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Maximum Likelihood Estimation for Generalized Pareto Distribution under Progressive Censoring with Binomial Removals

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ABSTRACT

The paper deals with the estimation problem for the generalized Pareto distribution based on progressive type-II censoring with random removals. The number of components removed at each failure time is assumed to follow a binomial distribution. Maximum likelihood estimators and the asymptotic variance-covariance matrix of the estimates are obtained. Finally, a numerical example is given to illustrate the obtained results.

Keywords: Pareto Distribution; Binomial Removal; Progressive Censoring; Maximum Likelihood Estimator

1. Introduction

The generalized Pareto distribution is also known as the Lomax distribution with two parameters, or the Pareto of the second type. It can be considered as a mixture distribution. Suppose that a random variable X has an exponential distribution with some parameter λ. Further, suppose that λ itself has a gamma distribution, and then the resulting unconditional distribution of X is called the Lomax distribution. This distribution has been extensively used for reliability modeling and life testing, for details see e.g. Balkema and de Haan [1]. It also has been used as an alternative to the exponential distribution when the data are heavy tailed, see Bryson [2]. It has applications in economics, actuarial modeling, queuing problems and biological sciences, for more details we refer to Johnson *et al*. [3].

A random variable X is said to have the generalized Pareto distribution with two parameters, abbreviated as $X \sim GP(\alpha,\beta)$, if it has the probability density function (pdf)

$$f(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{-(\alpha+1)}, x>0, \alpha,\beta>0. \quad (1)$$

Here α and β are the shape and scale parameters, respectively. The survival function (sf) associated to (1) is given by

$$\bar{F}(x;\alpha,\beta) = (1+\beta x)^{\alpha}, x>0 \quad (2)$$

The hazard function is

$$h(x;\alpha,\beta) = \alpha\beta/(1+\beta x), x>0. \quad (3)$$

All standard (further probabilistic properties of this distribution are given, for example, in Arnold [4]).

In life testing, the experimenter does not always observe the failure times of all components placed on the test. In such cases, the censored sampling arises. Furthermore, there are some cases in which components are lost or removed from the test before failure. This would lead to progressive censoring. Progressive censoring schemes are very useful in clinical trials and life-testing experiments. Balakrishnan and Aggarwala [5] provided a wealth of information on inferences under progressive censoring sampling. The progressively type-II censored life test is described as follows. The experimenter puts *n* components on test at time zero. The first failure is observed at X_1 and then R_1 of surviving components is randomly selected and removed. When the *i*th failure component is observed at X_i, R_i of surviving components are randomly selected and removed, $i = 2,3,\cdots,m.$ The experiment terminates when the *m*th failure component is observed at X_m and $R_m = n-m-\sum_{i=1}^{m} R_i$ all removed. In this censoring scheme $R_1, R_2,\cdots,R_m$ are all prefixed. However, in some practical experiments, these numbers cannot be pre-fixed and they occur at random. Yuen and Tse [6], Tse and Yuen [7] and Tse *et al.* [8] considered the estimation problem for Weibull distribution under type-II progressive censoring with binomial removals. Shuo [9] studied the estimation problem for two-parameter Pareto distribution based on progressive censoring with uniform removals. Wu [10] provided estimation for the two-parameter Pareto distribution under progressive censoring with uniform removals. Wu *et al.* [11] studied the Burr type XII distribution based on progressively censored samples with random removals.

This paper is concerned with the estimation problem of the unknown parameters for the generalized Pareto distribution based on progressive type-II censoring with random removals. The number of components removed at each failure time is assumed to follow a binomial distribution. In Section 2, we derive the maximum likelyhood estimators. The asymptotic variance-covariance matrix of the estimates is obtained in Section 3. In Section 3, numerical examples are given to illustrate the obtained results.

2. Maximum Likelihood Estimation

Let $X_1 < X_2 < \cdots < X_m$ be the ordered failure times of mcomponents, where $m < n$ is pre-specified before the test. At the *i*th failure, R_i components are removed from the test. For progressive censoring with pre-specified number of number of removals $R = \left(R_1 = r_1, \cdots, R_{m-1} = r_{m-1}\right)$, the likelihood function can be defined as follows: (see Cohen [12])

$$L_1\left(x;\alpha,\beta \middle| R\right) = C\left(r\right)\prod_{i=1}^{m} f\left(x_i\right)\left[1-F\left(x_i\right)\right]^{r_i} \tag{4}$$

where

$$C\left(r\right) = n\left(n-r_1-1\right)\cdots\left(n-\sum\nolimits_{i=1}^{m-1}\left(r_i+1\right)\right. \tag{5}$$

