Advances in Pure Mathematics, 2012, 2, 291-295
http://dx.doi.org/10.4236/apm.2012.24038 Published Online July 2012 (http://www.SciRP.org/journal/apm)

A Certain Subclass of Analytic Functions

Young Jae Sim, Oh Sang Kwon*
Department of Mathematics, Kyungsung University, Busan, Korea (South)
Email: {yjsim, *oskwon}@ks.ac.kr

Received February 20, 2012; revised April 20, 2012; accepted April 28, 2012

ABSTRACT

In the present paper, we introduce a class of analytic functions in the open unit disc by using the analytic function $q_\alpha(z) = 3\big/\left(3+(\alpha-3)z-\alpha z^2\right)$, which was investigated by Sokół [1]. We find some properties including the growth theorem or the coefficient problem of this class and we find some relation with this new class and the class of convex functions.

Keywords: Univalent Functions; Convex Functions; Subordination; Order of Convexity

1. Introduction

Let H denote the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let A denote the subclass of H consisting of functions normalized by $f(0) = 0$ and $f'(0) = 1$. The set of all functions $f \in A$ that are convex univalent in $\mathbb{U}$ by K. Recall that a set E is said to be convex if and only if the linear segment joining any two points of E lies entirely in E. Let the function f be analytic univalent in the unit disc $\mathbb{U}$ on the complex plane $\mathbb{C}$ with the normalization. Then f maps $\mathbb{U}$ onto a convex domain E if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0 \ (z \in U).$$

Robertson introduced in [2], the class $K(\alpha)$ of convex functions of order $\alpha\,(\alpha \le 1)$, which is defined by

$$K(\alpha) = \left\{ f \in A : \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{U}\right\}.$$

If $\alpha \in [0,1)$, then a function of this set is univalent and if $\alpha < 0$ it may fail to be univalent. We denote $K(0) = K$. Let S be denote the subset of A which is composed of univalent functions. We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if, $f(z) = F(w(z))$ for some Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$. The class of convex functions K can be defined in several ways, for example we say that f is convex if it satisfies the condition

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}. \tag{1}$$

Many subclass of K have been defined by the condition (1) with a convex univalent function p, given arbitrary, instead of the functions $(1+z)/(1-z)$. Janowski considered the function p, which maps the unit disc onto a disc in [3,4]. An interesting case when the function p is convex but is not univalent was considered in [5]. A function p that is not univalent and is not convex and maps unit circle onto a concave set was considered in [1].

Now, we shall introduce the class of analytic functions used in the sequel.

Definition 1.1. The function $f \in A$ belongs to the class $SQ(\alpha)$, $\alpha \in (-3,1]$, if it satisfies the condition

$$\sqrt{f'(z)} \prec q_\alpha(z) = \frac{3}{3+(\alpha-3)z-\alpha z^2} \tag{2}$$

Let the function q_α be given by (2). We note that

$$q_\alpha(z) = 3\big/\left(3+(\alpha-3)z-\alpha z\right) = \frac{3}{3+\alpha}\left[\frac{1}{1-z}+\frac{\alpha}{\alpha z+3}\right]$$
$$= 1+\frac{(3-\alpha)^2}{3(3+\alpha)}z+\cdots$$

Sokół investigated in [1] that the image of the unit circle $|z| = 1$ under the function q_α is a curve described by

$$\Gamma : (x-a)\left(x^2+y^2\right)-k\left(x-\frac{1}{2}\right)^2 = 0,$$

where

$$x = \operatorname{Re}\left\{q_\alpha\left(e^{i\phi}\right)\right\} \text{ and } y = \operatorname{Im}\left\{q_\alpha\left(e^{i\phi}\right)\right\},$$

with $\phi \in (0, 2\pi)$ and $a = \dfrac{9(1+\alpha)}{2(3+\alpha)^2}$ and

*Corresponding author.

$$k = \frac{54}{(3+\alpha)^2(3-\alpha)}.$$

Thus the curve Γ is symmetric with respect to real axis and $q_\alpha\left(e^{i\phi}\right)$ satisfies

$$\frac{9(1+\alpha)}{2(3+\alpha)^2} < \mathrm{Re}\left(q_\alpha\left(e^{i\phi}\right)\right) \le \frac{3}{2(3-\alpha)}, \qquad (3)$$

where $\phi \in (0, 2\pi)$.

Especially, if $\alpha = 0$, then $q_0(z) = 1/(1-z)$, which maps $\mathbb{U}$ onto the right of line $x = 1/2$. And we note that if $-3 < \alpha_2 < \alpha_1 \le -1$, then $q_{\alpha_1} \prec q_{\alpha_2}$.

2. Some Properties of Functions in $SQ(\alpha)$

Now we shall find some properties of functions in the class $SQ(\alpha)$.

