Self-accelerating two-step Steffensen-type methods with memory and their applications on the solution of nonlinear BVPs

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Abstract—In this paper, seven self-accelerating iterative methods with memory are derived from an optimal two-step Steffensen-type method without memory for solving nonlinear equations, their orders of convergence are proved to be increased from 4 to $2+\sqrt{6}$, $(5+\sqrt{17})/2$, 5 and $(5+\sqrt{33})/2$, numerical examples are demonstrat-ed demonstrated to verify the theoretical results, and applications for solving systems of nonlinear equations and BVPs of nonlinear ODEs are illustrated.

Keywords-Nonlinear equation; Newton's method; Steffensen-type method; Derivative free; Super convergence

1. Introduction

Considering iterative methods to solve a nonlinear equation $f(x)=0$, the most famous method is Newton's method (NM, see [1]):

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, n=1,2,..., \tag{1}$$

where x_0 is an initial guess of the root. If the derivative $f'(x_n)$ is replaced by the divided difference $f[x_n, x_n+f(x_n)]=\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}$,

Newton's method becomes Steffensen's method (SM, see [1]). Steffensen's method is a tough competitor of Newton's method since it is derivative free. Because Kung and Traub conjectured in 1974 that a multipoint iteration based on m evaluations without memory has optimal order 2^{m-1} of convergence (see [2]), NM and SM are one-step methods of optimal order without memory. The efficiency index of them is $\sqrt{2}=1.4142$.

Further, a parametric Steffensen's method (PSM) was suggested in Section 8.4 in [3]:

$$x_{n+1}=x_n-\frac{f(x_n)}{f[x_n,x_n+\beta_n f(x_n)]}, n=0,1,2,... \tag{2}$$

where βn are arbitrary parameters. PSM is convergent with the asymptotic convergence constant $\frac{(1+\beta_n f'(a))f''(a)}{2f'(a)}$. If $\beta_n\approx-\frac{1}{f'(a)}$,

PSM can have smaller error. Therefore, by defining $\beta_n=-1/f[x_{n-1},x_{n-1}+\beta_{n-1}f(x_{n-1})]$ at the previous iteration recursively as the iteration proceeds, a self-accelerating Steffensen's method (SASM) was proposed in Section 8.6 in [3]:

$$\begin{cases} x_{n+1}=x_n-\dfrac{f(x_n)}{f[x_n,x_n+\beta_n f(x_n)]}, \\ \beta_n=-\dfrac{1}{f[x_{n-1},z_{n-1}]}, \end{cases} \tag{3}$$

where $\overline{\beta}_0=-sign(f'(x_0))$, or $-1/f[x_0,x_0+f(x_0)]$, etc. SASM is a derivative-free one-step method with memory. It only uses two new evaluations of the function per step to achieve convergence of order $1+\sqrt{2}\approx2.4142$, and has efficiency index $\sqrt{1+\sqrt{2}}\approx1.5538$.

Some other optimal multipoint Steffensen-type methods without memory were derived in [4-7]. General optimal derivative-free iterative methods for solving nonlinear equations were introduced in [2] and [7]. Two-step self-accelerating Steffensen-type methods were investigated in [8-10]. Steffensen-type methods and their applications in the solution of non-linear systems and nonlinear differential equations were discussed in the literature (see [1, 3, 4, 8, 9]).

This work suggests seven self-accelerating iterative methods with memory for solving nonlinear equations in section 2, proves their high orders of super convergence in section 3, demonstrates examples and applications in section 4, and makes conclusions in section 5.

2. The self-accelerating methods

By using second-order Newtonian interpolation $N_2(x)$ after an iteration of SM to approximate $f(x)$ and find the root of $N_2(x)=0$ to approximate the exact root, we obtain a parametric Steffensen-type method (PSTM):

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f[x_n,z_n]}, \\ x_{n+1}=y_n-\dfrac{f(y_n)}{f[x_n,y_n]+f[y_n,z_n]-f[x_n,z_n]}. \end{cases} \tag{4}$$

where $z_n=x_n+\beta_n f(x_n)$. PSTM is an optimal two-step meth-od without memory (see [5, 7]).

