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Generalization of the Second Order Vector Potential Formulation for Arbitrary Non-Orthogonal Curvilinear Coordinates Systems from the Covariant Form of Maxwell's Equations

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ABSTRACT

A great number of semi-analytical models, notably the representation of electromagnetic fields by integral equations are based on the second order vector potential (SOVP) formalism which introduces two scalar potentials in order to obtain analytical expressions of the electromagnetic fields from the two potentials. However, the scalar decomposition is often known for canonical coordinate systems. This paper aims in introducing a specific SOVP formulation dedicated to arbitrary non-orthogonal curvilinear coordinates systems. The electromagnetic field representation which is derived in this paper constitutes the key stone for the development of semi-analytical models for solving some eddy currents modelling problems and electromagnetic radiation problems considering at least two homogeneous media separated by a rough interface. This SOVP formulation is derived from the tensor formalism and Maxwell's equations written in a non-orthogonal coordinates system adapted to a surface characterized by a 2D arbitrary aperiodic profile.

Keywords: Second Order Vector Potential (SOVP); Curvilinear Coordinate System; Eddy Current Non-Destructive Testing (ECNDT)

1. Introduction

A great number of semi-analytical models for simulating Eddy Current Non-Destructive Testing (ECNDT) of conductive test pieces have been developed since the research works of several pioneers [1-7]. Most of canonical ECNDT configurations can be simulated today by numerically implementing some closed-form expression of the solution of the forward problem to be solved [8,9]. Most of these semi-analytical models lead to fast numerical models and are thus very useful for running analysis or parametric studies. Most of these semi-analytical models are based on the scalar decomposition of the electromagnetic field in some specific curvilinear orthogonal coordinate systems. The numerical models coming from these analytical models are therefore limited to canonical geometries. The purpose of this paper is to present a more general scalar potential representation of the electromagnetic field which can be applied for a non orthogonal coordinate system in order to prepare the development of some semi-analytical model which has the capability to compute the quasi-static electromagnetic fields due to an eddy current probe scanning a conducting half-space. The shape of the boundary surface is complex since two regions are separated by a rough surface. The framework of this project aims in generalizing the previous work [10] which has been firstly introduced for 2D eddy current problems.

The scalarization of the electromagnetic field is well known to researchers for a long time [11] and it has been extensively used for different applications in electromagnetism such as the radiation and scattering theory [12,13] and in the analysis of eddy currents [14]. A great number of authors have made use of the scalar potential formulation of the electromagnetic field since it is the starting point which allows to derive some analytical expressions of the field arising from specific canonical geometries implying an arbitrary time harmonic current above a conducting half space [3,7], or above a slab of finite thickness [4]. This scalar formulation, also called the second order vector potential formulation has been also used in the cylindrical coordinate system for an arbitrary current source inside a borehole [15,16] or in the spherical coordinate system [17,18]. This paper concerns more particularly the calculation of the electromagnetic field in the quasi-static limit when the geometry of the separating surface between two homogeneous media is described by an arbitrary non-orthogonal coordinate system.

This paper is organized as follows. In Section 2, the second-order vector potential formulation is briefly introduced. Since a Laplacian operator is only applied on a scalar potential, this formulation can minimize the number of unknowns and consequently the computer storage when calculating the electromagnetic fields in 3D scattering electromagnetic problems and 3D eddy currents problems. However, this formulation has been investigated for a finite number of orthogonal coordinates systems. The third section describes an extended formulation based on the covariant form of Maxwell's equations. The curvilinear coordinate method is summarized for obtaining the relationship between the components of the electromagnetic field and two longitudinal components. In Section 4, the second-order vector formulation is finally derived and some examples of different coordinate system are given in Section 5 in order to give some illustration of the formulas. Finally, Section 6 gives the conclusion and future works.

2. The Second Order Vector Potential Formulation for Eddy Current Problems

In curvilinear coordinate systems, the components of the magnetic vector potential cannot be separated due to the coupling between them [12]. Thus, the problem of the computation of the vector field leads to a great number of coupled unknowns since it is not possible to obtain separable Helmholtz equations. In order to overcome this drawback, it is usual to split the vector field into a longitudinal part and a transversal part. The longitudinal part is obtained from the calculation of the gradient of a scalar potential, this part is irrotational (rotational free). The transversal part is derived from another vector potential; this part is called solenoidal (divergence free).

$$\boldsymbol{A} = \boldsymbol{A}_L + \boldsymbol{A}_T$$

$$\boldsymbol{A}_L = \nabla\phi \quad \text{with} \quad \nabla\times\boldsymbol{A}_L = 0 \tag{1}$$

$$\boldsymbol{A}_T = \boldsymbol{A}_{T_1} + \boldsymbol{A}_{T_2} \quad \text{with} \quad \nabla\cdot\boldsymbol{A}_T = 0 \tag{2}$$

This vector potential can also derived from two other scalar potentials and a fixed unit vector judiciously chosen [11] according to the coordinate system used. The second order vector potential results in the separation of the Helmholtz equation in several coordinate systems [19].

$$\boldsymbol{A}_T = \nabla\times W = \nabla\times\left(uW_1 + u\times\nabla W_2\right) \tag{3}$$

According to this decomposition, the magnetic vector potential depends on three scalar quantities ϕ, W_1 and W_2. Since the magnetic vector potential $\boldsymbol{A}$ is derived from the curl of $\boldsymbol{W}$, this implies the coulomb gauge. This vector potential is also called the Second Order Vector Potential (SOVP). The longitudinal part of the electromagnetic field is not necessary for representing the magnetic flux density since the definition of $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $\nabla\times(\nabla\phi) = 0$. Only two scalar potential are finally necessary to represent the magnetic flux density $\boldsymbol{B} = \nabla\times\nabla\times\boldsymbol{W}$ and the current density can be derived by $\boldsymbol{J} = \frac{1}{\mu}\nabla\times\boldsymbol{B}$.

