

Common Fixed Point Theorems for Weakly Compatible Mappings in Fuzzy Metric Spaces Using (JCLR) Property^{*}

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ABSTRACT

In this paper, we prove a common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space using the joint common limit in the range property of mappings called (JCLR) *property*. An example is also furnished which demonstrates the validity of main result. We also extend our main result to two finite families of self mappings. Our results improve and generalize results of Cho *et al.* [Y. J. Cho, S. Sedghi and N. Shobe, "Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces," Chaos, Solitons & Fractals, Vol. 39, No. 5, 2009, pp. 2233-2244.] and several known results existing in the literature.

Keywords: Fuzzy Metric Space; Weakly Compatible Mappings; (E.A) Property; (CLR) Property; (JCLR) Property

1. Introduction

In 1965, Zadeh [1] investigated the concept of a fuzzy set in his seminal paper. In the last two decades there has been a tremendous development and growth in fuzzy mathematics. The concept of fuzzy metric space was introduced by Kramosil and Michalek [2] in 1975, which opened an avenue for further development of analysis in such spaces. Further, George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [2] with a view to obtain a Hausdoroff topology which has very important applications in quantum particle physics, particularly in connection with both string and ε^{∞} theory (see, [4] and references mentioned therein). Fuzzy set theory also has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc.

In 2002, Aamri and El-Moutawakil [5] defined the notion of (E.A) property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that (E.A) property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the complexness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Subsequently, there are a number of results proved for contraction mappings satisfying (E.A) property in fuzzy metric spaces (see [6-11]). Most recently, Sintunavarat and Kumam [12] defined the notion of "common limit in the range" property (or (CLR) property) in fuzzy metric spaces and improved the results of Mihet [10]. In [12], it is observed that the notion of (CLR) property never requires the condition of the closedness of the subspace while (E.A) property requires this condition for the existence of the fixed point (also see [13]). Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions. For details, we refer to [14-25].

The aim of this paper is to introduce the notion of the joint common limit in the range of mappings property called (JCLR) property and prove a common fixed point theorem for a pair of weakly compatible mappings using (JCLR) property in fuzzy metric space. As an application to our main result, we present a common fixed point theorem for two finite families of self mappings in fuzzy metric space using the notion of pairwise commuting due to

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Imdad *et al.* [15]. Our results improve and generalize the results of Cho *et al.* [26], Abbas *et al.* [7] and Kumar [8].

2. Preliminaries

Definition 2.1 [27] A binary operation

*: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous *t*-norm if it satisfies the following conditions:

1) * is associative and commutative,

2) * is continuous,

3) a * 1 = a for all $a \in [0,1]$,

4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of continuous t-norms are a * b = ab and $a * b = \min \{a, b\}$.

Definition 2.2 [3] A 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous *t*-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: For all $x, y, z \in X$, t, s > 0,

1) M(x, y, t) > 0,

2) M(x, y, t) = 1 if and only if x = y,

3) M(x, y, t) = M(y, x, t),

4)
$$M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$$

5) $M(x, y, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.

Then *M* is called a fuzzy metric on *X* and M(x, y, t) denotes the degree of nearness between *x* and *y* with respect to *t*.

Let (X, M, *) be a fuzzy metric space. For t > 0, the open ball $\mathcal{B}(x, r, t)$ with center $x \in X$ and radius 0 < r < 1 is defined by

$$\mathcal{B}(x,r,t) = \{ y \in X : M(x,y,t) > 1-r \}.$$

Now let (X, M, *) be a fuzzy metric space and τ the set of all $A \subset X$ with $x \in A$ if and only if there exist t > 0 and 0 < r < 1 such that $\mathcal{B}(x, r, t) \subset A$. Then τ is a topology on X induced by the fuzzy metric M.

In the following example (see [3]), we know that every metric induces a fuzzy metric:

Example 2.1 Let (X,d) be a metric space. Denote a * b = ab (or $a * b = \min\{a,b\}$) for all $a, b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.

Definition 2.3 Let (X, M, *) be a fuzzy metric space. *M* is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converge to a point $(x, y, t) \in X^2 \times (0, \infty)$, *i.e.*,

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_y, x, t) = 1$$

and

$$\lim M(x, y, t_n) = M(x, y, t).$$

Lemma 2.1 [28] Let (X, M, *) be a fuzzy metric space. Then M(x, y, t) is non-decreasing for all $x, y \in X$.

