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# Some Classes of Operators Related to *p*-Hyponormal Operator

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## ABSTRACT

We introduce a new family of classes of operators termed as $^*p$-paranormal operator, classes $^*A(p,p)$; $p > 0$ and $^*A(p,q)$; $p, q > 0$, parallel to $p$-paranormal operator and classes $A(p,p)$; $p > 0$ and $A(p,q)$; $p, q > 0$ introduced by M. Fujii, D. Jung, S. H. Lee, M. Y. Lee and R. Nakamoto [1]. We present a necessary and sufficient condition for $p$-hyponormal operator $T \in B(H)$ to be $^*p$-paranormal and the monotonicity of $^*A(p,q)$. We also present an alternative proof of a result of M. Fujii, *et al.* [1, Theorem 3.4].

**Keywords:** $p$-Hyponormal Operator; Monotonicity; Class of Operators $^*A(p,q)$; $^*$Paranormal Operator; $^*p$-Paranormal Operator

## 1. Introduction

Let $B(H)$ denote the algebra of bounded liner operators on a Hilbert space $H$. An operator $T \in B(H)$ is positive if $(Tx,x) \geq 0$ for all $x \in H$. An operator $T \in B(H)$ is hyponormal if $T^*T \geq TT^*$ and $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p > 0$. By the well known Lowner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for $\alpha \in (0,1)$", every $p$-hyponormal operator is $q$-hyponormal for $p \geq q \geq 0$. The Furuta's inequalities [2] are as follows:

If $A \geq B \geq 0$ then for each $r_0 \geq 0$

$$\left(B^{r_0} A^{p_0} B^{r_0}\right)^{1/q_0} \geq \left(B^{r_0} B^{p_0} B^{r_0}\right)^{1/q_0} \tag{1.1}$$

$$\left(A^{r_0} A^{p_0} A^{r_0}\right)^{1/q_0} \geq \left(A^{r_0} B^{p_0} A^{r_0}\right)^{1/q_0} \tag{1.2}$$

hold for $p_0 \geq 0$ and $q_0 \geq 1$ with $(1+2r_0)q_0 \geq p_0 + 2r_0$.

An operator $T \in B(H)$ is

1) paranormal if $\|T^2x\|\|x\| \geq \|Tx\|^2$ for all $x \in H$;

2) $^*$paranormal if $\|T^2x\|\|x\| \geq \|T^*x\|^2$ for all $x \in H$.

## 2. Preliminaries and Background

M. Fujii, D. Jung, S. H. Lee, M. Y. Lee and R. Nakamoto [1] introduced the following classes of operators:

An operator $T \in B(H)$ is $p$-paranormal for $p > 0$, if

$$\left\||T|^p U |T|^p x\right\|\|x\| \geq \left\||T|^p x\right\|^2 \tag{2.1}$$

holds for all $x \in H$, where $U$ is the partial isometry appearing in the polar decomposition $T = U|T|$ of $T$ with $|T| = (T^*T)^{1/2}$.

For $p > 0$, an operator $T \in B(H)$ is of class $A(p,p)$ if it satisfies an operator inequality

$$\left(|T^*|^p |T|^{2p} |T^*|^p\right)^{1/2} \geq |T^*|^{2p}. \tag{2.2}$$

For $p, q > 0$, an operator $T \in B(H)$ is of class $A(p,q)$ if it satisfies an operator inequality

$$\left(|T^*|^q |T|^{2p} |T^*|^q\right)^{q/p+q} \geq |T^*|^{2q}. \tag{2.3}$$

In this sequel we introduce $^*p$-paranormal operator, classes of operators $^*A(p,p)$ for $p > 0$ and $^*A(p,q)$ for $p, q > 0$ as follows:

A $p$-hyponormal operator is $^*p$-paranormal if

$$\left\||T|^p U |T|^p x\right\|\|x\| \geq \left\||T^*|^p x\right\|^2. \tag{2.4}$$

For $p > 0$ a $p$-hyponormal operator $T \in {}^*A(p,p)$ if it satisfies an operator inequality

$$|T|^{2p} \geq \left(|T|^p |T^*|^{2p} |T|^p\right)^{1/2}. \tag{2.5}$$

More generally, we define the class $^*A(p,q)$ for $p$, $q$

 

$> 0$ by an operator inequality

$$|T|^{2q} \geq \left(|T|^{q}|T^{*}|^{2p}|T|^{q}\right)^{q/p+q}. \tag{2.6}$$

**Remark (2.1).** If $T$ is $p$-hyponormal then using Furuta inequality (1.1) (§1) it can be proved easily that $T \in A(p,p)$.