In such a way, this formulation has been extensively used to derive some analytical solutions for eddy current modeling problems. The goal of this paper is to generalize this formulation for a non-orthogonal curvilinear coordinates system which could describe the arbitrary shape of a separating rough surface between two regions of the space.

3. Problem Formulation

Let us consider an isotropic, homogeneous conducting half-space characterized by its conductivity σ and its magnetic permeability μ and the permittivity ε. The global planar surface delimiting this half-space is locally corrugated according to a cylindrical surface profile. In a Cartesian coordinate system (x, y, z), the surface is assumed to be invariant along the z axis and is described by a parametric function $a(x)$. Eddy currents are induced in the conducting region due to 3D arbitrary current sources in air above the half-space. The goal of this paper is to introduce a generalized scalar potentials formalism to build up a fast semi-analytical model which is able to compute the electromagnetic fields considering the quasi-static regime. This formalism is based on the introduction of a curvilinear coordinate system which conforms the rough surface. Writing the boundary conditions is thus easier since one of the new system of curvilinear coordinates is set to zero for each point on the boundary surface. Moreover, the tangential components and the normal components of the electromagnetic field are easily written by using the covariant and contravariant basis respectively. In the next section, the covariant form of Maxwell's equations governing the electromagnetic field are summarized and discussed.

3.1. The Covariant Form of Maxwell's Equations

In curvilinear coordinates systems, the Maxwell's equation are based on the tensorial formalism. By assuming a standard time dependence $\left(e^{-i\omega t}\right)$, the covariant form is given by:

$$\begin{aligned}
&\partial_j B^j = 0 \\
&\partial_j D^j = 0 \quad (\text{no charge sources}) \\
&\xi^{ijk}\partial_j E_k = -\frac{\partial B^i}{\partial t} = +i\omega B^i \quad \text{with} \quad i,j,k = 1,2,3 \\
&\xi^{ijk}\partial_j H_k = J^i + \frac{\partial D^i}{\partial t} = \left(J_s^i + J^i\right) - i\omega D^i
\end{aligned} \tag{4}$$

where ξ^{ijk} stands for the Lévi-Cività indicator [20] and $i,j,k \in 1,2,3$ are the indices associated to the components of the fields on the the three coordinate axes. The notation ∂_j means $\partial/\partial x^j$. This formalism is invariant to a change of referential. The components E_k and H_k are the covariant components of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. The components B^i, J^i and D^i are the contravariant components of the vectors $\boldsymbol{B}$, $\boldsymbol{J}$ and $\boldsymbol{D}$ respectively. They are themselves related to the covariant components of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ by the constitutive relations:

$$\begin{aligned}
B^i &= \mu H^i = \mu\sqrt{g}g^{ij}H_j = \mu^{ij}H_j \\
J^i &= \sigma E^i = \sigma\sqrt{g}g^{ij}E_j = \sigma^{ij}E_j \\
D^i &= \varepsilon E^i = \varepsilon\sqrt{g}g^{ij}E_j = \varepsilon^{ij}E_j
\end{aligned} \tag{5}$$

where g^{ij} are the contravariant components of the metric tensor of the coordinate system. The covariant components of the metric tensor verify the condition $g^{ik}g^{kj} = \delta_{ij}$ and $\sqrt{g} = \det\left(g^{ij}\right)$. The pseudo-tensors μ^{ij}, σ^{ij} and ε^{ij} depend on the choice of the metric system and they contain the physical and geometrical information of the problem. In the quasi-static regime, the permittivity of the conducting material is neglected and the wave number becomes $k_c^2 = i\omega\mu\sigma$. To exhibit a symmetry in the Maxwell's equation, let us introduce some notations. The complex impedance of the material is defined so that $k_c\underline{Z} = \omega\mu$ and $\underline{Z}\sigma = -ik_c$. By denoting $G_j = i\underline{Z}H_j$, Maxwell's equations may be simply written by:

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E} = 0 \\
&\nabla\cdot\boldsymbol{B} = 0 \\
&\nabla\times\boldsymbol{E} = k_c\sqrt{g}g^{ij}\boldsymbol{G} \\
&\nabla\times\boldsymbol{G} = i\underline{Z}\sqrt{g}g^{ij}\boldsymbol{J}_s + k_c\sqrt{g}g^{ij}\boldsymbol{E}
\end{aligned} \tag{6}$$

The dot product and the cross product are used exactly as in the Cartesian case.

3.2. Boundary Conditions

The main interest of this approach based on the coordinate transformation is that boundary conditions may be written in an analytical form since the boundary surface conforms exactly with the curvilinear coordinate system. The new system of coordinates is chosen so that one variable, at least, is constant for each point conforming the interface between the two media. Let us consider the coordinates $\left(x^1,x^2,x^3\right)$ and let $\boldsymbol{u}$ a field vector. The boundary surface is defined by $x^3 = 0$ for instance. Then, the continuity of the tangential components of the field vector $\boldsymbol{u}$ may be expressed by the continuity of the covariant components u_1,u_2 while the continuity of the normal component of $\boldsymbol{u}$ may be translated in the continuity of the contravariant component u^3. These two conditions will be explicitly described in the following of this paper. The main goal of this paper is to define two scalar potentials, usually the transverse electric and the transverse magnetic potentials for solving 3D eddy currents problems by using the curvilinear coordinate method.