Lemma 2.2 [29] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0,1)$ such that

$$M(x, y, kt) \ge M(x, y, t),$$

for all $x, y \in X$ and t > 0, then x = y.

Definition 2.4 [30] Two self mappings f and g of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, *i.e.* if fz = gz some $z \in X$, then fgz = gfz.

Remark 2.1 [30] Two compatible self mappings are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.

Definition 2.5 [7] A pair of self mappings f and g of a fuzzy metric space (X, M, *) are said to satisfy the (E.A) property, if there exists a sequence $\{x_n\}$ in X for some $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Remark 2.2 It is noted that weak compatibility and (E.A) property are independent to each other (see [31], *Example* 2.1, *Example* 2.2).

In 2011, Sintunavarat and Kumam [12] defined the notion of "common limit in the range" property in fuzzy metric space as follows:

Definition 2.6 A pair (f,g) of self mappings of a fuzzy metric space (X, M, *) is said to satisfy the "common limit in the range of g" property (shortly, (CLRg) property) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$.

Now, we show examples of self mappings f and g which are satisfying the (CLRg) property.

Example 2.2 Let (X, M, *) be a fuzzy metric space with $X = [0, \infty)$ and

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Define self mappings f and g on X by

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f(x) = 2x+3 and g(x) = 5x for all $x \in X$. Let a sequence $\{x_n\} = \left\{1+\frac{1}{n}\right\}_{n \in \mathbb{N}}$ in X, we have

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=5=g(1)\in X,$$

which shows that f and g satisfy the (CLRg) property.

Example 2.3 The conclusion of Example 2.2 remains true if the self mappings f and g is defined on X by $f(x) = \frac{x}{6}$ and $g(x) = \frac{2x}{5}$ for all $x \in X$. Let a sequence $\{x_n\} = \left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$ in X. Since

$$\lim fx_n = \lim gx_n = 0 = g(0) \in X,$$

therefore f and g satisfy the (CLRg) property.

The following definition is on the lines due to Imdad *et al.* [32].

Definition 2.7 [32] Two families of self mappings $\{f_i\}_{i=1}^m$ and $\{g_k\}_{k=1}^n$ are said to be pairwise commuting if

1) $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \dots, m\}$, 2) $g_k g_l = g_l g_k$ for all $k, l \in \{1, 2, \dots, m\}$, 3) $f_i g_k = g_k f_i$ for all $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, m\}$.

Throughout this paper, (X, M, *) is considered to be a fuzzy metric space with condition

 $\lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X.$

3. Main Results

In this section, we first introduce the notion of "the joint common limit in the range property" of two pairs of self mappings.

Definition 3.1 Let (X, M, *) be a fuzzy metric space and $f, g, a, b: X \to X$. The pair (f, b) and (a, g) are said to satisfy the "joint common limit in the range of *b* and *g*" property (shortly, (JCLRbg) property) if there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} bx_n = \lim_{n \to \infty} ay_n = \lim_{n \to \infty} gy_n = bu = gu, \quad (1)$$

for some $u \in X$.

Remark 3.1 If a = f, b = g and $\{x_n\} = \{y_n\}$ in (1), then we get the definition of (CLRg).

Throughout this section, Φ denotes the set of all continuous and increasing functions $\phi:[0,1]^5 \rightarrow [0,1]$ in any coordinate and $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$.

Following are examples of some function $\phi \in \Phi$:

1)
$$\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{x_i\})^n$$
 for some $0 < h < 1$.

- 2) $\phi(x_1, x_2, x_3, x_4, x_5) = x_1^h$ for some 0 < h < 1.
- 3) $\phi(x_1, x_2, x_3, x_4, x_5) = (x_1 * x_2 * x_3 * x_4 * x_5)^h$ for some

0 < h < 1 and for all *t*-norm * such that t * t = t.

Now, we state and prove main results in this paper.

Theorem 3.1 Let (X, M, *) be a fuzzy metric space, where * is a continuous t-norm and f, g, a and b be mappings from X into itself. Further, let the pair (f,b)and (a,g) are weakly compatible and there exists a

constant
$$k \in \left(0, \frac{1}{2}\right)$$
 such that
 $M(fx, ay, kt) \ge \phi(M(bx, gy, t), M(fx, bx, t),$
 $M(ay, gy, t), M(fx, gy, \alpha t),$ (2)
 $M(ay, bx, 2t - \alpha t))$

holds for all $x, y \in X$, $\alpha \in (0,2)$, t > 0 and $\phi \in \Phi$. If (f,b) and (a,g) satisfy the (JCLRbg) property, then f, g, a and b have a unique common fixed point in X.

