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Bounds for Goal Achieving Probabilities of Mean-Variance Strategies with a No Bankruptcy Constraint

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ABSTRACT

We establish, through solving semi-infinite programming problems, bounds on the probability of safely reaching a desired level of wealth on a finite horizon, when an investor starts with an optimal mean-variance financial investment strategy under a non-negative wealth restriction.

Keywords: First Passage-Time; Mean-Variance Portfolios; Semi-Infinite Programming

1. Introduction

In probability theory, the first passage-time problem is the study of the first moment when a stochastic process reaches a certain threshold. This problem often arises in financial mathematics and particularly in portfolio management. For example, consider a risky strategy on an horizon [0,*T*], the investor may encounter a specific instant *t* when the amount of wealth $x(t)$ be sufficient enough so that he may, at this point, safely reinvest all of his money in a simple bank account with (deterministic) interest rate $r(t)$ and the resulting terminal wealth $x(T)$ will attain his financial goal *z*. So we consider the following stopping time random variable :

$$\tau_z = \inf\left\{0 \le t \le T : x(t)\mathrm{e}^{\int_t^T r(s)\mathrm{d}s} = z\right\} \tag{1}$$

and we naturally want to compute the probability $P(\tau_z \le T)$ of such an event. If $x_0 > 0$ is his initial wealth then we will assume $z > x_0 \exp\left\{\int_0^T r(s)\mathrm{d}s\right\}$ so that the investor cannot achieve his financial goal by simply placing his initial investment in a bank account.

2. Market Model

In order to investigate this goal-achieving problem, we must first define a mathematical setting for the dynamics of the financial market. We will consider here the celebrated Black-Scholes model that we next describe. The first asset is a bank account whose price at time *t*, $P_0(t)$, is the solution to the following ordinary differential equation (ODE):

$$\mathrm{d}P_0(t) = r(t)P_0(t)\mathrm{d}t. \tag{2}$$

The next assets consist of *m* stocks whose prices $\{P_1(t),\cdots,P_m(t)\}$ at time *t* are the solutions to the following SDEs (stochastic differential equations):

$$\mathrm{d}P_i(t) = P_i(t)\left[b_i(t)\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\mathrm{d}W_j(t)\right] \tag{3}$$

where $\{W(t), t \ge 0\}$ is a standard *m*-dimensional Brownian motion.

We will assume that the interest rate $r(t)$, stock appreciation rates $b_i(t)$ and stock volatilities $\sigma_{ij}(t)$ are deterministic functions and that

$$\sigma(t) = \begin{bmatrix} \sigma_{11}(t) & \dots & \sigma_{1m}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{m1}(t) & \dots & \sigma_{mm}(t) \end{bmatrix} \tag{4}$$

is invertible.

Let $u(t) = (u_1(t),\cdots,u_m(t))^T, 0 \le t \le T$ be a financial strategy (or portfolio) where $u_i(t)$ is the amount placed in the i^{th} stock. If we assume that all strategies $u(t)$ are self-financed (no outside injection of funds to the investors) and with no transaction costs then the wealth dynamic at time *t* is given by the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}x(t) = &\{r(t)x(t) + B(t)u(t)\}\mathrm{d}t \\ &+ u(t)'\sigma(t)\mathrm{d}W(t) \end{aligned} \tag{5}$$

where $B(t) = (b_1(t) - r(t),\cdots,b_m(t) - r(t))^T$.

Finally, among all the possible strategies, we will focus on the one generated by a family of stochastic control problems defined by

$$\min VAR\left(x(T)\right) \text{ s.t. } E\left(x(T)\right)=z. \quad (6)$$

These are known as mean-variance problems and are considered the cornerstone of modern portfolio management theory which originated with the work of Nobel Prize laureate H. Markowitz.

3. Goal Achieving Probabilities

3.1. Case 1: Unconstrained and No Short-Selling Restriction

In this context, the optimal wealth process has the following form

$$x(t)=y_0\mathrm{e}^{\int_0^t\left(r(s)-\frac{3}{2}\|\alpha(s)\|^2\right)\mathrm{d}s-\int_0^t\alpha(s)'\mathrm{d}W(s)}+\beta\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s} \quad (7)$$

with $y_0<0$, $\beta>z$ and $\alpha(t)>0$ having specific values for the unconstrained and no-short selling (no borrowing stocks) case respectively. The computation of the probability $P(\tau_z\le T)$, following a stochastic time change, can be reduced to the calculation of the probability of the first passage time of a Brownian motion with drift through a fixed level, more precisely the probability is given by:

$$\begin{aligned}P(\tau_z\le T)=&\Phi\left(\frac{1}{2}\sqrt{\int_0^T\|\alpha(s)\|^2\mathrm{d}s}\right)\\&+\mathrm{e}^{3\int_0^T\|\alpha(s)\|^2\mathrm{d}s}\Phi\left(-\frac{5}{2}\sqrt{\int_0^T\|\alpha(s)\|^2\mathrm{d}s}\right)\end{aligned} \quad (8)$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-z^2/2}\mathrm{d}z \quad (9)$$

is the cumulative density function of a standardized normal distribution.