**Remark (2.2).** By inequality (2.6) we have

$$\{p\text{-hyponormal, } 0<p<1\}$$
$$\subset\{\text{class } {}^{*}A(p,q);\ p,q\in(0,1)\}.$$

The well known theorem of $T$. Ando [3] for paranormal operator is required in the proof of our main result.

**Theorem (2.3).** (Ando's Theorem): An operator $T$ is paranormal if and only if

$$T^{*2}T^{2}+2kT^{*}T+k^{2}\geq 0 \tag{2.7}$$

for all real $k$.

## 3. Main Results

M. Fujii, *et al*. [1] proved the following theorem [1; Theorem 3.4].

**Theorem (3.1).** If $T \in A(p,p)$ for $p > 0$ then $T$ is $p$-paranormal.

In the following first we present an alternative way in which Theo (3.1) is proved in [1]. For this we have considered a quadratic form analogous to inequation (2.7) (§2). We also present a necessary and sufficient condition for a p-hyponormal operator $T$ to be a ${}^{*}p$-paranormal operator and the monotonicity of class ${}^{*}A(p,q)$.

**Theorem (3.2).** A $p$-hyponormal operator $T \in B(H)$ is $p$-paranormal if and only if $|T|^{4p}+2k|T|^{2p}+k^{2}\geq 0$ for all $k\in R$ and $p > 0$.

**Proof.** Let $T = U|T|$ be $p$-hyponormal where $U$ is partial isometry, hence

$$U|T|^{p}=|T^{*}|^{p}U \text{ and } U^{*}|T^{*}|^{p}=|T|^{p}U^{*}.$$

We have

$$U|T|^{2p}=U|T|^{p}|T|^{p}=|T^{*}|^{p}U|T|^{p}$$
$$=|T^{*}|^{p}|T^{*}|^{p}U=|T^{*}|^{2p}U$$

and

$$U^{*}|T^{*}|^{2p}=U^{*}|T^{*}|^{p}|T^{*}|^{p}=|T|^{p}U^{*}|T^{*}|^{p}$$
$$=|T|^{p}|T|^{p}U^{*}=|T|^{2p}U^{*}.$$

Now,

$$|T|^{4p}+2k|T|^{2p}+k^{2}\geq 0$$

for all $k\in R$

$$\Leftrightarrow\left(\left(|T|^{4p}+2k|T|^{2p}+k^{2}\right)x,x\right)\geq 0$$

for all $k\in R$

$$\Leftrightarrow\left(|T|^{4p}x,x\right)+2k\left(|T|^{2p}x,x\right)+k^{2}(x,x)\geq 0$$

for all $k\in R$

$$\Leftrightarrow\left\||T|^{2p}x\right\|^{2}+2k\left\||T|^{p}x\right\|^{2}+k^{2}\|x\|^{2}\geq 0$$

for all $k\in R$.

We know that if $a > 0$, $b$ and $c$ are real numbers then $at^{2}+2bt+c\geq 0$ for every real $t$ if and only if $b^{2}-4ac\leq 0$. Hence

$$|T|^{4p}+2k|T|^{2p}+k^{2}\geq 0$$

for all $k\in R$

$$\Leftrightarrow 4\left\||T|^{p}x\right\|^{4}\leq 4\left\||T|^{2p}x\right\|^{2}\|x\|^{2}$$
$$\Leftrightarrow\left\||T|^{p}x\right\|^{4}\leq\left\||T|^{2p}x\right\|^{2}\|x\|^{2}$$
$$=\left(|T|^{2p}x,|T|^{2p}x\right)\|x\|^{2}$$
$$=\left(|T|^{2p}x,U^{*}U|T|^{2p}x\right)\|x\|^{2}$$
$$=\left(U|T|^{2p}x,U|T|^{2p}x\right)\|x\|^{2}$$
$$=\left(U|T|^{2p}x,|T^{*}|^{2p}Ux\right)\|x\|^{2}$$
$$\left(\because U|T|^{2p}=|T^{*}|^{2p}U\right)$$
$$=\left(U^{*}|T^{*}|^{2p}U|T|^{2p}x,x\right)\|x\|^{2}$$
$$=\left(U^{*}|T^{*}|^{2p}|T^{*}|^{2p}Ux,x\right)\|x\|^{2}$$
$$=\left(U^{*}|T^{*}|^{4p}Ux,x\right)\|x\|^{2}.$$