Proof. Since the pairs (f,b) and (a,g) satisfy the (JCLRbg) property, there exists a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} bx_n = \lim_{n\to\infty} ay_n = \lim_{n\to\infty} gy_n = bu = gu,$$

for some $u \in X$.

Now we assert that gu = au. Using (2), with $x = x_n$, y = u, for $\alpha = 1$, we get

$$M(fx_n, au, kt) \ge \phi(M(bx_n, gu, t), M(fx_n, bx_n, t))$$
$$M(au, gu, t), M(fx_n, gu, t),$$
$$M(au, bx_n, t)).$$

Taking the limit as $n \to \infty$, we have

$$M(gu, au, kt) \ge \phi(M(gu, gu, t), M(gu, gu, t),$$
$$M(au, gu, t), M(gu, gu, t),$$
$$M(au, gu, t)).$$

Since ϕ is increasing in each of its coordinate and $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$, we get

 $M(gu, au, kt) \ge M(gu, au, t)$. By Lemma 2.2, we have gu = au.

Next we show that fu = gu. Using (2), with x = u, $y = y_n$, for $\alpha = 1$, we get

$$M(fu,ay_n,kt) \ge \phi(M(bu,gy_n,t),M(fu,bu,t),$$
$$M(ay_n,gy_n,t),M(fu,gy_n,t),$$
$$M(ay_n,bu,t)).$$

Taking the limit as $n \to \infty$, we have

$$M(fu,gu,kt) \ge \phi(M(gu,gu,t),M(fu,gu,t),$$
$$M(gu,gu,t),M(fu,gu,t),$$
$$M(gu,gu,t),M(fu,gu,t),$$

Since ϕ is increasing in each of its coordinate and

 $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$, we get

 $M(fu, gu, kt) \ge M(fu, gu, t)$. By Lemma 2.2, we have fu = gu.

Now, we assume that z = fu = gu = au = bu. Since the pair (f,b) is weakly compatible, fbu = bfu and then fz = fbu = bfu = bz. It follows from (a,g) is weakly compatible, gau = agu and hence gz = gau = agu = az.

We show that z = fz. To prove this, using (2) with x = z, y = u, for $\alpha = 1$, we get

$$M(fz,au,kt) \ge \phi(M(bz,gu,t),M(fz,bz,t),$$
$$M(au,gu,t),M(fz,gu,t),$$
$$M(au,bz,t))$$

and so

$$M(fz, z, kt) \ge \phi(M(fz, z, t), M(fz, fz, t)),$$
$$M(z, z, t), M(fz, z, t),$$
$$M(z, fz, t).$$

Since ϕ is increasing in each of its coordinate and $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$,

M(fz, z, kt) > M(fz, z, t), which implies that fz = z. Hence z = fz = bz.

Next, we show that z = az. To prove this, using (2) with x = u, y = z, for $\alpha = 1$, we get

$$M(fu,az,kt) \ge \phi(M(bu,gz,t),M(fu,bu,t))$$
$$M(az,gz,t),M(fu,gz,t),$$
$$M(az,bu,t))$$

and so

$$M(z,az,kt) \ge \phi(M(z,az,t),M(z,z,t),$$
$$M(az,az,t),M(z,az,t),$$
$$M(az,z,t)).$$

Since ϕ is increasing in each of its coordinate and $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$,

M(z, az, kt) > M(az, az, t), which implies that z = az. Hence z = az = gz. Therefore, we conclude that z = fz = gz = az = bz this implies f, g, a and b have

common fixed point that is a point z. For uniqueness of common fixed point, we let w be another common fixed point of the mappings f, g, a and b.

On using (2) with
$$x = z$$
, $y = w$, for $\alpha = 1$, we have

$$M(f_z a_w k_t) \ge \phi(M(b_z a_w t) M(f_z b_z t))$$

$$M(Jz, aw, kt) \ge \phi(M(bz, gw, t), M(Jz, bz, t)),$$
$$M(aw, gw, t), M(fz, gw, t),$$
$$M(aw, bz, t))$$

and then

$$M(z, w, kt) \ge \phi(M(z, w, t), M(z, z, t), M(w, w, t),$$
$$M(z, w, t), M(w, z, t)).$$

Since ϕ is increasing in each of its coordinate and $\phi(t,t,t,t,t) > t$ for all $t \in [0,1)$,

M(z, w, kt) > M(z, w, t), which implies that z = w. Therefore *f*, *g*, *a* and *b* have a unique a common fixed point.