3.3. The Curvilinear Coordinate Method

The boundary surface separating two isotropic homogeneous regions may be described by some parametric equations. Starting from the Cartesian coordinate system (x,y,z), we are looking for a new coordinate system such that the boundary surface conforms with one surface of coordinates. The Cartesian coordinates are labeled using index notation so that $(x,y,z) = \left(x^1,x^2,x^3\right)$. Now, let us define a set of curvilinear coordinates $\left(\theta^1,\theta^2,\theta^3\right)$:

$$\begin{aligned}
x^1 &= f\left(\theta^1,\theta^2,\theta^3\right) \\
x^2 &= g\left(\theta^1,\theta^2,\theta^3\right) \\
x^3 &= h\left(\theta^1,\theta^2,\theta^3\right)
\end{aligned} \tag{7}$$

Two natural sets of base vectors are associated with a curvilinear system: the covariant vectors $\boldsymbol{u}_i$ that are tangent to the coordinate lines and the contravariant vectors $\boldsymbol{u}^i$ that are normal to the coordinate surfaces.

The Jacobian matrix J defines the transformation from the Cartesian coordinates $\left(x^1,x^2,x^3\right)$ to the curvilinear coordinates $\left(\theta^1,\theta^2,\theta^3\right)$:

$$J = \left[\frac{\partial\underline{x}}{\partial\underline{\theta}}\right] = \begin{bmatrix} \dfrac{\partial f}{\partial\theta^1} & \dfrac{\partial f}{\partial\theta^2} & \dfrac{\partial f}{\partial\theta^3} \\ \dfrac{\partial g}{\partial\theta^1} & \dfrac{\partial g}{\partial\theta^2} & \dfrac{\partial g}{\partial\theta^3} \\ \dfrac{\partial h}{\partial\theta^1} & \dfrac{\partial h}{\partial\theta^2} & \dfrac{\partial h}{\partial\theta^3} \end{bmatrix}. \tag{8}$$

In the following of the paper, let us denote $\partial_i F = \dfrac{\partial F}{\partial\theta^i}$.

The matrix representation of the covariant components of the metric tensor is given by:

$$\left[g_{ij}\right] = \left[\frac{\partial\underline{x}}{\partial\underline{\theta}}\right]^T\left[\frac{\partial\underline{x}}{\partial\underline{\theta}}\right] \tag{9}$$

and we denote by g the determinant $g = \left|g_{ij}\right|$. Similarly, the matrix representation of the contravariant components of the metric tensor are given by:

$$\left[g^{ij}\right] = \left[\frac{\partial\theta}{\partial \underline{x}}\right]^T \left[\frac{\partial\theta}{\partial \underline{x}}\right] \tag{10}$$

Assume that this matrix may be expressed by:

$$\left[g^{ij}\right] = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{11}$$

Due to the relationship between these two representations of the metric tensors $\left[g^{ij}\right] = \left[g_{ij}\right]^{-1}$, the determinant of the matrix $\left[g^{ij}\right]$ is equal to $1/g$ and (see the formula below):

To fit boundary conditions, it is convenient to split the electromagnetic field into two components:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_t + E_u \boldsymbol{u} \\ \boldsymbol{G} &= \boldsymbol{G}_t + G_u \boldsymbol{u} \end{aligned} \tag{13}$$

Any vector $\boldsymbol{u}$ may be chosen for this decomposition though there are suitable choices [11]. In what follows in the paper, $\boldsymbol{u}$ is a vector in the directions of the coordinates $\boldsymbol{x}^n, n = 1,2,3$.

3.4. Decomposition Transversal/Longitudinal Components

Let us consider the longitudinal component E_u and G_u and the transversal fields $\boldsymbol{E}_t$ and $\boldsymbol{G}_t$. The nabla operator can be decomposed:

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \left(\nabla_t + \partial_n \boldsymbol{x}^n\right) \times \left(\boldsymbol{E}_t + E_n \boldsymbol{x}^n\right) \\ &= \left(\nabla_t \times \boldsymbol{E}_t\right) + \nabla_t \times \left(E_n\, \boldsymbol{x}^n\right) + \left(\boldsymbol{x}^n \partial_n\right) \times \boldsymbol{E}_t \end{aligned} \tag{14}$$

By applying the operator $\boldsymbol{x}^n \times$ to this equation and taking into account Equation (6), one obtains:

$$\begin{aligned} \boldsymbol{x}^n \times \left(\nabla \times \boldsymbol{E}\right) &= \nabla_t E_n - \partial_n \boldsymbol{E}_t \\ &= k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{G}_t + k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} G_n \boldsymbol{x}^n \end{aligned} \tag{15}$$

The same equation may be applied to the field vector $\boldsymbol{G}$ by substituting $\boldsymbol{E} \to \boldsymbol{B}$ and $E_n \to G_n$. These coupled equations may be expressed:

$$\begin{aligned} \nabla_t E_n - k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n G_n &= \partial_n \boldsymbol{E}_t + k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{G}_t \\ \nabla_t G_n - k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n E_n &= \partial_n \boldsymbol{G}_t + k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{E}_t \end{aligned} \tag{16}$$