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Theorem 2.1 (see [7]) Let $f: D \to R$ be a sufficiently differentiable function with a simple root $a \in D, D \subset R$ be an open set, x_0 be close enough to a, then the method (4) is at least of fourth-order, and satisfies the error equation

$$e_{n+1} = (1+\beta_n f'(a))^2 c_2[c_2^2 - c_3]e_n^4 + O(e_n^5), \quad (5)$$

where $c_k = \frac{f^{(k)}(a)}{k!\,f'(a)}, e_n = x_n - a, n = 0,1,2,...$

From PSTM and its error equation, in order to achieve super fourth-order convergence, the free parameters β_n should tend to $-1/f'(a)$. Therefore, we propose six self-accelerating Steffensen-type methods (SASTMs) with memory as follows: compute (4) with

$$(i)\beta_n \equiv \beta_n^1 = -\frac{1}{f[x_{n-1}, z_{n-1}]} \qquad \text{(SASTM1)} \qquad (6)$$

$$(ii)\beta_n \equiv \beta_n^2 = -\frac{1}{f[x_{n-1}, y_{n-1}]} \qquad \text{(SASTM2)} \qquad (7)$$

$$(iii)\beta_n \equiv \beta_n^3 = -\frac{1}{f[y_{n-1}, z_{n-1}]} \qquad \text{(SASTM3)} \qquad (8)$$

$$(iv)\beta_n \equiv \beta_n^4 = -\frac{1}{f[x_{n-1}, x_n]} \qquad \text{(SASTM4)} \qquad (9)$$

$$(v)\beta_n \equiv \beta_n^5 = -\frac{1}{f[z_{n-1}, x_n]} \qquad \text{(SASTM5)} \qquad (10)$$

$$(vi)\beta_n \equiv \beta_n^6 = -\frac{1}{f[y_{n-1}, x_n]} \qquad \text{(SASTM6)} \qquad (11)$$

By Newton's interpolation formula on points x_n, y_{n-1} and x_{n-1} as Follows:

$$N(x) = f(x_n) + f[x_n, y_{n-1}](x - x_n) + f[x_n, y_{n-1}, x_{n-1}](x - x_n)(x - y_{n-1})$$

using $\beta_n = -1/N'(x_n)$ we propose the seventh self-accelerating Steffensen-type methods: compute (4) with

$$(vii)\beta_n \equiv \beta_n^7 = -\frac{1}{f[x_n, y_{n-1}] + f[x_n, x_{n-1}] - f[y_{n-1}, x_{n-1}]}.$$

(SASTM7) (12)

3. The convergence

Theorem 3.1. Let $f: D \to R$ be sufficiently differentiable near a simple root $a \in D, D \subset R$ be an open set,be close enough to a, then SASTM1, SASTM2 and SASTM4 achieve the convergence of order $2+\sqrt{6}$, SASTM3 and SASTM5 achieve the convergence of order $(5+\sqrt{17})/2$,SASTM6 achieves fifth-order convergence, and SASTM7 achieves the convergence of order $(5+\sqrt{33})/2$.

Proof. Using Taylor formula and denoting $e_n^z = z_n - a$ and $e_n^y = y_n - a$, for SASTM1, we have

$$e_n^z = (1+\beta_n f'(a))e_n + o(e_n),$$

$$f[x_n, z_n] = f'(a) + \frac{f''(a)}{2}(2+\beta_n f'(a))e_n + o(e_n).$$

Hence,

$$\beta_n^1 = -\frac{1}{f'(a)}[1-(2+\beta_{n-1}^1 f'(a))c_2 e_{n-1}] + o(e_{n-1}) \cdot$$

By substituting it into Eq. (5), we obtain

$$e_{n+1} = (2+\beta_{n-1}^1 f'(a))^2 c_2^3[c_2^2 - c_3]e_n^4 e_{n-1}^2 + o(e_{n-1}^2 e_n^4).$$

For SASTM2, we have

$$e_n^y = (1+\beta_n f'(a))c_2 e_n^2 + o(e_n^2),$$

$$f[x_n, y_n] = f'(a) + \frac{1}{2}f''(a)e_n + o(e_n),$$

$$\beta_n^2 = -\frac{1}{f'(a)}[1 - c_2 e_{n-1}] + o(e_{n-1}) \cdot$$

By substituting β_n^2 into Eq. (5), we obtain

$$e_{n+1} = c_2^3[c_2^2 - c_3]e_n^4 e_{n-1}^2 + o(e_{n-1}^2 e_n^4).$$

For SASTM4, we have

$$f[x_{n-1}, x_n] = f'(a) + \frac{1}{2}f''(a)e_{n-1} + o(e_{n-1}),$$

and obtain the same error equation as the above. By solving $r^2 = 4r + 2$, the convergence of order $2+\sqrt{6} \approx 4.4495$ is proved for SASTM1, 2 and 4.