Remark 3.2 From the result, it is asserted that (JCLRgb) property never requires any condition closedness of the subspace, continuity of one or more mappings and containment of ranges amongst involved mappings.

Remark 3.3 Theorem 3.1 improves and generalizes the results of Abbas *et al.* ([7], Theorem 2.1) and Kumar ([8], Theorem 2.3) without any requirement of containment amongst range sets of the involved mappings and closedness of the underlying subspace.

Remark 3.4 Since the condition of *t*-norm with t*t=t for all $t \in [0,1]$ is replaced by arbitrary continuous *t*-norm, Theorem 3.1 also improves the result of Cho *et al.* ([26], Theorem 3.1) without any requirement of completeness of the whole space, continuity of one or more mappings and containment of ranges amongst involved mappings.

Corollary 3.1 Let (X, M, *) be a fuzzy metric space, where * is a continuous t-norm and f, g, a and b be mappings from X into itself. Further, let the pair (f, b)and (a, g) are weakly compatible and there exists a

constant
$$k \in \left(0, \frac{1}{2}\right)$$
 such that

$$M\left(fx, ay, kt\right)$$

$$\geq \left[a_{1}(t)M\left(bx, gy, t\right) + a_{2}(t)M\left(fx, bx, t\right) + a_{3}(t)M\left(ay, gy, t\right) + a_{4}(t)M\left(fx, gy, \alpha t\right) + a_{5}(t)M\left(ay, bx, 2t - \alpha t\right)\right]^{1/2}$$
(3)

holds for all $x, y \in X$, $\alpha \in (0,2)$, t > 0 and $a_i : \mathbb{R}^+ \to (0,1]$ such that $\sum_{i=1}^5 a_i(t) = 1$. If (f,b) and (a,g) satisfy the (JCLRbg) property, then f, g, a and b have a unique common fixed point in X.

Proof. By Theorem 3.1, if we define

 $\begin{pmatrix} 1 \end{pmatrix}$

$$\phi(x_1, x_2, x_3, x_4, x_5)$$

= $(a_1(t)x_1 + a_2(t)x_2 + a_3(t)x_3 + a_4(t)x_4 + a_5(t)x_5)^{\frac{1}{2}},$

then the result follows.

Remark 3.5 Corollary 3.1 improves the result of Cho *et al.* ([26], Corollary 3.4) without any requirement of completeness of the whole space, continuity of one or more mappings and containment of ranges amongst involved mappings while the condition of *t*-norm t * t = t for all $t \in [0,1]$ is replaced by arbitrary continuous *t*-norm.

Corollary 3.2 Let (X, M, *) be a fuzzy metric space, where * is a continuous t-norm and f and g be mappings from X into itself. Further, let the pair (f,g) is weakly compatible and there exists a constant $k \in \left(0, \frac{1}{2}\right)$ such that

$$M(fx, fy, kt) \ge \phi(M(gx, gy, t), M(fx, gx, t),$$
$$M(fy, gy, t), M(fx, gy, \alpha t), \qquad (4)$$
$$M(fy, gx, 2t - \alpha t))$$

holds for all $x, y \in X$, $\alpha \in (0,2)$, t > 0 and $\phi \in \Phi$. If (f,g) satisfies the (CLRg) property, then *f* and *g* have a unique common fixed point in *X*.

Proof. Take a = f and b = g in Theorem 3.1, then we get the result.

Our next theorem is proved for a pair of weakly compatible mappings in fuzzy metric space (X, M, *) using (E.A) property under additional condition closedness of the subspace.

Theorem 3.2 Let (X, M, *) be a fuzzy metric space, where * is a continuous t-norm. Further, let the pair (f,g) of self mappings is weakly compatible satisfying inequality (4) of Corollary 3.2. If f and g satisfy the (E.A) property and the range of g is a closed subspace of X, then f and g have a unique common fixed point in X.

Proof. Since the pair (f,g) satisfies the (E.A) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=z,$$

for some $z \in X$. It follows from g(X) being a closed subspace of *X* that there exists $u \in X$ in which z = gu. Therefore *f* and *g* satisfy the (CLRg) property. From Corollary 3.2, the result follows.

In what follows, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results.