Using relations (1), (2) and (3), the log-likelihood function is given by

$$\begin{aligned}\ln L_1\left(x;\alpha,\beta\right) &= \ln C\left(r\right) + m\ln\left(\alpha\beta\right)\\ &-\left(\alpha+1\right)\sum\nolimits_{i=1}^{m}\ln\left(1+\beta x_i\right)\\ &-\alpha\sum\nolimits_{i=1}^{m}\ln\left(1+\beta x_i\right)\end{aligned}$$

Looking at relation (4), it is noted that the relation is derived conditional on r_i. Each r_i can be of any integer value between 0 and $n-m-\sum_{j=1}^{i-1} r_i$ We assume that r_i is a random number and is assumed to follow a binomial distribution with parameter p. This means the probability that each component leaves will remain the same, say p, and the probability of r_i component leaving after the *i*th failure occurs is

$$P\left(R_1 = r_1\right) = \binom{n-m}{r_i} p^{r_1}\left(1-p\right)^{n-m-r_1} \tag{6}$$

and

$$\begin{aligned}&P\left(R_i = r_i \middle| R_i - 1 = r_i - 1, \cdots, R_1 = r_1\right)\\ &= \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i}\left(1-p\right)^{n-m-\sum\limits_{j=1}^{i-1} r_1}\end{aligned}$$

where $0 \le r_i \le n-m-\left(r_1+r_2+\cdots+r_{i-1}\right)$, for $i = 1, \cdots, m-1$. Furthermore, we assume that R_i is independent of X_i for all *i*. The joint likelihood funtion of $X = \left(X_1, X_2, \cdots, X_m\right)$ and $R = \left(R_1, R_2, \cdots, R_m\right)$ can be found

$$L\left(x;\alpha,\beta,p\right) = L_1\left(x;\alpha,\beta \middle| R\right)P\left(R,p\right) \tag{7}$$

where $P\left(R,p\right)$ is the joint probability distribution of $R = \left(r_1, r_2, \cdots, r_m\right)$ and is given by

$$\begin{aligned}&P\left(R,p\right)\\ &= P\left(R_{m-1} = r_{m-1} \middle| R_{m-2} = r_{m-2}, \cdots, R_1 = r_1\right)\\ &\times P\left(R_{m-2} = r_{m-2} \middle| R_{m-3} = r_{m-3}, \cdots, R_1 = r_1\right)\times\cdots\\ &\times P\left(R_2 = r_2 \middle| R_1 = r_1\right)P\left(R_1 = r_1\right)\\ &= \frac{\left(n-m\right)!}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)\prod_{i=1}^{m-1} r_i!} p^{\sum\limits_{i=1}^{m-1} r_i}\left(1-p\right)^{(m-1)(n-m)-\sum\limits_{i=1}^{m-1}(m-i)r_i}.\end{aligned}$$

2.1. Point Estimation

The maximum likelihood estimators of α and β are found by maximizing $L\left(x;\alpha,\beta,p\right)$, since $P\left(R,p\right)$ does not involve the parameters α and β. Thus, the maximum likelihood estimates (MLEs), $\left(\hat{\alpha},\hat{\beta}\right)$ of $\left(\alpha,\beta\right)$ can be found by solving the following equations:

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} &= \frac{m}{\alpha} - \sum_{i=1}^{m}\ln\left(1+\beta x_i\right) - \sum_{i=1}^{m} r_i \ln\left(1+\beta x_i\right)\\ &= 0.\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta} &= \frac{m}{\beta} - \left(\alpha+1\right)\sum_{i=1}^{m}\frac{x_i}{1+\beta x_i} - \alpha\sum_{i=1}^{m}\frac{r_i x_i}{1+\beta x_i}\\ &= 0\end{aligned} \tag{9}$$

From (8) and (9), we obtain $\hat{\alpha}$ the estimators of α:

$$\hat{\alpha} = \frac{m}{\sum_{i=1}^{m}\left(1+r_i\right)\ln\left(1+\beta x_i\right)} \tag{10}$$

and β can be obtained as the solution of the non-linear equation

$$\begin{aligned}f\left(\beta\right) &= \frac{m}{\beta} - \frac{m}{\sum_{i=1}^{m}\left(1+r_i\right)\ln\left(1+\beta x_i\right)}\sum_{i=1}^{m}\frac{\left(1+r_i\right)x_i}{1+\beta x_i}\\ &-\sum_{i=1}^{m}\frac{x_i}{1+\beta x_i}\end{aligned} \tag{11}$$

