***Applied Mathematics*, 2012, 3, 1409-1414**
http://dx.doi.org/10.4236/am.2012.330198 Published Online October 2012 (http://www.SciRP.org/journal/am)

A New Lagrangian Multiplier Method on Constrained Optimization*

Youlin Shang#, Shengli Guo, Xiangyi Jiang
Department of Mathematics, Henan University of Science and Technology, Luoyang, China
Email: #mathshang@sina.com

Received July 8, 2012; revised August 8, 2012; accepted August 15, 2012

ABSTRACT

In this paper, a new augmented Lagrangian function with 4-piecewise linear NCP function is introduced for solving nonlinear programming problems with equality constrained and inequality constrained. It is proved that a solution of the original constrained problem and corresponding values of Lagrange multipliers can be found by solving an unconstrained minimization of the augmented Lagrange function. Meanwhile, a new Lagrangian multiplier method corresponding with new augmented Lagrangian function is proposed. And this method is implementable and convergent.

Keywords: Nonlinear Programming; NCP Function; Lagrange Function; Multiplier; Convergence

1. Introduction

Considering the following nonlinear inequality constrained optimization Problem (NLP):

$$\begin{aligned} &\min f(x) \\ &s.t \quad H(x)=0 \quad G(x)\le 0, \end{aligned} \tag{1}$$

where $f: R^n \to R$ and

$$G(x)=\left(g(x_1),g(x_2),\cdots,g(x_m)\right)^T : R^n \to R^m$$

$$H(x)=\left(h_1(x),h_2(x),\cdots,h_p(x)\right)^T : R^n \to R^p$$

are continuously differentiable functions.

We denote by

$$\begin{aligned} D=\{x\in R^n \,|\, &g(x_i)\le 0,\ i=(1,2,\cdots,m), \\ &h(x_j)=0,\ j=(1,2,\cdots,p)\} \end{aligned}$$

the feasible set of the problem (NLP).

The Lagrangian function associated with the problem (NLP) is the function

$$L(x,\omega,\lambda)=f(x)+\omega^T H(x)+\lambda^T G(x),$$

where

$$\omega=(\omega_1,\cdots,\omega_P)^T\in R^P, \lambda=(\lambda_1,\cdots,\lambda_m)^T\in R^m$$

are the multiplier vectors, For simplicity, we use (x,ω,λ) to denote the column vector $\left(x^T,\omega^T,\lambda^T\right)^T$

Defintion 1.1. A point $(\bar{x},\bar{\omega},\bar{\lambda})\in R^n\times R^p\times R^m$ is called a Karush-Kuhn-Tucker (KKT) point or a KKT pair of Problem (NLP), if it satisfies the following conditions:

$$\begin{aligned} &\Delta_x L(\bar{x},\bar{\omega},\bar{\lambda})=0, G(\bar{x})\le 0, H(\bar{x})=0, \\ &\bar{\lambda}\ge 0, \lambda_i g_i(\bar{x})=0, \forall i\in I, \end{aligned} \tag{2}$$

where $I=\{1\le i\le m\}$, we also say $\bar{x}$ is a KKT point if there exists a $(\bar{\omega},\bar{\lambda})$ such that $(\bar{x},\bar{\omega},\bar{\lambda})$ satisfies (2).

For the nonlinear inequality constrained optimization problem (NLP), there are many practical methods to solve it, such as augmented Lagrangian function method [1-6], Trust-region filter method [7,8], QP-free feasible method [9,10], Newton iterative method [11,12], etc. As we know, Lagrange multiplier method is one of the efficient methods to solve problem (NLP). Pillo and Grippo in [1-3] proposed a class of augmented Lagrange function methods which have nice equivalence between the unconstrained optimization and the primal constrained problem and get good convergence properties of the related algorithm. However, a max function is used for these methods which may be not differentiable at infinite numbers of points. To overcome this shortcoming, Pu in [4] proposed a augmented Lagrange function with Fischer-Burmeister nonlinear NCP function and Lagrange multiplier methods. Pu and Ding in [6] proposed a Lagrange multiplier methods with 3-piecewise linear NCP function. In this paper, a new class augmented Lagrange function with 4-piecewise linear NCP function and some Lagrange multiplier methods are proposed for the minimization of a smooth function subject to smooth inequality

*This paper was partially supported by the National Natural Science Foundation of China under Grant No. 10971053, and NNSF of Henan Province under Grant No. 094300510050.
#Corresponding author.

constraints and equality constrains.

