result for each $p>0$. The extension to $p=0$ obtains by letting $p\to 0+$.

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