Let us consider a compact form:

$$\begin{aligned} &\begin{bmatrix} \partial_n \boldsymbol{I} & k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \\ k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} & \partial_n \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_t \\ \boldsymbol{G}_t \end{bmatrix} \\ &= \begin{bmatrix} \nabla_t & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n & \nabla_t \end{bmatrix} \begin{bmatrix} E_n \\ G_n \end{bmatrix} \end{aligned} \tag{17}$$

where $\boldsymbol{I}$ means an identity dyadic operator. The left-hand side of (17) may be multiplied by the matrix:

$$\begin{bmatrix} \partial_n \boldsymbol{I} & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} & \partial_n \boldsymbol{I} \end{bmatrix} \tag{18}$$

such as:

$$\begin{aligned} &\begin{bmatrix} \partial_n \boldsymbol{I} & -k_c\, \boldsymbol{x}^n \times \sqrt{g} g^{ij} \\ -k_c\, \boldsymbol{x}^n \times \sqrt{g}\, g^{ij} & \partial_n \boldsymbol{I} \end{bmatrix} \\ &\times \begin{bmatrix} \partial_n \boldsymbol{I} & -k_c\, \boldsymbol{x}^n \times \sqrt{g} g^{ij} \\ -k_c\, \boldsymbol{x}^n \times \sqrt{g}\, g^{ij} & \partial_n \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_t \\ \boldsymbol{G}_t \end{bmatrix} \\ &= \begin{bmatrix} C_{11} & k_c\, C_{12} \\ k_c\, C_{12} & C_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_t \\ \boldsymbol{G}_t \end{bmatrix} \end{aligned} \tag{19}$$

with

$$\begin{aligned} C_{11} &= \left[\partial_n \partial_n \boldsymbol{I} - k_c^2 \left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\right] \\ C_{12} &= \partial_n \left[\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right] - \left[\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right] \partial_n \end{aligned} \tag{20}$$

So, in the general case, the transverse fields $\boldsymbol{E}_t$ and $\boldsymbol{G}_t$ are coupled. Similarly, the right side of (17) is multiplied by the same matrix (18):

$$\begin{aligned} &\begin{bmatrix} \partial_n \boldsymbol{I} & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} & \partial_n \boldsymbol{I} \end{bmatrix} \\ &\times \begin{bmatrix} \nabla_t & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n & \nabla_t \end{bmatrix} \begin{bmatrix} E_n \\ G_n \end{bmatrix} \\ &= \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{11} \end{bmatrix} \begin{bmatrix} E_n \\ G_n \end{bmatrix} \end{aligned} \tag{21}$$

with

$$\begin{aligned} M_{11} &= \partial_n \nabla_t + k_c^2 \left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right) \boldsymbol{x}^n \\ M_{12} &= -k_c \left(\boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla_t + \partial_n \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n\right) \end{aligned} \tag{22}$$

In particular, the second term of the right side of the matrix M_{12} can be transformed:

$$\left[g_{ij}\right] = \left[g^{ij}\right]^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = g \begin{bmatrix} g^{22}g^{33} - g^{23}g^{32} & g^{13}g^{32} - g^{12}g^{33} & g^{12}g^{23} - g^{13}g^{22} \\ g^{31}g^{23} - g^{21}g^{33} & g^{11}g^{33} - g^{13}g^{31} & g^{13}g^{21} - g^{11}g^{23} \\ g^{21}g^{32} - g^{31}g^{22} & g^{31}g^{12} - g^{11}g^{32} & g^{11}g^{22} - g^{12}g^{21} \end{bmatrix} \tag{12}$$

$$\begin{aligned} M_{12} &= -k_c\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla_t + \partial_n \boldsymbol{x}^n \times \sqrt{g} g^{ij} \boldsymbol{x}^n\right) \\ &= -k_c\left[\boldsymbol{x}^n \times \sqrt{g} g^{ij}\left(\nabla_t + \partial_n \boldsymbol{x}^n\right)\right] \\ &= -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \end{aligned} \tag{23}$$

The first term on the left side of the matrix can be written:

$$\begin{aligned} &\partial_n \nabla_t + k_c^2\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right) \boldsymbol{x}^n \\ &= \left(\partial_n \nabla - \partial_n \partial_n \boldsymbol{x}^n\right) + k_c^2\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right) \boldsymbol{x}^n \\ &= \partial_n \nabla - \left[\partial_n \partial_n \boldsymbol{I} - k_c^2\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\right] \boldsymbol{x}^n \end{aligned} \tag{24}$$

The first term on the right side of the Equation (24) may be translated on the left side so that an extended formulation can be obtained for all the components of the electromagnetic field in terms of the two longitudinal components:

$$\begin{aligned} &\begin{bmatrix} C_{11} & k_c C_{12} \\ k_c C_{12} & C_{11} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{G} \end{bmatrix} \\ &= \begin{bmatrix} \partial_n \nabla & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla & \partial_n \nabla \end{bmatrix} \begin{bmatrix} E_n \\ G_n \end{bmatrix} \end{aligned} \tag{25}$$

Finally, the term C_{11} on the left side of (25) may be substituted by:

$$\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right)\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij}\right) \boldsymbol{F} = \left[g_{ij}\right] \boldsymbol{x}^n\left(\boldsymbol{x}^n \cdot \boldsymbol{F}\right) - g_{nn} \boldsymbol{F} \tag{26}$$

and the expressions of the electromagnetic fields may be obtained from the longitudinal components by the simplified equation:

$$\begin{aligned} &\begin{bmatrix} \left(\partial_n \partial_n + k_c^2 g_{nn}\right) & k_c C_{12} \\ k_c C_{12} & \left(\partial_n \partial_n + k_c^2 g_{nn}\right) \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{G} \end{bmatrix} \\ &= \begin{bmatrix} \partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla & \partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \end{bmatrix} \begin{bmatrix} E_n \\ G_n \end{bmatrix} \end{aligned} \tag{27}$$

In the general case, the matrix operator cannot be translated on the right side of the equation but this form can help us to find a scalar decomposition. This is the main goal of the next section.