The paper is organized as follows: In the next section we give some definitions and properties about NCP function, and then define a new augmented Lagrange function with 4-piecewise NCP function. In Section three, we give the algorithm. In Section four, we prove convergence of the algorithm. Some conclusions are given in Section five.

2. Preliminaries

In this section, we recall some definitions and define a new Lagrange multiplier function with 4-piecewise NCP function.

Definition 2.1 (NCP pair and SNCP pair). We call a pair (*a*, *b*) to be an NCP pair if $a \ge 0, b \ge 0$ and $ab = 0$; and call (*a*, *b*) to be an SNCP pair if (*a*, *b*) is a pair and $a^2 + b^2 \ne 0$.

Definition 2.2 (NCP function). A function $\phi : R^2 \to R$ is called an NCP function if $\phi(a,b) = 0$ if and only (a,b) is an NCP pair.

In this paper, we propose a new 4-piecewise linear NCP function $\psi(a,b)$ is as follows:

$$\psi(a,b) = \begin{cases} k^2 a, & if\ b \ge k|a|, \\ 2kb - b^2/a, & if\ a > |b|/k \\ 2k^2 a + 2kb + b^2/a, & if\ a < -|b|/k \\ k^2 a + 4kb, & if\ b \le -k|a| < 0 \end{cases} \tag{3}$$

If $(a,b) \ne (0,0)$, then

$$\nabla\psi(a,b) = \begin{cases} \begin{pmatrix} k^2 \\ 0 \end{pmatrix}, & if\ b \ge k|a| \\ \begin{pmatrix} b^2/a^2 \\ 2k - 2b/a \end{pmatrix}, & if\ a > |b|/k \\ \begin{pmatrix} 2k^2 - b^2/a^2 \\ 2k + 2b/a \end{pmatrix}, & if\ a < -|b|/k \\ \begin{pmatrix} k^2 \\ 4k \end{pmatrix}, & if\ b \le -k|a| \end{cases} \tag{4}$$

and

$$A_\psi = \partial\psi(0,0) = \left\{ \begin{pmatrix} k^2 t^2 \\ 2k(1-t) \end{pmatrix} \cup \begin{pmatrix} 2k^2(1-t^2) \\ 2k(1-t) \end{pmatrix} \middle| \ |t| \le 1 \right\}, \tag{5}$$

It is easy to check the following propositions:

1) $\psi(a,b) = 0 \Leftrightarrow a \ge 0, b \ge 0, ab = 0$;

2) The square of ψ is continuously differentiable;

3) ψ is twice continuously differentiable everywhere except at the origin but it is strongly semi-smooth at the origin.

Let

$$\phi_i(x,\lambda,C) = \psi(\lambda_i, -Cg_i(x)), 1 \le i \le m,$$

where $C > 0$ is a parameter. $\phi_i(x,\lambda,C) = 0$ if and only if $g_i(x) \le 0$, $\lambda_i \ge 0$ and $\lambda_i g_i(x) = 0$ for any $C > 0$.

We construct function:

$$\Phi(x,\lambda,C) = (\phi_1(x,\lambda_1,C), \cdots, \phi_m(x,\lambda_m,C))$$

Clearly, the KKT point condition (2) is equivalently reformulate as the condition:

$$\Phi(x,\lambda,C) = 0,\ H(x) = 0,\ \nabla L(x,\omega,\lambda) = 0.$$

If $(g_i(x), \lambda_i) \ne (0,0)$, then $\phi_i(x,\lambda,C)$ is continuously differentiable at $(x,\lambda) \in R^{n+m}$. We have