Therefore, $\hat{\beta}$ can be obtained as solution of the nonlinear equation of the form $g\left(\beta\right) = \beta$, where

$$\begin{aligned}g(\beta) = m\Bigg[&\frac{m}{\sum_{i=1}^{m}\left(1+r_i\right)\ln\left(1+\beta x_i\right)}\sum_{i=1}^{m}\frac{\left(1+r_i\right)x_i}{1+\beta x_i}\\ &+\sum_{i=1}^{m}\frac{x_i}{1+\beta x_i}\Bigg]^{-1}\end{aligned} \tag{12}$$

Since $\hat{\beta}$ is a fixed point solution of the non-linear Equation (12), therefore, it can be obtained using an iterative scheme as follows

$$g\left(\beta_j\right)=\beta_{j+1},$$

where $\hat{\beta}$ is the jth iterate of β. The iteration procedure should be stopped when $\left|\beta_j-\beta_{j+1}\right|$ is sufficiently small. Once we obtain $\hat{\beta}$ then $\hat{\alpha}$ can be obtained from (10). The MLEs of reliability and hazard function are

$$\hat{\bar{F}}\left(x;\alpha,\beta\right)=\left(1+\hat{\beta}x\right)^{-\hat{\alpha}}$$

and

$$\hat{h}\left(x;\alpha,\beta\right)=\frac{\hat{\alpha}\hat{\beta}}{\left(1+\hat{\beta}x\right)}.$$

Similarly, the MLE $\hat{p}$ of p can be found by maximizing $P\left(R,p\right)$. Independently, the MLE of parameter p is the solution of

$$\frac{\partial\ln P\left(R,p\right)}{\partial p}=\frac{\sum_{i=1}^{m-1}r_i}{p}-\frac{\left(m-1\right)\left(n-m\right)-\sum_{i=1}^{m-1}\left(m-i\right)r_i}{1-p}$$

Hence, the MLE of parameter p is

$$\hat{p}=\frac{\sum_{i=1}^{m-1}r_i}{\sum_{i=1}^{m-1}r_i+\left(m-1\right)\left(n-m\right)-\sum_{i=1}^{m-1}\left(m-i\right)r_i}$$

2.2. Interval Estimation

We obtain approximate confidence intervals of the parameters α and β based on the asymptotic distribution of the maximum likelihood estimator of the parameters. Hence, we employ the asymptotic normal approximation to obtain confidence intervals for the unknown parameters. We now obtain the asymptotic Fisher's information matrix. The observed information matrix of $\boldsymbol{\theta}=\left(\alpha,\beta,p\right)$, denoted by $\boldsymbol{I}=\left[I_{ij}\right]_{i,j=1,2,3}$ is

$$\boldsymbol{I}=-\begin{bmatrix}\frac{\partial^2\ln L}{\partial\alpha^2} & \frac{\partial^2\ln L}{\partial\alpha\partial\beta} & \frac{\partial^2\ln L}{\partial\alpha\partial p}\\ \frac{\partial^2\ln L}{\partial\beta\partial\alpha} & \frac{\partial^2\ln L}{\partial\beta^2} & \frac{\partial^2\ln L}{\partial\beta\partial p}\\ \frac{\partial^2\ln L}{\partial p\partial} & \frac{\partial^2\ln L}{\partial p\partial\beta} & \frac{\partial^2\ln L}{\partial p^2}\end{bmatrix}=\begin{bmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{bmatrix}$$

where

$$I_{11}=\frac{m}{\alpha^2},I_{13}=I_{31}=I_{23}=I_{32}=0$$

$$I_{12}=I_{21}=\sum_{i=1}^{m}\left(\frac{x_i}{1+\beta x_i}\left(1+r\right)_i\right)$$

$$I_{22}=\frac{m}{\beta^2}-\sum_{i=1}^{m}\left(\frac{x_i^{\,2}}{\left(1+\beta x_i\right)^2}\left(\alpha\left(1+r_i\right)+1\right)\right)$$

$$I_{33}=\frac{\sum_{i=1}^{m-1}r_i}{p^2}-\frac{\left(m-1\right)\left(n-m\right)-\sum_{i=1}^{m-1}\left(m-i\right)r_i}{\left(1-p\right)^2}$$