4. The Scalar Decomposition

In the following of the paper, all developments are performed according to the vector $\boldsymbol{x}^3$ but any vector may be chosen. Let us introduce two auxiliary scalar functions W_1 and W_2 such that:

$$\begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{G} \end{bmatrix} = \begin{bmatrix} \partial_n \nabla + k_c^2 g_{ij}\ \boldsymbol{x}^n & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla & \partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{28}$$

When there is no source term, Maxwell's equation are written in a compact form:

$$\begin{bmatrix} -k_c \sqrt{g} g^{ij} & \nabla \times \boldsymbol{I} \\ \nabla \times \boldsymbol{I} & -k_c \sqrt{g} g^{ij} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{G} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

so, by introducing (28) in (29),

$$\begin{aligned} &\begin{bmatrix} -k_c \sqrt{g} g^{ij} & \nabla \times \boldsymbol{I} \\ \nabla \times \boldsymbol{I} & -k_c \sqrt{g} g^{ij} \end{bmatrix} \\ &\times \begin{bmatrix} \partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n & -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \\ -k_c \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla & \partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{30}$$

which becomes:

$$\begin{bmatrix} -k_c \boldsymbol{a} & k_c^2 \boldsymbol{b} \\ k_c^2 \boldsymbol{b} & -k_c \boldsymbol{a} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{31}$$

with:

$$\boldsymbol{a} = \sqrt{g} g^{ij}\left(\partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n\right) + \nabla \times \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \tag{32}$$

$$\boldsymbol{b} = \sqrt{g} g^{ij}\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla\right) + \nabla \times g_{ij} \boldsymbol{x}^n \tag{33}$$

Since the product of the covariant metric tensor by the contravariant metric tensor is equal to identity, $\left[g^{ij}\right]\left[g_{ij}\right] = I_d$ and due to the property

$$\nabla \times \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla) \boldsymbol{a} - (\boldsymbol{a} \cdot \nabla) \boldsymbol{b}, \tag{34}$$

the operator $\boldsymbol{a}$ may be transformed:

$$\begin{aligned} \boldsymbol{a} &= \sqrt{g} g^{ij} \partial_n \nabla + k_c^2 \sqrt{g} \boldsymbol{x}^n + \nabla \times \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \\ &= \sqrt{g} g^{ij} \partial_n \nabla + k_c^2 \sqrt{g} \boldsymbol{x}^n + \boldsymbol{x}^n\left(\nabla \cdot \sqrt{g} g^{ij} \nabla\right) - \partial_n \sqrt{g} g^{ij} \nabla \\ &= \sqrt{g} g^{ij} \partial_n \nabla + k_c^2 \sqrt{g} \boldsymbol{x}^n + \boldsymbol{x}^n\left(\nabla \cdot \sqrt{g} g^{ij} \nabla\right) - \sqrt{g} g^{ij} \partial_n \nabla \\ &= \left(k_c^2 \sqrt{g} + \nabla \cdot \sqrt{g} g^{ij} \nabla\right) \boldsymbol{x}^n \end{aligned} \tag{35}$$

Moreover, after some tedious calculations, if $\boldsymbol{x}^n = \boldsymbol{x}^3$, it is possible to show that:

$$\begin{aligned} \boldsymbol{b} &= \sqrt{g} g^{ij}\left(\boldsymbol{x}^3 \times \sqrt{g} g^{ij} \nabla\right) + \nabla \times g_{ij} \boldsymbol{x}^3 \\ &= \left(g\left[g^{ij}\right] \begin{bmatrix} -g^{21} & -g^{22} & -g^{23} \\ g^{11} & g^{12} & g^{13} \\ 0 & 0 & 0 \end{bmatrix} \nabla\right) \\ &\quad + \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix} \end{aligned} \tag{36}$$

Since the tensor matric is symmetric, one obtains:

$$\boldsymbol{b} = \begin{bmatrix} 0 & -g_{33} & g_{23} \\ g_{33} & 0 & -g_{13} \\ -g_{23} & g_{13} & 0 \end{bmatrix} \nabla + \begin{bmatrix} -\partial_3 g_{23} + \partial_2 g_{33} \\ \partial_3 g_{13} - \partial_1 g_{33} \\ -\partial_2 g_{13} + \partial_1 g_{23} \end{bmatrix}$$
$$= \begin{bmatrix} -\partial_2 g_{33} + \partial_3 g_{23} \\ \partial_1 g_{33} - \partial_3 g_{13} \\ -\partial_1 g_{23} + \partial_2 g_{13} \end{bmatrix} + \begin{bmatrix} -\partial_3 g_{23} + \partial_2 g_{33} \\ \partial_3 g_{13} - \partial_1 g_{33} \\ -\partial_2 g_{13} + \partial_1 g_{23} \end{bmatrix} = 0 \tag{37}$$

The same result may be also obtained for $\boldsymbol{x}^n = \boldsymbol{x}^1$ and $\boldsymbol{x}^n = \boldsymbol{x}^2$. Finally, the two scalar potentials W_1 and W_2 are governed by the same propagation equation:

$$\mathcal{L}(\nabla) = \left(k_c^2 + \frac{1}{\sqrt{g}} \nabla \cdot \sqrt{g} g^{ij} \nabla \right) = 0 \tag{38}$$