$$\nabla\psi(\lambda_i, -Cg_i(x)) = \begin{cases} \begin{pmatrix} 0 \\ k^2 \end{pmatrix}, \\ \text{if } -Cg_i(x) \ge k|\lambda_i| \\ \begin{pmatrix} \left(-2kC - \dfrac{2C^2 g_i(x)}{\lambda_i}\right)\nabla g_i(x) \\ \left(C^2 g_i^2(x)/\lambda_i^2\right) e_i \end{pmatrix}, \\ \text{if } \lambda_i > C|g_i(x)|/k \\ \begin{pmatrix} \left(-2kC + \dfrac{2C^2 g_i(x)}{\lambda_i}\right)\nabla g_i(x) \\ \left(2k^2 - C^2 g_i^2(x)/\lambda_i^2\right) e_i \end{pmatrix}, \\ \text{if } \lambda_i < -C|g_i(x)|/k \\ \begin{pmatrix} -4kC\nabla g_i(x) \\ k^2 e_i \end{pmatrix}, \\ \text{if } -Cg_i(x) \le -k|\lambda_i| < 0 \end{cases} \tag{6}$$

where $e_i = (0,\cdots,0,1,\cdots,0)^T \in R^m$ is the *i*th column of the unit matrix, its *j*th element is 1, and other elements are 0, in this paper take $k = 1$.

If $(g_i(x), \lambda_i) = (0,0)$, and then $\phi_i(x,\lambda,C)$ is strongly semi-smooth and direction differentiable at $(x,\lambda) \in R^{n+m}$. We have

$$\partial\psi(x,\lambda_i) = \left\{ \begin{pmatrix} k^2 t^2 e_i \\ 2Ck(1-t)\nabla g_i(x) \end{pmatrix} \cup \begin{pmatrix} 2k^2(1-t^2) e_i \\ 2Ck(1-t)\nabla g_i(x) \end{pmatrix} \middle| \ |t| \le 1 \right\}, \tag{7}$$

For Problem (NLP), we define a Di Pillo and Grippo type Lagrange multiplier function with 4-piecewise linear NCP function is as following:

$$
\begin{aligned}
&S(x,\omega,\lambda,C,D)\\
&= f(x)+\omega^T H(x)+\sum_{j=1}^{p} D\left(h_j(x)\right)^2 \Big/ 2\\
&+\sum_{i=1}^{m}\left[\left(\phi_i(x,\lambda,C)-\lambda_i/2\right)^2+\left(\lambda_i/2\right)^2\right]\Big/(2C)\\
&+\left\|\nabla G(x)^T \nabla_x L(x,\omega,\lambda)\right\|^2\Big/2,
\end{aligned}
\tag{8}
$$

where

$\lambda=(\lambda_1,\cdots,\lambda_m)^T\in R^m$, $\omega=(\omega_1,\cdots,\omega_P)^T\in R^p$ are the Lagrange multiplier, C and D are positive parameters.

In this section, we gave some assumptions as follows:

Assumpion 1 f, $h_j(x)$, $j=1,\cdots,p$, $g_i(x)$ $i=1,\cdots,m$ are twice Lipschitz continuously differentiable.

Define index set I_0 and I_1 as follows:

$$I_0(x,\ \lambda)=\left\{i\,\middle|\,\left(g_i(x),\lambda_i\right)=(0,0), i=1,2,\cdots,m\right\}$$

$$I_1(x,\ \lambda)=\left\{i\,\middle|\,\left(g_i(x),\lambda_i\right)\neq(0,0), i=1,2,\cdots,m\right\}$$

for any $i\in I_0$, according to definition of ϕ_i, have

$$\nabla_x\left[\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)^2+\left(\lambda_i/2\right)^2\right]=0$$

for any $i\in I_1$, we have

1) if $\lambda_i > C\left|g_i(x)\right|$, we have

$$
\begin{aligned}
&\nabla_x\left[\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)^2+\left(\lambda_i/2\right)^2\right]\Big/2C\\
&=\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)\left(-2-2Cg_i(x)/\lambda_i\right)\nabla g_i(x)\\
&=\left(-2-2Cg_i(x)/\lambda_i\right)\phi_i(x,\lambda_i,C)\nabla g_i(x)+\\
&\quad\lambda_i\nabla g_i(x)+2Cg_i(x)\nabla g_i(x)
\end{aligned}
\tag{9}
$$