Example 3.1 Let X = [2,19) with the metric d defined by d(x, y) = |x - y| and for each $t \in [0,1]$ define

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly (X, M, *) be a fuzzy metric space with t-norm defined by $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Consider a function $\phi : [0,1]^5 \rightarrow [0,1]$ defined by $\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{x_i\})^{\frac{1}{2}}$. Then we have $M(fx, fy, t) \ge \phi(x_1, x_2, x_3, x_4, x_5)$. Define the self mappings f and g on X by

$$fx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (3,19); \\ 15 & \text{if } x \in (2,3], \end{cases}$$

and

$$gx = \begin{cases} 2 & \text{if } x = 2; \\ 12 & \text{if } x \in (2,3]; \\ \frac{x+1}{2} & \text{if } x \in (3,19) \end{cases}$$

ſ

Taking $\{x_n\} = \left\{3 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$ or $\{x_n\} = \{2\}$, it is clear

that the pair (f,g) satisfies the (CLRg) property since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2 = g(2) \in X.$$

It is noted that $f(X) = \{2,15\} \not\subseteq [2,10) \cup \{12\} = g(X)$. Thus, all the conditions of Corollary 3.2 are satisfied for a fixed constant $k \in \left(0, \frac{1}{2}\right)$ and 2 is a unique common fixed point of the pair (f,g). Also, all the involved mappings are even discontinuous at their unique common fixed point 2. Here, it may be pointed out that g(X) is not a closed subspace of *X*.

Example 3.2 In the setting of Example 3.1, replace the mapping *g* by the following, besides retaining the rest:

$$gx = \begin{cases} 2 & \text{if } x = 2; \\ 10 & \text{if } x \in (2,3]; \\ \frac{x+1}{2} & \text{if } x \in (3,19), \end{cases}$$

Taking $\{x_n\} = \left\{3 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$ or $\{x_n\} = \{2\}$, it is clear

that the pair (f,g) satisfies the (E.A) property since

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = 2 \in X.$$

It is noted that $f(X) = \{2,15\} \not\subseteq [2,10] = g(X)$. Thus, all the conditions of Theorem 3.2 are satisfied and 2 is a unique common fixed point of the mappings f and g. Notice that all the involved mappings are even discontinuous at their unique common fixed point 2. Here, it is worth noting that g(X) is a closed subspace of X.

Now, we utilize Definition 2.7 which is a natural extension of commutativity condition to two finite families of self mappings. Our next theorem extends Corollary 3.2 in the following sense:

Theorem 3.3 Let $\{f_i\}_{i=1}^m$ and $\{g_j\}_{j=1}^n$ be two finite families of self mappings in fuzzy metric space (X, M, *), where * is a continuous t-norm such that

 $f = f_1 f_2 \cdots f_m$ and $g = g_1 g_2 \cdots g_n$ which satisfy the inequalities (4) of Corollary 3.2. If the pair (f,g) shares (CLRg) property, then f and g have a unique point of coincidence.

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Moreover, $\{f_i\}_{i=1}^m$ and $\{g_j\}_{j=1}^n$ have a unique common

fixed point provided the pair of families $(\{f_i\}, \{g_j\})$

commutes pairwise, where $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

Proof. The proof of this theorem can be completed on the lines of Theorem 3.1 contained in Imdad *et al.* [15], hence details are avoided.

Putting $f_1 = f_2 = \dots = f_m = f$ and $g_1 = g_2 = \dots = g_n = g$ in Theorem 3.3, we get the following result:

Corollary 3.3 Let f and g be two self mappings of a fuzzy metric space (X, M, *), where * is a continuous t-norm. Further, let the pair (f^m, g^n) shares (CLRg) property. Then there exists a constant $k \in \left(0, \frac{1}{2}\right)$ such

that

$$M(f^{m}x, f^{m}y, kt) \ge \phi(M(g^{n}x, g^{n}y, t), M(f^{m}x, g^{n}x, t),$$
$$M(f^{m}y, g^{n}y, t), M(f^{m}x, g^{n}y, \alpha t),$$
$$M(f^{m}y, g^{n}x, 2t - \alpha t)$$

holds for all $x, y \in X$, $\alpha \in (0,2)$, t > 0, $\phi \in \Phi$ and m and n are fixed positive integers, then f and g have a unique common fixed point provided the pair (f^m, g^n) commutes pairwise.

Remark 3.6 Theorem 3.2, Theorem 3.3 and Corollary 3.3 can also be outlined in respect of Corollary 3.1.

Remark 3.7 Using Example 2.2, we can obtain several fixed point theorems in fuzzy metric spaces in respect of Theorems 3.2 and 3.3 and Corollaries 3.2, 3.1 and 3.3.

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