Detailed proofs can be found in Li and Zhou [1] and Scott and Watier [2].

3.2. Case 2: No Bankruptcy Restriction

In this case, unfortunately, the optimal wealth process has a more complex expression, according to Bielecki *et al*. [3] it is given by

$$x^{NB}(t)=\lambda\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}f\left(t,\int_0^t\theta(s)\mathrm{d}W(s)\right) \quad (10)$$

where

$$\begin{aligned}f(t,Z)=&\Phi\left(-d_-\left(t,y(t,Z)\right)\right)\\&-\frac{y(t,Z)}{\lambda}\mathrm{e}^{\int_t^T r(s)\mathrm{d}s}\Phi\left(-d_+\left(t,y(t,Z)\right)\right)\end{aligned} \quad (11)$$

$$\begin{aligned}&d_+\left(t,y(t,Z)\right)\\&=\frac{\ln\left(\frac{y(t,Z)}{\lambda}\right)+\int_t^T\left(r(s)+\frac{1}{2}|\theta(s)|^2\right)\mathrm{d}s}{\sqrt{\int_t^T|\theta(s)|^2\mathrm{d}s}}\end{aligned} \quad (12)$$

$$d_-\left(t,y(t,Z)\right)=d_+\left(t,y(t,Z)\right)-\sqrt{\int_t^T|\theta(s)|^2\mathrm{d}s} \quad (13)$$

$$y(t,Z)=\mu\mathrm{e}^{-\int_0^T\left(2r(s)-|\theta(s)|^2\right)\mathrm{d}s}\mathrm{e}^{\int_0^t\left(r(s)-\frac{3}{2}|\theta(s)|^2\right)\mathrm{d}s}\mathrm{e}^{-Z} \quad (14)$$

and $\lambda>z$ and $\mu>0$ are Lagrange multipliers obtained by solving the nonlinear system of equations:

$$E\left[\left(\lambda-\mu\rho(T)\right)^+\right]=z \quad (15)$$

$$E\left[\rho(T)\left(\lambda-\mu\rho(T)\right)^+\right]=x_0 \quad (16)$$

with

$$\rho(T)=\mathrm{e}^{-\int_0^T\left(r(s)+\frac{1}{2}\|\theta(s)\|^2\right)\mathrm{d}s-\int_0^t\theta^T(s)\mathrm{d}W(s)} \quad (17)$$

Evidently, an explicit form for the corresponding goal-achieving probability $P\left(\tau_z^{NB}\le T\right)$ as in the cases discussed in Section 3.1 appears unrealistic. However, we will show that we can obtain precise bounds for this probability through solving (deterministic) semi-infinite programming (SIP) problems.

The basic idea is to convert the original passage-time problem of this complex stochastic process with a fixed barrier into an equivalent passage-time problem for a simple Gaussian Markovian process but with a time-varying boundary.

To this end, the following result will be useful.

Let $A>0$, then

$$g(x)=\Phi(x+A)-\mathrm{e}^{-Ax-\frac{1}{2}A^2}\Phi(x) \quad (18)$$

is a strictly increasing function on the real line that takes on values in $]0,1[$.

The proof is straightforward since clearly $\lim_{x\to-\infty}g(x)=0$ and $\lim_{x\to\infty}g(x)=1$, while

$$\frac{\partial g(x)}{\partial x}=A\mathrm{e}^{-Ax-\frac{1}{2}A^2}\Phi(x)>0 \quad (19)$$

From this property we have that, for each fixed $t\in[0,T]$,

$$\begin{aligned}&f\left(t,\int_0^t\theta(s)\mathrm{d}W(s)\right)=\frac{z}{\lambda}\\&\Leftrightarrow\left(t,\int_0^t\theta(s)\mathrm{d}W(s)\right)=f^{-1}\left(\frac{z}{\lambda}\right)\end{aligned} \quad (20)$$

therefore, if

$$\tau_z^{\text{invf}} = \inf\left\{0 \le t \le T : \left(t, \int_0^t \theta(s)\,\mathrm{d}W(s)\right) = f^{-1}\left(\frac{z}{\lambda}\right)\right\} \quad (21)$$

then

$$P\left(\tau_z^{NB} \le T\right) = P\left(\tau_z^{invf} \le T\right). \quad (22)$$