For SASTM3 and 5, if z_n converges to a with orderand x_n Converges to a with order r as:

$$e_n^z = C_n e_n^p + o(e_n^p) \text{ and } e_{n+1} = D_n e_n^r + o(e_n^r),$$

then

$$e_n^z = C_n D_{n-1}^p e_{n-1}^{rp} + o(e_{n-1}^{rp}),$$

$$e_{n+1} = D_n D_{n-1}^r e_{n-1}^{r^2} + o(e_{n-1}^{r^2}).$$

So, we have

$$e_n^y = e_n - \frac{f[x_n, a]e_n}{f[x_n, z_n]} = \frac{f[x_n, z_n, a]}{f[x_n, z_n]}e_n^z e_n,$$

$$e_{n+1} = e_n^y - \frac{f[y_n, a]e_n^y}{f[x_n, y_n] + f[x_n, y_n, z_n](y_n - x_n)}$$

$$= e_n^y \frac{f[x_n, y_n, a] + f[x_n, y_n, z_n](e_n^y - e_n)}{f[x_n, y_n] + f[x_n, y_n, z_n](e_n^y - e_n)}$$

$$= e_n^y \frac{f[x_n, y_n, z_n]e_n^y - f[x_n, y_n, z_n, a]e_n^z e_n}{f[x_n, y_n] + f[x_n, y_n, z_n](e_n^y - e_n)}$$

$$= \frac{f[x_n, z_n, a]}{f[x_n, z_n]}(e_n^z e_n)^2$$

$$\times \frac{f[x_n, y_n, z_n]\frac{f[x_n, z_n, a]}{f[x_n, z_n]} - f[x_n, y_n, z_n, a]}{f[x_n, y_n] + f[x_n, y_n, z_n](e_n^y - e_n)}.$$

For SASTM3,

$$e_n^z = \frac{f[y_{n-1}, z_{n-1}] - f[x_n, a]}{f[y_{n-1}, z_{n-1}]} e_n$$

$$= c_2 e_{n-1}^z e_n + o(e_{n-1}^z e_n)$$

$$= c_2 C_{n-1} D_{n-1} e_{n-1}^{p+r} + o(e_{n-1}^{p+r}),$$

$$e_{n+1} = c_2 (c_2 C_{n-1} D_{n-1}^2 e_{n-1}^{p+2r})^2 (c_2^2 - c_3) + o(e_{n-1}^{2p+4r})$$

$$= c_2^3 (c_2^2 - c_3) C_{n-1}^2 D_{n-1}^4 e_{n-1}^{2p+4r} + o(e_{n-1}^{2p+4r}).$$

For SASTM5, we have the same results

$$e_n^z = c_2 C_{n-1} D_{n-1} e_{n-1}^{p+r} + o(e_{n-1}^{p+r}),$$

$$e_{n+1} = c_2^3 (c_2^2 - c_3) C_{n-1}^2 D_{n-1}^4 e_{n-1}^{2p+4r} + o(e_{n-1}^{2p+4r}).$$

Comparing the exponents of e_{n-1} in the expression of and e_{n+1} respectively, we have respectively, we have

$$\begin{cases} rp = p + r, \\ r^2 = 2p + 4r. \end{cases}$$

From its non-trivial solution $p = (1 + \sqrt{17}) / 2$ and $r = (5 + \sqrt{17}) / 2$, we prove that SASTM3 and 5 achieve the same convergence of order $(5 + \sqrt{17}) / 2 \approx 4.5616$.