The gradient of $S(x,\omega,\lambda,C,D)$ is

$$
\begin{aligned}
&\nabla_x S(x,\omega,\lambda,C,D)\\
&=\nabla f(x)+\left(\nabla H(x)\right)\omega^T+\sum_{j=1}^{p} D\left(h_j(x)\nabla h_j(x)\right)\\
&+\sum_{i\in I_1}\nabla\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)^2\Big/2C\\
&+\nabla G(x)\nabla^2 L(x,\omega,\lambda)\nabla^T G(x)\nabla L(x,\omega,\lambda)\\
&=\nabla L(x,\omega,\lambda)+\left(\nabla H(x)\right)\omega^T+\sum_{j=1}^{p} D\left(h_j(x)\nabla h_j(x)\right)\\
&+\sum_{i\in I_1}\left(-2-2Cg_i(x)/\lambda_i\right)\phi_i(x,\lambda_i,C)\nabla g_i(x)\\
&+\sum_{i=1}^{m}2Cg_i(x)\nabla g_i(x)\\
&+\nabla G(x)\nabla^2 L(x,\omega,\lambda)\nabla^T G(x)\nabla L(x,\omega,\lambda)
\end{aligned}
\tag{10}
$$

The Henssian matrix of $S(x,\omega,\lambda,C,D)$ at KKT point $(\bar{x},\bar{\omega},\bar{\lambda})$ is

$$
\begin{aligned}
&\nabla^2 S(\bar{x},\bar{\omega},\bar{\lambda},C,D)\\
&=\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})+\sum_{j=1}^{p} D\left(\nabla h_j(\bar{x})\nabla h_j(\bar{x})^T\right)\\
&+\sum_{i=1}^{m}6C\nabla g_i(\bar{x})\left(\nabla g_i(\bar{x})\right)^T,\\
&+\nabla G(\bar{x})\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})\left(\nabla G(\bar{x})\right)^T\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})
\end{aligned}
\tag{11}
$$

2) if $\lambda_i < -C\left|g_i(x)\right|$, then

$$
\begin{aligned}
&\nabla_x\left[\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)^2+\left(\lambda_i/2\right)^2\right]\Big/2C\\
&=\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)\left(-2+2Cg_i(x)/\lambda_i\right)\nabla g_i(x),\\
&=\left(-2+2Cg_i(x)/\lambda_i\right)\phi_i(x,\lambda_i,C)\nabla g_i(x)\\
&\quad+\lambda_i\nabla g_i(x)-2Cg_i(x)\nabla g_i(x)
\end{aligned}
\tag{12}
$$

The gradient of $S(x,\omega,\lambda,C,D)$ is

$$
\begin{aligned}
&\nabla_x S(x,\omega,\lambda,C,D)\\
&=\nabla f(x)+\left(\nabla H(x)\right)\omega^T+\sum_{j=1}^{p} C\left(h_j(x)\nabla h_j(x)\right)\\
&+\sum_{i\in I_1}\nabla\left(\phi_i(x,\lambda_i,C)-\lambda_i/2\right)^2\Big/2C\\
&+\nabla G(x)\nabla^2 L(x,\omega,\lambda)\nabla^T G(x)\nabla L(x,\omega,\lambda)\\
&=\nabla L(x,\omega,\lambda)+\left(\nabla H(x)\right)\omega^T+\sum_{j}^{p} C\left(h_j(x)\nabla h_j(x)\right)\\
&+\sum_{i\in I_1}\left(-2+2Cg_i(x)/\lambda_i\right)\phi_i(x,\lambda_i,C)\nabla g_i(x)\\
&-\sum_{i=1}^{m}2Cg_i(x)\nabla g_i(x)\\
&+\nabla G(x)\nabla^2 L(x,\omega,\lambda)\nabla^T G(x)\nabla L(x,\omega,\lambda)
\end{aligned}
\tag{13}
$$