Due to the intricate nature of the time-varying boundary obtained, there is again little hope to find an explicit formula. But suppose we can get simpler boundaries h_l and h_u such that $h_l(t) \le f^{-1}(t, z/\lambda) \le h_u(t)$ then clearly by defining

$$\tau_z^{h_l} = \inf\left\{0 \le t \le T : \int_0^t \theta(s)\,\mathrm{d}W(s) = h_l(t)\right\} \quad (23)$$

$$\tau_z^{h_u} = \inf\left\{0 \le t \le T : \int_0^t \theta(s)\,\mathrm{d}W(s) = h_u(t)\right\} \quad (24)$$

we would have

$$P\left(\tau_z^{h_u} \le T\right) \le P\left(\tau_z^{invf} \le T\right) \le P\left(\tau_z^{h_l} \le T\right). \quad (25)$$

The next task at hand is to find suitable boundaries, for this, we need to recall first passage-time results for Gaussian Markovian processes through a specific family of time-varying boundaries known as Daniels' curves (see Dinardo *et al*. [4]).

Consider the stochastic process

$$\left\{\int_0^t \theta(s)\,\mathrm{d}W(s), 0 \le t \le T\right\}$$

then the first passage-time probability through a boundary of the form

$$S(t) = \frac{\alpha}{2} - \left(\frac{\int_0^t \|\theta(s)\|^2\,\mathrm{d}s}{\alpha}\right)\ln\left[\frac{c_1 + \sqrt{\Delta(t)}}{2}\right] \quad (26)$$

where

$$\Delta(t) = c_1^2 + 4c_2 \exp\left\{-\frac{\alpha^2}{\int_0^t \|\theta(s)\|^2\,\mathrm{d}s}\right\} \quad (27)$$

$\alpha > 0, c_1 > 0, c_2 \in R$ and $\lim_{t\to T} \Delta(t) > 0$, is given in explicit form by

$$P(\tau \le T) = \Phi\left(\frac{-S(T)}{\sqrt{\int_0^T \|\theta(s)\|^2\,\mathrm{d}s}}\right) + c_1\Phi\left(\frac{S(T)-\alpha}{\sqrt{\int_0^T \|\theta(s)\|^2\,\mathrm{d}s}}\right) + c_2\Phi\left(\frac{S(T)-2\alpha}{\sqrt{\int_0^T \|\theta(s)\|^2\,\mathrm{d}s}}\right). \quad (28)$$

Therefore the family of Daniels curves appears to be excellent candidates for obtaining explicit upper and lower bounds for our original goal-achieving problem. Finally, in order to generate the tightest bounds possible, we are naturally led to solve the following SIP problems:

$$\sup_{\alpha>0,c_1>0,c_2\in R}\left\{\Phi\left(\frac{-S(T)}{\sqrt{h(T)}}\right) + c_1\Phi\left(\frac{S(T)-\alpha}{\sqrt{h(T)}}\right) + c_2\Phi\left(\frac{S(T)-2\alpha}{\sqrt{h(T)}}\right)\right\} \quad (29)$$

$$\text{s.t.}\, S(t) \ge f^{-1}\left(t, \frac{z}{\lambda}\right) \text{for all } t \in [0,T]$$

and

$$\inf_{\alpha>0,c_1>0,c_2\in R}\left\{\Phi\left(\frac{-S(T)}{\sqrt{h(T)}}\right) + c_1\Phi\left(\frac{S(T)-\alpha}{\sqrt{h(T)}}\right) + c_2\Phi\left(\frac{S(T)-2\alpha}{\sqrt{h(T)}}\right)\right\} \quad (30)$$

$$\text{s.t.}\, S(t) \le f^{-1}\left(t, \frac{z}{\lambda}\right) \text{ for all } t \in [0,T]$$

For inquiries on efficient techniques for solving these SIP problems we refer the reader to Lopez and Still [5] and Reemtsen and Rückmann [6].

4. Numerical Examples

In order to illustrate that the solutions to the 3-parameter SIP problems can produce tight bounds, let us reprise the one stock market model example in Bielecki *et al*. that is $r(t) = 0.06,\ b(t) = 0.12,\ \sigma(t) = 0.15,\ x_0 = 1,\ T = 1$ but with different wealth objective z. **Table 1** sums up the results.

Finally, we can easily show that the 80% rule (*i.e.* $P(\tau_z \le T) > 0.80$, for all possible values of the market parameters) obtained by Li and Zhou and, Scott and Watier unfortunately does not hold in general for a no-bankruptcy optimal mean-variance strategy. For example, if we set $z = 2.0$, by solving (29), we have $P(\tau_z \le T) < 0.65$.

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Table 1. Goal achieving probability bounds.

Probabilities	Wealth objective				
	1.10	1.15	1.20	1.25	1.30
Lower bounds	0.83565	0.83534	0.83338	0.82899	0.82183
Upper bounds	0.83566	0.83539	0.83351	0.82928	0.82325

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