According to the tensor analysis, the Laplacian operator of a scalar Φ may be written:

$$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial \theta_j} \right) = 0 \tag{39}$$

So, Equation (38) is the diffusion Helmholtz equation expressed in the curvilinear coordinate system. This equation will be then described for some examples in Section 5. Finally, we can verify if the two first equations of Maxwell's Equation in (4) are satisfied:

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \left(\bar{\bar{\mu}} \boldsymbol{H} \right) = \frac{\mu}{i\underline{Z}} \nabla \cdot \left(\sqrt{g} g^{ij} \boldsymbol{G} \right) \tag{40}$$

By using (36) and (38), after some calculations:

$$\begin{aligned} \left(\sqrt{g} g^{ij} \boldsymbol{G} \right) &= +k_c \nabla \times g_{ij} \boldsymbol{x}^n W_1 - \nabla \times \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla W_2 \\ &= \nabla \times \left(k_c g_{ij} \boldsymbol{x}^n W_1 + \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla W_2 \right) \end{aligned} \tag{41}$$

Due to the vector identity $\nabla \cdot (\nabla \times \boldsymbol{a}) = 0$, one obtains $\nabla \cdot \boldsymbol{B} = 0$. Likewise, the condition $\nabla \cdot \boldsymbol{D} = 0$ is also verified. Finally, let us define two scalar potentials so that:

$$\begin{aligned} W_1 &= -i\omega\mu\Gamma, \\ W_2 &= i\underline{Z}\Pi \end{aligned} \tag{42}$$

we can write the scalar decomposition:

$$\begin{aligned} \boldsymbol{E} &= -i\omega\mu \left[\left(\partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \right) \Gamma + \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \Pi \right] \\ \boldsymbol{H} &= k_c^2 \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \Gamma + \left(\partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \right) \Pi \end{aligned} \tag{43}$$

Since $\boldsymbol{b}$ = 0 in Equation (31), $\boldsymbol{a}$ occurs and by using (32), we can write two substitutions:

$$\begin{aligned} &\left(\partial_n \nabla + k_c^2 g_{ij} \boldsymbol{x}^n \right) \\ &= -\frac{1}{\sqrt{g}} g_{ij} \left(\nabla \times \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \right) \end{aligned} \tag{44}$$

$$\begin{aligned} &\left(\boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \right) \\ &= -\frac{1}{\sqrt{g}} g_{ij} \nabla \times g_{ij} \boldsymbol{x}^n \end{aligned} \tag{45}$$

So, the covariant components of the electromagnetic fields are expressed in the scalar potential decomposition:

$$\boldsymbol{E} = +i\omega\mu \frac{1}{\sqrt{g}} g_{ij} \left[\nabla \times \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \Gamma + \nabla \times g_{ij} \boldsymbol{x}^n \Pi \right] \tag{46}$$

$$\boldsymbol{H} = -\frac{1}{\sqrt{g}} g_{ij} \nabla \times \left[k_c^2 g_{ij} \boldsymbol{x}^n \Gamma + \boldsymbol{x}^n \times \sqrt{g} g^{ij} \nabla \Pi \right] \tag{47}$$

In the following of the paper, some particular curvilinear coordinate systems are introduced in order to verify the validity of this decomposition with respect to the works in the literature.

5. Examples of Different Curvilinear Coordinate Systems

In this section, several coordinate systems are introduced in order to compare the formulation to other calculations exciting in the literature on the second order potential. Finally, the last Section 5.4 provides a new writing concerning the translation coordinate system.

5.1. Application to the Planar Coordinate System

In the Cartesian coordinate system, the tensor metric is equal to Identity and the scalar decomposition becomes:

$$\boldsymbol{E} = i\omega\mu\nabla \times \left[\boldsymbol{x}^3 \times \nabla\Gamma + \boldsymbol{x}^3 \Pi \right] \tag{48}$$

$$= i\omega\mu\nabla \times \left[\boldsymbol{x}^3 \Pi - \nabla \times \boldsymbol{x}^3 \Gamma \right] \tag{49}$$

$$\boldsymbol{H} = \nabla \times \left(k_c^2 \boldsymbol{x}^3 \Gamma - \nabla \times \boldsymbol{x}^3 \Pi \right) \tag{50}$$

This formula may be compared to the equation described in [3,11,21]. Π is often called the Transverse Electric (TE) potential while Γ is the Transverse Magnetic (TM) potential.

5.2. Application to the Cylindrical Coordinate System

In a cylindrical coordinate system, let us consider $\left(\theta^1, \theta^2, \theta^3 \right) = (r, \theta, z)$.

$$x = r\cos\theta \tag{51}$$

$$y = r\sin\theta \tag{52}$$

$$z = z \tag{53}$$

The Jacobian matrix J defining the transformation from the Cartesian coordinates system to the cylindrical coordinates system is given by:

$$J=\left[\frac{\partial \underline{x}}{\partial \underline{\theta}}\right]=\begin{bmatrix}\frac{\partial f}{\partial\theta^1} & \frac{\partial f}{\partial\theta^2} & 0\\ \frac{\partial g}{\partial\theta^1} & \frac{\partial g}{\partial\theta^2} & 0\\ \frac{\partial h}{\partial\theta^1} & \frac{\partial h}{\partial\theta^2} & 1\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -r\sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \tag{54}$$

The matrix representation of the covariant components of the metric tensor is given by:

$$\left[g_{ij}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{55}$$

and we denote by g the determinant $g=\left|g_{ij}\right|=r^2$. The matrix representation of the contravariant components of the metric tensor are given by:

$$\left[g^{ij}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{r^2} & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & g^{22} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{56}$$