The Henssian matrix of $S(x,\omega,\lambda,C,D)$ at KKT point $(\bar{x},\bar{\omega},\bar{\lambda})$ is

$$
\begin{aligned}
&\nabla^2 S(\bar{x},\bar{\omega},\bar{\lambda},C,D)\\
&=\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})+\sum_{j=1}^{p} D\left(\nabla h_j(\bar{x})\nabla h_j(\bar{x})^T\right)\\
&+\sum_{i=1}^{m}4C\nabla g_i(\bar{x})\left(\nabla g_i(\bar{x})\right)^T,\\
&+\nabla G(\bar{x})\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})\left(\nabla G(\bar{x})\right)^T\nabla^2 L(\bar{x},\bar{\omega},\bar{\lambda})
\end{aligned}
\tag{14}
$$

Definition 2.3 A point (x,ω,λ) is said to satisfy the strong second-order sufficiency condition for problem (NLP) if it satisfies the first-order KKT condition and if $d^T\nabla_{xx}^2 L(x,\omega,\lambda)d>0$ for all

$$d\in p(x)=\left\{d\,\middle|\,d^T\nabla h_j(x)=0, j=1,2,\cdots,p;\right.$$
$$\left.d^T\nabla g_i(x)=0, i\in\left\{i\,\middle|\,i=1,2,\cdots,m,\lambda_i>0\right\}\right\},$$

and $d \neq 0$.

Assumption 2 At any KKT point $(\bar{x}, \bar{\omega}, \bar{\lambda})$ satisfied strong second-order sufficiency condition.

Lemma If $A_{n\times n}$ is a positive semi-definite matrix, for any $d \in R^n$, $Ad = 0$, matrix $B_{n\times n}$ satisfied $d^T Bd > 0$, then exist m_1, for any $m > m_1$, $B + mA$ is positive definite matrix (see [4]).

Theorem 2.1 If $(\bar{x}, \bar{\omega}, \bar{\lambda})$ is KKT point of problem (1), then for sufficiently large *C* and *D*, $S(x, \omega, \lambda, C, D)$ is strong convex function at point $(\bar{x}, \bar{\omega}, \bar{\lambda})$.

Proof: Let $B = \nabla^2 L(x, \omega, \lambda)$

$$A = \sum_{j=1}^{p}\left(\nabla h_j(\bar{x})\nabla h_j(\bar{x})^T\right) + \sum_{i=1}^{m}\nabla g_i(x)(\nabla g_i(x))^T$$

for $d \in p(x)$, we have

$$\begin{aligned} Ad &= \sum_{j=1}^{p}\left(\nabla h_j(\bar{x})\nabla h_j(\bar{x})^T\right)d + \sum_{i=1}^{m}\nabla g_i(x)(\nabla g_i(x))^T d \\ &= \sum_{j=1}^{p}\left(\nabla h_j(\bar{x})d^T\nabla h_j(\bar{x})\right) + \sum_{i=1}^{m}\nabla g_i(x)d^T(\nabla g_i(x)) \\ &= 0 \end{aligned}$$

from **A2**, we have $d^T Bd > 0$. Furthermore there is m_1 if $\min\{6C, 4C, D\} > m_1$, for any $m_2 > m_1$, $B + m_2 A$ is positive definite matrix. And then for any $x \neq 0$ and sufficiently large *C* and *D* have

$$\begin{aligned} & x^T\nabla^2 S(\bar{x}, \bar{\omega}, \bar{\lambda}, C, D)x \\ &= x^T\nabla^2 L(\bar{x}, \bar{\omega}, \bar{\lambda})x + \sum_{j=1}^{p} D\left(x^T\nabla h_j(\bar{x})\right)^2 \\ &+ \sum_{i=1}^{m} 6C\left(x^T\nabla g_i(\bar{x})\right)^2 > x^T\nabla^2 L(\bar{x}, \bar{\omega}, \bar{\lambda})x \\ &+ m_1 x^T Ax > 0 \end{aligned}$$

by its continuously, we may obtained that there is $\eta > 0$, for all

$$\begin{aligned} (x, \omega, \lambda) &\in B_\eta(\bar{x}, \bar{\omega}, \bar{\lambda}, C, D) \\ &= \left\{\left\|(x, \omega, \lambda) - (\bar{x}, \bar{\omega}, \bar{\lambda})\right\| \leq \eta\right\}, \end{aligned}$$

we have $x^T S(x, \omega, \lambda, C, D)x > \varepsilon > 0$ the theorem hold.