For SASTM6 and 7, if

$$e_n^y = C_n e_n^p + o(e_n^p) \text{ and } e_{n+1} = D_n e_n^r + o(e_n^r),$$

then

$$e_n^y = C_n D_{n-1}^p e_{n-1}^{rp} + o(e_{n-1}^{rp}),$$

$$e_{n+1} = D_n D_{n-1}^r e_{n-1}^{r^2} + o(e_{n-1}^{r^2}).$$

So, for SASTM6,

$$e_n^z = c_2 C_{n-1} D_{n-1} e_{n-1}^{p+r} + o(e_{n-1}^{p+r}),$$

$$e_n^y = c_2^2 C_{n-1} D_{n-1}^2 e_{n-1}^{p+2r} + o(e_{n-1}^{p+2r}),$$

$$e_{n+1} = c_2^3 (c_2^2 - c_3) C_{n-1}^2 D_{n-1}^4 e_{n-1}^{2p+4r} + o(e_{n-1}^{2p+4r}),$$

and establish

$$\begin{cases} rp = p + 2r, \\ r^2 = 2p + 4r. \end{cases}$$

From its non-trivial solution $p = 5/2$ and $r = 5$, we prove that SASTM6 achieves fifth-order convergence.

For SASTM7, we have

$$e_n^z = \frac{f[x_n, y_{n-1}, a]e_{n-1}^y + f[x_n, y_{n-1}, x_{n-1}](e_n - e_{n-1}^y)}{f[x_n, y_{n-1}] + f[x_n, x_{n-1}] - f[y_{n-1}, x_{n-1}]} e_n$$

$$= \frac{f[x_n, y_{n-1}, x_{n-1}]e_n - f[x_n, y_{n-1}, x_{n-1}, a]e_{n-1}^y e_{n-1}}{f[x_n, y_{n-1}] + f[x_n, x_{n-1}] - f[y_{n-1}, x_{n-1}]} e_n$$

$$= -c_3 C_{n-1} D_{n-1} e_{n-1}^{p+r+1} + o(e_{n-1}^{p+r+1}),$$

$$e_n^y = -c_2 c_3 C_{n-1} D_{n-1}^2 e_{n-1}^{p+2r+1} + o(e_{n-1}^{p+2r+1}),$$

$$e_{n+1} = c_2 c_3^2 (c_2^2 - c_3) C_{n-1}^2 D_{n-1}^4 e_{n-1}^{2p+4r+2} + o(e_{n-1}^{2p+4r+2}),$$

and establish

$$\begin{cases} rp = p + 2r + 1, \\ r^2 = 2p + 4r + 2. \end{cases}$$

From its non-trivial solution $p = (5 + \sqrt{33})/4$ and $r = (5 + \sqrt{33})/2 \approx 5.3723$, we prove that SASTM7 achieves the convergence of order $(5 + \sqrt{33})/2$.

Each of SASTMs is a two-step derivative-free method with memory and only uses three new evaluations of the function per step to achieve super fourth-order convergence. SASTM1, 2 and 4 have the same efficiency index $\sqrt[3]{2 + \sqrt{6}} \approx 1.6448$. SASTM3 and 5 have the same efficiency index $\sqrt[3]{(5 + \sqrt{17})/2} \approx 1.6585$. SASTM 6 and 7 have the efficiency indices $\sqrt[3]{5} \approx 1.7100$ and $\sqrt[3]{(5 + \sqrt{33})/2} \approx 1.7514$, respectively. Whereas, two self-accelerating methods proposed by Petković-Ilić-Džunić in[10] each uses three new evaluations of the function per iteration to achieve the super fourth-order convergence of order $2 + \sqrt{6}$ and its efficiency index is only $\sqrt[3]{2 + \sqrt{6}} \approx 1.6448$.

4. Numerical examples

In the examples,NM,SM,PSM,SASM, PSTM and SASTMs are compared with each other. The computational order of convergence is defined as:

$$COC = \frac{\log(|e_n| / |e_{n-1}|)}{\log(|e_{n-1}| / |e_{n-2}|)}.$$

where $\beta_0^i = 1, i = 1, 2, \ldots, 7.$

Example 1. Numerical results in Table I agree with theoretical results in the theorems.