The variance-covariance matrix may be approximated as

$$\boldsymbol{V}=\begin{bmatrix}V_{11} & V_{12} & 0\\ V_{21} & V_{22} & 0\\ 0 & 0 & V_{33}\end{bmatrix}=\begin{bmatrix}I_{11} & I_{12} & 0\\ I_{21} & I_{22} & 0\\ 0 & 0 & I_{33}\end{bmatrix}^{-1}$$

The asymptotic distribution of the maximum likelihood can be written as follows see e.g. Miller [13].

$$\left[\left(\hat{\alpha}-\alpha\right),\left(\hat{\beta}-\beta\right),\left(\hat{p}-p\right)\right]\sim N_3\left(0,V\right)\tag{14}$$

Since V involves the parameters α, β and p, we replace the parameters by the corresponding MLEs in order to obtain an estimate of V, which is denoted by $\hat{V}$. By using (14), approximate $100\left(1-\vartheta\right)\%$ confidence intervals for α, β and p are determined, respectively, as

$$\hat{\alpha}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{11}},\hat{\beta}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{22}},\hat{p}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{33}}$$

where Z_{ϑ} is the upper 100ϑ-th percentile of the standard normal distribution.

3. Numerical Study

In our study, we firstly generate the numbers of progresssive censoring with binomial removals, $r_i, i=1,2,\cdots,m$, and then we get the progressive censoring with binomial removals samples from GP distribution by the Monte-Carlo method. The steps are:

1) **Step 1** Generate a group value

$$r_i\sim Bin\left(n-m-\sum_{j=1}^{i-1}r_j\right)\text{ and }r_m=n-m-\sum_{i=1}^{m-1}r_i$$

2) **Step 2** Generate m independent $U\left(0,1\right)$ random variables $W_1,W_2,\cdots,W_m$.

3) **Step 3** For given values of the progressive censoring scheme r_i, set $V_i=W_i^{1/\left(i+r_m+\cdots+r_{m-i+1}\right)}$.

4) **Step 4** Set $U_i=1-V_mV_{m-1}\cdots V_{m-i+1}$, then $U_1,U_2,\cdots,U_m$ are progressive censoring with binomial removals samples of size m from $U\left(0,1\right)$.

5) **Step 5** Finally, for given values of parameters α and β, we set $x_i=F-1\left(U_i\right)=\left[\left(1-U_i\right)-1/\alpha-1\right]/\beta$. Then, $\left(x_1,x_2,\cdots,x_m\right)$ is the required from progressive censoring with binomial removals sample of size m from

the GP distribution.

Table 1 shows the numerical results and the MLE of the parameters α, β and p when the actual values of the parameters are $\alpha=\beta=1$, and $p=0.5$. Note that in the table, we use for example 0*3 to abbreviate (0, 0, 0).

4. Conclusion

Here we have derived the maximum likelihood estimators and their respective variance-covariance matrix for the generalized Pareto distribution based on progressive type-II censoring with random removals. The asymptotic distribution of the maximum likelihood estimator can be used to construct confidence intervals for the involved parameters.

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Table 1. The maximum likelihood estimator of *α*, *β* and *p*.

n	m	r	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$
15	3	6, 4, 2	0.5556	1.1424	1.0139
	5	5, 4, 0, 1, 0	0.5882	1.6692	1.0768
	7	4, 2*2, 0*4	0.5714	0.8226	1.0682
	9	4, 1*2, 0*6	0.6667	0.9096	1.1834
	11	1, 3, 0*9	0.5714	1.3048	1.0335
	13	0, 2, 0*11	0.5001	1.3379	1.0670
20	9	5, 3, 2, 0, 1, 0*4	0.5000	0.8163	1.0202
	11	5, 0, 3, 0, 1, 0*6	0.4737	1.0506	1.0682
	13	5, 0*3, 2, 0*8	0.4667	0.7410	1.0834
	15	3, 1, 0, 1, 0*11	0.5556	0.9885	1.0701
	17	2, 1, 0*15	0.7500	0.8736	1.0130
	19	0, 1, 0*17	0.5020	0.8891	0.9479
30	19	6, 2, 1, 2, 0*15	0.5238	0.7631	1.2202
	21	4, 3, 1*2, 0*17	0.5294	0.9390	1.1281
	23	3*2, 0*4, 1, 0*16	0.4375	0.8137	0.9466
	25	3, 2, 0*23	0.7143	1.0339	1.0394
	27	2, 1, 0*25	0.7500	0.8906	1.0044
	29	0, 1, 0*27	0.5000	0.9166	1.0962

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