3. Lagrange Multiplier Algorithm

Step 0 Choose parameters $C^0 > 0$, $D^0 > 0$, $0 \leq \eta \ll 1$, $0 < \theta_1 < 1 < \theta_2$, given point $x^0 \in R^n$, and

$$\omega^0 = (\omega_1^0, \cdots, \omega_p^0) \in R^p,$$

$$\lambda^0 = (\lambda_1^0, \cdots, \lambda_m^0) \in R^m,$$

Let $k = 0$.

Step 1 Solve following, we will obtain x^{k+1}.

$$\begin{aligned} & \min S(x, \omega, \lambda, C, D)^{def} \\ &= f(x) + \omega^T H(x) \\ &+ \sum_{j=1}^{p} D(h_j(x))^2 \Big/ 2 \\ &+ \frac{\sum_{i=1}^{m}\left[\left(\phi_i(x, \lambda_i, C) - \lambda_i/2\right)^2 + (\lambda_i/2)^2\right]}{2C} \\ &+ \left\|\nabla G(x)^T \nabla_x L(x, \omega, \lambda)\right\|^2 \Big/ 2 \end{aligned}$$

if $\left\|H(x^{k+1})\right\|_\infty \leq \eta$ and $\left\|\Phi(x^{k+1}, \lambda^k, C^k)\right\|_\infty \leq \eta$ then stop.

Step 2 For $j = 1, 2, \cdots, p$, $\left|h_j(x^{k+1})\right| \leq \theta_1\left|h_j(x^k)\right|$, then $D^{k+1} = D^k$ or $D^{k+1} = \theta_2 D^k$, for $i = 1, 2, \cdots, m$, if

$$\left|\phi_i(x^{k+1}, \lambda^k, C^k)\right| \leq \theta_1\left|\phi_i(x^k, \lambda^k, C^k)\right|,$$

then $C^{k+1} = C^k$, or $C^{k+1} = \theta_2 C^k$

Step 3 Compute ω^{k+1} and λ^{k+1}

$$\omega_j^{k+1} = \omega_j^k + Dh_j(x^{k+1}) \quad \lambda_i^{k+1} = \lambda_i^k + Cg_i(x^{k+1})$$

Step 4 Let *k* = *k* + 1, go to Step 1.

4. Convergence of the Algorithm

In this section, we make a assumption follow as:

Assumption 3 For any ω^k, λ^k, C^k, D^k, $S(x^{k+1}, \omega^k, \lambda^k, C^k, D^k)$ exists a minimizer point x^{k+1}.

Theorem 4.1 Assume feasible set of problem (NLP) is non-empty set and $f(x)$ is bounded, then algorithm is bound to stop after finite steps iteration.

Proof: Assume that the algorithm can not stop after finite steps iteration, by the sack of convenience, we define index set as following

$$J_e = \left(i \middle| \lim_{x\to\infty} h_i(x^k) \neq 0\right), \bar{J}_e = \left(i \middle| \lim_{x\to\infty} D^k = \infty\right)$$

$$J_i = \left(i \middle| \lim_{x\to\infty} \phi_i(x^k) \neq 0\right), \bar{J}_i = \left(i \middle| \lim_{x\to\infty} C^k = \infty\right)$$

according to assumption A3, it is clearly that $J_e \cup J_i$ or $\overline{J_e} \cap \overline{J_i}$ are non-empty set. for any *k*, obtain

$$\begin{aligned} & \mathrm{S}(x^{k+1}, \omega^k, \lambda^k, C^k, D^k) = f(x^{k+1}) \\ &+ \sum_{i=1}^{p} D^k\left[\left(h_i(x^{k+1}) + \omega_i^k/D^k\right)^2 - \left(\omega_i^k/D^k\right)^2\right] \Big/ 2 \\ &+ \sum_{i=1}^{m} C^k\left[\left(\phi(x^{k+1}, \lambda_i^k, C^k)/C^k - \lambda_i^k/2C^k\right)^2 + \left(\lambda_i^k/2C^k\right)^2\right] \Big/ 2 \\ &+ \left\|\nabla G(x)\nabla L(x^{k+1}, \omega^k, \lambda^k)\right\|^2 \Big/ 2 \end{aligned}$$