Since $T$ be $p$-hyponormal, by Remark (2.1) (§2) $T\in A(p,p)$ *i.e.*

$$|T^{*}|^{2p}\leq\left(|T^{*}|^{p}|T|^{2p}|T^{*}|^{p}\right)^{1/2}$$
$$\Rightarrow|T^{*}|^{4p}\leq|T^{*}|^{p}|T|^{2p}|T^{*}|^{p}.$$

Hence

$$\left\||T|^{p}x\right\|^{4}\leq\left(U^{*}|T^{*}|^{4p}Ux,x\right)\|x\|^{2}$$
$$\Leftrightarrow\left\||T|^{p}x\right\|^{4}\leq\left(U^{*}|T^{*}|^{p}|T|^{2p}|T^{*}|^{p}Ux,x\right)\|x\|^{2}$$
$$=\left(|T|^{p}U^{*}|T|^{2p}U|T|^{p}x,x\right)\|x\|^{2}$$

$$\left[\because U^*\left|T^*\right|^p=|T|^p U^* \text{ and } \left|T^*\right|^p U=U|T|^p\right]$$

$$=\left(\left(|T|^p U^*|T|^p\right)\left(|T|^p U|T|^p\right)x,x\right)\|x\|^2$$

$$=\left(|T|^p U|T|^p x,|T|^p U|T|^p x\right)\|x\|^2$$

$$=\left\||T|^p U|T|^p x\right\|^2\|x\|^2$$

$$\Leftrightarrow \left\||T|^p x\right\|^2 \le \left\||T|^p U|T|^p x\right\|\|x\|$$

*i.e.* if and only if $T$ is $p$-paranormal.

**Remark (3.3).** Theorem (3.2) is independent of $\boldsymbol{x}\in H$ being taken as unit vector where as M. Fujii, *et al*. [1] have considered $\boldsymbol{x}\in H$ as unit vector in the result [1, Theo. 3.4].

The following result presents a necessary and sufficient condition for $p$-hyponormal operator $T$ to be a $^*p$-paranormal operator.

**Theorem (3.4).** A $p$-hyponormal operator $T$ is $^*p$-paranormal if and only if

$$|T|^{4p}+2k\left|T^*\right|^{2p}+k^2\ge 0 \quad \text{for all} \quad k\in R.$$

**Proof.** Let $T=U|T|$ be $p$-hyponormal operator where $U$ is a partial isometry also let $S=U|T|^p$ so that $U|T|^p=\left|T^*\right|^p U$, $U^*\left|T^*\right|^p=|T|^p U^*, U|T|^{2p}=\left|T^*\right|^{2p}U$ and $U^*\left|T^*\right|^{2p}=|T|^{2p}U^*$. Now

$$|T|^{4p}+2k\left|T^*\right|^{2p}+k^2\ge 0$$

for all $k\in R$

$$\Leftrightarrow \left(|T|^{4p}x,x\right)+2k\left(\left|T^*\right|^{2p}x,x\right)+k^2(x,x)\ge 0$$

for all $k\in R$

$$\Leftrightarrow \left\||T|^{2p}x\right\|^2+2k\left\|\left|T^*\right|^p x\right\|^2+k^2\|x\|^2\ge 0$$

for all $k\in R$

$$\Leftrightarrow \left\|\left|T^*\right|^p x\right\|^4 \le \left\||T|^{2p}x\right\|^2\|x\|^2$$

$$=\left(|T|^{2p}x,|T|^{2p}x\right)\|x\|^2$$

$$=\left(U|T|^{2p}x,U|T|^{2p}x\right)\|x\|^2$$

$$=\left(U|T|^{2p}x,\left|T^*\right|^{2p}Ux\right)\|x\|^2$$

$$=\left(U^*\left|T^*\right|^{2p}U|T|^{2p}x,x\right)\|x\|^2$$

$$=\left(U^*\left|T^*\right|^{2p}\left|T^*\right|^{2p}Ux,x\right)\|x\|^2$$

*i.e.*, $$\left\|\left|T^*\right|^p x\right\|^4 \le \left(U^*\left|T^*\right|^{4p}Ux,x\right)\|x\|^2. \qquad (3.2)$$