The transverse fields $\boldsymbol{E}_t$ and $\boldsymbol{G}_t$ are not coupled since the components of the metric tensor does not depend on the variable $\boldsymbol{x}^3$. From the scalar decomposition, the covariant components of the electrical field are given by:

$$\boldsymbol{E}=+i\omega\mu\frac{1}{\sqrt{g}}g_{ij}\nabla\times\left(\boldsymbol{x}^3\times\sqrt{g}g^{ij}\nabla\Gamma+g_{ij}\boldsymbol{x}^3\Pi\right) \tag{57}$$

$$\boldsymbol{H}=-\frac{1}{\sqrt{g}}g_{ij}\nabla\times\left(k_c^2 g_{ij}\boldsymbol{x}^3\Gamma+\boldsymbol{x}^3\times\sqrt{g}g^{ij}\nabla\Pi\right) \tag{58}$$

$$E_1=E_r=i\omega\mu\left[\frac{1}{r}\frac{\partial\Pi}{\partial\theta}-\frac{\partial^2\Gamma}{\partial z\partial r}\right] \tag{59}$$

$$E_2=i\omega\mu\left[\frac{\partial\Pi}{\partial r}-\frac{1}{r}\frac{\partial^2\Gamma}{\partial z\partial\theta}\right] \tag{60}$$

$$E_3=i\omega\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Gamma}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\Gamma}{\partial^2\theta}\right] \tag{61}$$

The covariant vectors are related to the unit vectors in the Cartesian coordinate system:

$$\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}=J\begin{bmatrix}t_x\\ t_y\\ t_z\end{bmatrix} \tag{62}$$

The unit vectors of the cylindrical coordinates are related to the covariant vectors:

$$\begin{bmatrix}e_r\\ e_\theta\\ e_z\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}t_x\\ t_y\\ t_z\end{bmatrix}$$
$$=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}J^{-1}\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{r} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}t_x\\ t_y\\ t_z\end{bmatrix} \tag{63}$$

and finally

$$\begin{aligned}E_r&=E_1=i\omega\mu\left[\frac{1}{r}\frac{\partial\Pi}{\partial\theta}-\frac{\partial^2\Gamma}{\partial z\partial r}\right]\\ E_\theta&=\frac{1}{r}E_2=i\omega\mu\left[\frac{\partial\Pi}{\partial r}-\frac{1}{r}\frac{\partial^2\Gamma}{\partial z\partial\theta}\right]\\ E_z&=E_3=i\omega\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Gamma}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\Gamma}{\partial^2\theta}\right]\end{aligned} \tag{64}$$

These last equations are similar to those existing in the literature [15]. The propagation Equation (38) is given by:

$$\frac{1}{r}\partial_1\left(r\partial_1\right)+\frac{1}{r^2}\partial_2+\partial_3+k_c^2=0 \tag{65}$$

This equation is quite similar to the scalar Helmholtz equation expressed in the cylindrical coordinate system:

$$\nabla^2\Gamma+k_c^2\Gamma=0 \tag{66}$$

5.3. Application to the Spherical Coordinate System

In a spherical coordinate system, let us consider $\left(\theta^1,\theta^2,\theta^3\right)=\left(r,\theta,\phi\right)$.

$$x=r\sin\theta\cos\phi=\theta^1\sin\theta^2\cos\theta^3 \tag{67}$$

$$y=r\sin\theta\sin\varphi=\theta^1\sin\theta^2\sin\theta^3 \tag{68}$$

$$z=r\cos\theta=\theta^1\cos\theta^3 \tag{69}$$

The Jacobian matrix J defining the transformation from the Cartesian coordinates system to the spherical coordinates system is given by:

$$J=\left[\frac{\partial \underline{x}}{\partial \underline{\theta}}\right]=\begin{bmatrix}\frac{\partial f}{\partial\theta^1} & \frac{\partial f}{\partial\theta^2} & 0\\ \frac{\partial g}{\partial\theta^1} & \frac{\partial g}{\partial\theta^2} & 0\\ \frac{\partial h}{\partial\theta^1} & \frac{\partial h}{\partial\theta^2} & 1\end{bmatrix} \tag{70}$$
$$=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta\\ r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0\end{bmatrix}$$

The matrix representation of the covariant components of the metric tensor is given by:

$$\left[g_{ij}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta\end{bmatrix} \tag{71}$$

and $\sqrt{g}=r^2\sin\theta$. The matrix representation of the contravariant components of the metric tensor are given by:

$$\left[g^{ij}\right]=\begin{bmatrix}0\\ -\dfrac{r}{\sin\theta}\partial_3\\ r\sin\theta\partial_2\end{bmatrix} \tag{72}$$

in this case, we choose a longitudinal orientation along $\boldsymbol{r}=r\boldsymbol{u}_r$. The scalar decomposition is given by:

$$\boldsymbol{E}=+i\omega\mu\frac{1}{\sqrt{g}}g_{ij}\nabla\times\left(\boldsymbol{x}^1\times\sqrt{g}g^{ij}\nabla\Gamma+g_{ij}\boldsymbol{x}^1\Pi\right) \tag{73}$$

$$\boldsymbol{H}=-\frac{1}{\sqrt{g}}g_{ij}\nabla\times\left(k_c^2g_{ij}\boldsymbol{x}^1\Gamma+\boldsymbol{x}^1\times\sqrt{g}g^{ij}\nabla\Pi\right) \tag{74}$$

The covariant vectors are related to the unit vectors in the Cartesian coordinate system:

$$\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}=J\begin{bmatrix}t_x\\ t_y\\ t_z\end{bmatrix} \tag{75}$$