from above assumption, we obtain that for any a $\bar{k}$ there is $k > \bar{k}$, for any $i \in \overline{J_e}$ and $\tau > 0$, $D^{k+1} > D^k$ and $h_i(x^{k+1}) > \tau$, for sufficiently large *k*, it is not difficult to see that

$$D^k\left[\left(h_i\left(x^{k+1}\right)+\omega_i^k/D^k\right)^2-\left(\omega_i^k/D^k\right)^2\right]\Big/2>k^2\tau^2/2$$

Or for any $i\in\overline{J}$ and $\tau>0$, $C^{k+1}>C^k$ and $g_i\left(x^{k+1}\right)>\tau$, for sufficiently large k, have

$$C^k\left[\left(\phi_i\left(x^{k+1},\lambda_i^k,C^k\right)\big/C^k-\lambda_i^k/2C^k\right)^2+\left(\lambda_i^k/2C^k\right)^2\right]\Big/2$$
$$>k^2\tau^2/2$$

When $k\to\infty$ we can hold

$$\mathrm{S}\left(x^{k+1},\omega^k,\lambda^k,C^k,D^k\right)\to\infty$$

Which contradicts A3, the theorem holds.

Theorem 4.2 Let $X\times P\times M\subset R^{n+p+m}$ is a compact set, sequence $\left(x^{k+1},\omega^k,\lambda^k\right)$ are generated by the algorithm, and $\left(x^{k+1},\omega^k,\lambda^k\right)\in\operatorname{int}\left(X\times P\times M\right)$, in algorithm, 0 take the place of η, either algorithm stops at its kth and x^{k+1} is solution of problem(NLP), or for any an accumulation x^* of sequence $\left\{x^{k+1}\right\}$, x^* is solution of problem (NLP).

Proof: Because the algorithm stops at its kth, then we have

$$\begin{aligned}&\phi_i\left(x^{k+1},\lambda_i^k,C^k\right)=0,\ g_i\left(x^{k+1}\right)\le 0,\\&\lambda_i^k\ge 0,\ \lambda_i^k g_i\left(x^{k+1}\right)=0\end{aligned}\qquad(15)$$

for $g_i\left(x\right)=0$, $\lambda_i>0$, it is easy to see that for any $C>0$ have

$$\phi_i\left(x^{k+1},\lambda^k,C\right)=0\qquad\lambda_i^{k+1}=\lambda_i^k+Cg_i\left(x^{k+1}\right)=\lambda_i^k$$

It is from Step 2 of the algorithm that we have

$$\left\|\Phi\left(x,\lambda,C\right)\right\|_\infty=0,\left\|H\left(x\right)\right\|_\infty=0\qquad(16)$$

putting (15) (16) into (10) or (13), we can obtain

$$\begin{aligned}&\nabla S\left(x^{k+1},\omega^k,\lambda^k,C^k,D^k\right)\\&=\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)\\&\quad+\nabla G\left(x^{k+1}\right)\nabla^2L\left(x^{k+1},\omega^k,\lambda^k\right)\nabla^T G\left(x^{k+1}\right)\\&\quad\times\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)\\&=\left(E+\nabla G\left(x^{k+1}\right)\nabla^2L\left(x^{k+1},\omega^k,\lambda^k\right)\nabla^T G\left(x^{k+1}\right)\right)\\&\quad\times\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)=0\end{aligned}$$

for $g_i\left(x\right)<0$, $\lambda_i=0$, according to definition of $\Phi\left(x,\lambda,C\right)$, we can obtain, that

$$\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)=0$$

First part of the theorem holds, x^{k+1} is solution of problem (NLP).