TABLE I. $f(x) = x^2 - e^{-x} + 3x + 1, a = 0, x_0 = 0.2$

Method	n	1	2	3	4
NM	$\lvert e_n \rvert$	.533e-2	.356e-5	.158e-11	.312e-24
	COC		2.01691	2.00030	2.00000
SM	$\lvert e_n \rvert$	.282e-1	.513e-3	.165e-6	.170e-13
	COC		2.04367	2.00830	2.00009

PSM	$\lvert e_n \rvert$	.134e-1	.677e-4	.172e-8	.111e-17
$(\beta_n \equiv -1)$	COC		1.95757	2.00059	2.00000
SASM	$\lvert e_n \rvert$	.282e-1	.160e-4	.131e-12	.433e-32
	COC		3.81335	2.49109	2.40945
PSTM	$\lvert e_n \rvert$	.468e-4	.389e-18	.186e-74	.977e-300
$(\beta_n \equiv 1)$	COC		3.87748	4.00000	4.00000
PSTM	$\lvert e_n \rvert$	.595e-4	.366e-18	.525e-75	.222e-302
$(\beta_n \equiv -1)$	COC		4.02926	4.00000	4.00000
SASTM1	$\lvert e_n \rvert$	.468e-4	.414e-21	.441e-98	.331e-440
	COC		4.69620	4.51372	4.44480
SASTM2	$\lvert e_n \rvert$	.468e-4	.135e-22	.371e-104	.175e-467
	COC		5.10548	4.39945	4.45460
SASTM3	$\lvert e_n \rvert$	.468e-4	.317e-21	.228e-100	.956e-462
	COC		4.72820	4.60962	4.56612
SASTM4	$\lvert e_n \rvert$	.468e-4	.104e-22	.132e-104	.166e-469
	COC		5.13642	4.39102	4.45547
SASTM5	$\lvert e_n \rvert$	.468e-4	.302e-21	.180e-100	.324e-462
	COC		4.73383	4.60889	4.56606
SASTM6	$\lvert e_n \rvert$	.468e-4	.195e-24	.691e-127	.384e-639
	COC		5.61230	5.02717	5.00000
SASTM7	$\lvert e_n \rvert$	.468e-4	.810e-27	.207e-148	.679e-802
	COC		6.26819	5.34210	5.37438

Example 2. Consider to solve a boundary-value problem of ODEs as the following:

$$\begin{cases} y'' = \sqrt{1+y'^2}, \\ y(0) = 1, y(1) = \frac{1}{2}(e+e^{-1}). \end{cases}$$

Let $N=4$ and use the methods to solve the nonlinear system $F(\vec{s})=0$ with $\vec{s}_0 = 0$ by the multiple shooting method (see [4, 9]). The numerical results are showed in Table II.

TABLE II. FOR THE MULTIPLE SHOOTING METHOD

Method	n	1	2	3	4
SM	$\lVert F(\vec{s}_n) \rVert$	.560e-1	.149e-3	.994e-10	.203e-21
	$\lVert y-y_n \rVert$	.283e-1	.584e-4	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.585e-1	.143e-3	.459e-5	.458e-5
PSTM	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.124e-17	.219e-76	.336e-311
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM1	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.154e-4	.280e-9	.301e-15
	$\lVert y-y_n \rVert$	.142e-3	.230e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.129e-4	.458e-5	.458e-5
SASTM2	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.135e-21	.289e-99	.118e-446
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM3	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.764e-20	.236e-95	.121e-439
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM4	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.446e-22	.131e-102	.711e-462
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM5	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.996e-20	.897e-95	.564e-437
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM6	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.135e-21	.289e-99	.118e-446
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.459e-5	.459e-5	.458e-5
SASTM7	$\lVert F(\vec{s}_n) \rVert$	.271e-3	.446e-22	.131e-102	.711e-462
	$\lVert y-y_n \rVert$	.142e-3	.326e-5	.326e-5	.326e-5
	$\lVert y'-y'_n \rVert$	.346e-3	.458e-5	.458e-5	.458e-5

5. Conclusions

In this work, we derive seven self-accelerating Steffensen-type methods with memory for solving nonlinear equations and establish the comparative advantage of high efficiency theoretically and numerically. The suggested self-accelerating methods without derivative are convenient to be applied in the multiple shooting method for solving boundary-value problems of nonlinear ODEs, where derivatives are difficult to be obtained.

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