Since $T$ is $p$-hyponormal so $T\in A(p,p)$, *i.e.*

$$\left(\left|T^*\right|^p|T|^{2p}\left|T^*\right|^p\right)^{1/2}\ge\left|T^*\right|^{2p}$$

*i.e.* $$\left|T^*\right|^{4p}\le\left|T^*\right|^p|T|^{2p}\left|T^*\right|^p. \qquad (3.3)$$

From (3.2) and (3.3), we have

$$|T|^{4p}+2k\left|T^*\right|^{2p}+k^2\ge 0$$

for all $k\in R$

$$\Leftrightarrow \left\|\left|T^*\right|^p x\right\|^4 \le \left(U^*\left|T^*\right|^p|T|^{2p}\left|T^*\right|^p Ux,x\right)\|x\|^2$$

$$=\left(|T|^p U^*|T|^{2p}U|T|^p x,x\right)\|x\|^2$$

$$=\left(\left(|T|^p U^*|T|^p\right)\left(|T|^p U|T|^p\right)x,x\right)\|x\|^2$$

$$=\left(|T|^p U|T|^p x,|T|^p U|T|^p x\right)\|x\|^2$$

$$=\left\||T|^p U|T|^p x\right\|^2\|x\|^2$$

$$\Leftrightarrow \left\|\left|T^*\right|^p x\right\|^2 \le \left\||T|^p U|T|^p x\right\|\|x\|$$

*i.e.* if and only if $T$ is $^*p$-paranormal.

In the following we present monotonicity of $^*A(p,q)$. We need Furuta inequality [2,4] to prove the following theorem, see also [5,6].

**Theorem (3.5).** If $0<p<p'$ and $0<q$ then

$$^*A(q,p)\subseteq {}^*A(q,p').$$

**Proof.** Let $T\in{}^*A(t,s')$ where $0<s'<s$ and $0<t$ then by the definition of class $^*A(p,q)$ for $p, q>0$.

$$A=|T|^{2s'}\ge\left(|T|^{s'}\left|T^*\right|^{2t}|T|^{s'}\right)^{s'/(s'+t)}=B.$$

We apply it to (1.2) (§1), in the case when $r_0=\frac{s-s'}{2s'}$, $p_0=\frac{s'+t}{s'}$, $q_0=\frac{s+t}{s}$. We have

$$(1+2r_0)q_0=\left(1+\frac{s-s'}{s'}\right)\left(\frac{s+t}{s}\right)=\frac{s}{s'}\frac{(s+t)}{s}=\frac{s+t}{s'}$$

and

$$p_0+2r_0=\frac{s'+t}{s'}+\frac{s-s'}{s'}=\frac{s'+t+s-s'}{s'}=\frac{s+t}{s'}.$$

Hence $(1+2r_0)q_0=p_0+2r_0$, so that

$$\left[|T|^{\frac{2s'(s-s')}{2s'}}|T|^{\frac{2s'(s'+t)}{s'}}|T|^{\frac{2s'(s-s')}{2s'}}\right]^{s/(s+t)}$$

$$\ge\left[|T|^{\frac{2s'(s-s')}{2s'}}\left(|T|^{s'}\left|T^*\right|^{2t}|T|^{s'}\right)^{\frac{s'}{s'+t}\frac{s'+t}{s'}}|T|^{\frac{2s'(s-s')}{2s'}}\right]^{s/(s+t)}$$

*i.e.* $|T|^{2s} \geq \left(|T|^{s-s'+s'}|T^*|^{2t}|T|^{s'+s-s'}\right)^{s/(s+t)}$

*i.e.* $|T|^{2s} \geq \left(|T|^{s}|T^*|^{2t}|T|^{s}\right)^{s/(s+t)}$

*i.e.* $T \in {}^*A(t,s)$.

Hence

$$^*A(t,s') \subseteq {}^*A(t,s).$$

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