On the other hand, if the algorithm is not stop at kth, for any accumulation point $\left(x^*,\omega^*,\lambda^*\right)$ of sequence $\left(x^{k+1},\omega^k,\lambda^k\right)$, from theorem 4.1, we can obtain, for any positive number C, that

$$\phi\left(x^*,\lambda^*,C\right)=0,\left\|H\left(x^*\right)\right\|=0$$

for any $\overline{x}\in X$, have

$$\begin{aligned}&f\left(\overline{x}\right)\ge f\left(x^{k+1}\right)\\&+\sum_{i=1}^{p}D^k\left[\left(h_i\left(x^{k+1}\right)+\omega_i^k/D^k\right)^2-\left(\omega_i^k/D^k\right)^2\right]\Big/2\\&+\sum_{i=1}^{m}C^k\left[\left(\phi\left(x^{k+1},\lambda_i^k,C^k\right)\big/C^k-\lambda_i^k/2C^k\right)^2+\left(\lambda_i^k/2C^k\right)^2\right]\Big/2\\&+\left\|\nabla G\left(x^{k+1}\right)\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)\right\|^2\Big/2\\&=f\left(x^{k+1}\right)+\left\|\nabla G\left(x^{k+1}\right)\nabla L\left(x^{k+1},\omega^k,\lambda^k\right)\right\|^2\Big/2+\mathrm{o}(1)\end{aligned}$$

Let $k\to\infty$,have

$$f\left(\overline{x}\right)\ge f\left(x^*\right)+\left\|\nabla G\left(x^*\right)\nabla L\left(x^*,\omega^*,\lambda^*\right)\right\|^2\Big/2\ge f\left(x^*\right)$$

Clearly, second part of the theorem holds. x^* is solution of problem (NLP).

5. Conclusion

A new Lagrange multiplier function with 4-piecewise linear NCP function is proposed in this paper which has a nice equivalence between its solution and solution of original problem. We can solve it to obtain solution of original constrained problem, the algorithm corresponding with it be endowed with convergence.

6. Acknowledgements

This paper was partially supported by the NNSF of China under Grant No. 10971053, and NNSF of Henan under Grant No. 094300510050.

REFERENCES

[1] G. Pillo and L. Grippo, "A New Class of Augmented Lagrangian in Nonlinear Programming," *SIAM Journal on Control and Optimization*, Vol. 17, No. 5, 1979, pp. 617-628.

[2] G. Pillo and L. Grippo, "An Augmented Lageangian function for Inequlity Constrains in Nonlinear Programming Problems," *Optimization Theorem and Application*, Vol. 36, No. 4, 1982, pp. 495-520. doi:10.1007/BF00940544

[3] G. Pillo and S. Lucidi, "An augmented Lagrange Function with Improved Exactness Properties," *SIAM Journal on Control and Optimization*, Vol. 12, No. 2, 2001, pp. 376-406.

[4] D. G. Pu and Z. Jin, "A New Lagrange Multiplier Method," *Tongji University Journal*, Vol. 9, No. 38, 2010, pp. 1384-1391.

[5] K. D. Li and D. G. Pu, "A class of New Lagrange Multiplier Methods," *OR Transaction*, Vol. 10, No. 4, 2006, pp. 9-24.

[6] X.-W. Du and Y.-L. Shang, "Exact Augmented Lagrange Function for Nonlinear Programming Problems with Inequality and Equality Constraints," *Applied Mathematics and Mechanics*, Vol. 26, No. 12, 2005, pp. 1651-1656.

[7] K. Su, "Trust-Region Filter Method with NCP Function," *Journal of Systems Science and Mathematical Science*, Vol. 28, No. 12, 2008, pp. 1525-1534

[8] D. G. Pu, "A New Filter Method," *OR Transaction*, Vol. 1, No. 15, 2011, pp. 46-58.

[9] W. Hua and D. G. Pu, "A Modified QP-Free Feasible Method," *Applied Mathmatical Modelling*, Vol. 35, No. 4, 2011, pp. 1696-1708. doi:10.1016/j.apm.2010.10.002

[10] K. Li and D. G. Pu, "Filter QP-Free Method with 3-Piece-Wise Linear NCP Function," *OR Transaction*, Vol. 12, No. 2, 2008, pp. 49-57.

[11] Y. X. Yuan, "Numerical Method for Nonlinear Programming," Shanghai Science and Technology Press, Shanghai, 1993.

[12] B. L. Chen, "The Theory of Optimization and Calculate," 2nd Edition, Tsinghua University Press, Beijing, 2005.