The unit vectors of the spherical coordinates are related to the covariant vectors:

$$\begin{aligned}\begin{bmatrix}e_r\\ e_\theta\\ e_\varphi\end{bmatrix}&=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi\\ -\sin\phi & \cos\phi & 0\end{bmatrix}J^{-1}\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}\\ &=\begin{bmatrix}1 & 0 & 0\\ 0 & \dfrac{1}{r} & 0\\ 0 & 0 & \dfrac{1}{r\sin\theta}\end{bmatrix}\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix}\end{aligned} \tag{76}$$

and finally

$$\begin{aligned}E_r&=E_1=i\omega\mu\left[\frac{\cos\theta}{r\sin\theta}\frac{\partial\Gamma}{\partial\theta}+\frac{1}{r\sin^2\theta}\frac{\partial^2\Gamma}{\partial\phi^2}-\frac{1}{r}\frac{\partial^2\Gamma}{\partial\theta^2}\right]\\ E_\theta&=\frac{1}{r}E_2=-i\omega\mu\left[\frac{\partial^2\Gamma}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial\Gamma}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\Pi}{\partial\phi}\right]\\ E_\phi&=\frac{1}{r\sin\theta}E_3=-i\omega\mu\left[\frac{1}{\sin\theta}\left[\frac{\partial^2}{\partial r\partial\phi}+\frac{1}{r}\frac{\partial\Gamma}{\partial\phi}\right]-\frac{\partial\Pi}{\partial\theta}\right]\end{aligned} \tag{77}$$

These last equations are rigorously similar to those existing in the literature [17]. The propagation Equation (38) is given by:

$$\begin{aligned}&\frac{1}{r^2\sin\theta}\partial_1\left(r^2\sin\theta\partial_1\right)+\frac{1}{r^2\sin\theta}\partial_2\left(\sin\theta\partial_2\right)\\ &+\frac{1}{r^2\sin\theta}\partial_3\left(\frac{1}{\sin\theta}\partial_3\right)+k_c^2=0\end{aligned} \tag{78}$$

This equation is quite similar to the scalar Helmholtz equation expressed in the spherical coordinate system:

$$\nabla^2\Gamma+k_c^2\Gamma=0 \tag{79}$$

5.4. Application to the Translation Coordinate System

Starting from the Cartesian coordinate system (x,y,z), a 2D boundary surface may be described by a parametric form $a(x,y)$ and let us consider the translation coordinate system so that:

$$x=x^1 \tag{80}$$

$$y=x^2 \tag{81}$$

$$z=x^3+a\left(x^1,x^2\right) \tag{82}$$

The height z of each point $P(x,y,z)$ conforming the surface is translated in a simplified condition $x^3=0$. The jacobian matrix is given by:

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \dot{a}_x & \dot{a}_y & 1\end{bmatrix} \tag{83}$$

with $\dot{a}_x=\partial^1a(x,y)$ and $\dot{a}_y=\partial^2a(x,y)$. The metric tensor is given by:

$$\left[g_{ij}\right]=\begin{bmatrix}1+\dot{a}_x^2 & \dot{a}_x\,\dot{a}_y & \dot{a}_x\\ \dot{a}_x\,\dot{a}_y & 1+\dot{a}_y^2 & \dot{a}_y\\ \dot{a}_x & \dot{a}_y & 1\end{bmatrix} \tag{84}$$

and

$$\left[g^{ij}\right]=\begin{bmatrix}1 & 0 & g^{13}\\ 0 & 1 & g^{23}\\ g^{13} & g^{23} & g^{33}\end{bmatrix} \tag{85}$$

with:

$$g^{13}=-\dot{a}_x$$

$$g^{23}=-\dot{a}_y$$

$$g^{33}=1+\dot{a}_x^2+\dot{a}_y^2$$

The components of the metric tensor does not depend on the variable x^3. The field vector $\boldsymbol{E}_t$ and $\boldsymbol{G}_t$ are not coupled since the operator $\boldsymbol{G}_t$ in Equation (20) is null. The propagation Equation (38) is given by:

$$\begin{aligned}&\partial_1^2+\partial_2^2+k_c^2+g^{33}\partial_3^2+\partial_3\left(\partial_1 g^{13}+g^{13}\partial_1\right)\\&+\partial_3\left(\partial_2 g^{23}+g^{23}\partial_2\right)=0\end{aligned} \tag{86}$$

This last equation may be compared to the equations described by [22,23].

6. Conclusion and Future Work

In this paper, a generalized second order potential formulation (SOVP) is proposed for solving scattering or radiation problems described in an arbitrary non-orthogonal curvilinear coordinate system. This formulation takes advantages from the tensor analysis but no expertise is finally required for developing the expressions of the electromagnetic field in terms of two scalar potentials, usually the transverse electric potential and the transverse magnetic potential. For writing the components of the electrical field and the magnetic field for any curvilinear coordinate system, it is necessary to write the metric tensor which is easily defined in the paper and to use the vector cross product as usual in a Cartesian coordinate system. This SOVP formulation represents the key stone for implementing new numerical models dedicated to eddy current calculations based on the covariant form of the Maxwell' equations. By using a specific curvilinear coordinate system matching the geometry of the boundary surface, it is possible to write easily and analytically the boundary conditions implying the covariant and contravariant components of the electromagnetic field. In future work, a numerical method will be developed for calculating eddy currents induced in a conducting workpiece due to a 3D eddy current probe scanning the boundary surface described by an arbitrary and irregular geometry.

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