Theorem 2.1. If a function *f* belongs to the class $SQ(\alpha)$, $\alpha \in (-3,1]$, then there exists a function $g \in A$ such that

$$\sqrt{g'(z)} \prec 1/(1-z)$$

and a function $h \in A$ such that

$$\sqrt{h'(z)} \prec 3/(3+\alpha z)$$

and

$$f'(z) = g'(z)h'(z).$$

Proof. Let *f* be in $SQ(\alpha)$. Then there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that

$$\sqrt{f'(z)} = \frac{3}{(1-w(z))(3+\alpha w(z))}. \qquad (4)$$

From (4) we have

$$\frac{f''(z)}{2f'(z)} = \frac{w'(z)}{1-w(z)} - \frac{\alpha w'(z)}{3+\alpha w(z)}.$$

Define *g* and *h* so that

$$\frac{g''(z)}{2g'(z)} = \frac{w'(z)}{1-w(z)}$$

and

$$\frac{h''(z)}{2h'(z)} = -\frac{\alpha w'(z)}{3+\alpha w'(z)},$$

respectively. Then

$$\sqrt{g'(z)} \prec 1/(1-z),$$

$$\sqrt{h'(z)} \prec 3/(3+\alpha z)$$

and

$$\frac{f''(z)}{2f'(z)} = \frac{g''(z)}{2g'(z)} + \frac{h''(z)}{2h'(z)}.$$

Hence $f'(z) = g'(z)h'(z)$, which proves Theorem 2.1.

Theorem 2.2. If $f \in SQ(\alpha)$, $\alpha \in (-3,1]$ and $|z| = r$, $0 \le r < 1$, then

$$\frac{1}{(1+r)^2\left(1+(|\alpha|/3)r\right)^2} \le |f'(z)| \le \frac{1}{(1-r)^2\left(1-(|\alpha|/3)r\right)^2} \qquad (5)$$

Proof. Suppose that $f \in SQ(\alpha)$. Then

$$f'(z) = g'(z)h'(z)$$

For some *g* and *h* such that

$$\sqrt{g'(z)} \prec 1/(1-z)$$

and

$$\sqrt{h'(z)} \prec 3/(3+\alpha z),$$

respectively. And above subordination equations imply that

$$\frac{1}{(1+r)^2} \le |g'(z)| \le \frac{1}{(1-r)^2}$$

and

$$\frac{1}{\left(1+(|\alpha|/3)r\right)^2} \le |h'(z)| \le \frac{1}{\left(1-(|\alpha|/3)r\right)^2},$$

respectively. Since $f'(z) = g'(z)h'(z)$, the modulus of $f'(z)$ satisfies the inequality (5).

Next, we shall solve some coefficient problem for a special function to be in the class $SQ(\alpha)$.

Theorem 2.3. The function $g_n(z) = z + cz^n$ belongs to the class $SQ(\alpha)$, whenever

$$|c| < \frac{27 - 24\alpha + 4\alpha^2}{4n(3-\alpha)^2}.$$

Proof. Since $g_n'(z) = 1 + ncz^{n-1}$, if we put

$$G(z) = \sqrt{g_n'(z)},$$

then

$$G^2(z) - 1 = ncz^{n-1}.$$

Hence for $z \in \mathbb{U}$,

$$\left|G^2(z) - 1\right| < n|c|.$$

Since $\mathrm{Re}(G(z)) > \sqrt{1-n|c|}$, if

$$\frac{3}{2(3-\alpha)}<\sqrt{1-n|c|}, \tag{6}$$

Then $g_n(z)\in SQ(\alpha)$ and we can easily derive that the inequality (6) is equivalent to

$$|c|<\frac{27-24\alpha+4\alpha^2}{4n(3-\alpha)^2}.$$

3. The Relations of the Classes *SQ* and *K*

It is well-known that the following implication holds:

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\Rightarrow \mathrm{Re}\sqrt{f'(z)}>\frac{1}{2}. \tag{7}$$

More generally, the above implication (7) is can be generalized as following:

$$\frac{zf''(z)}{f'(z)}\prec\frac{zk''(z)}{k'(z)}\Rightarrow f'(z)\prec k'(z).$$

Evidently, the implication (7) implies the relation $K(0)\subset SQ(1/2)$. In this chapter, we find some general relation between the classes $K(\alpha)$ and $SQ(\alpha)$.

Let us denote by Q the class of functions f that are analytic and injective on $\overline{\mathbb{U}}-E(f)$, where

$$E(f)=\left\{\varsigma:\varsigma\in\partial\mathbb{U}\text{ and }\lim_{z\to\varsigma}f(z)=\infty\right\}$$

and are such that

$$f'(\varsigma)\neq 0\left(\varsigma\in\partial\mathbb{U}-E(f)\right).$$

Lemma 3.1. [6] Let $p\in Q$ with $p(0)=a$ and let

$$q(z)=a+a_nz^n+\cdots$$

Be analytic in $\mathbb{U}$ with

$$q(z)\not\equiv a\text{ and }n\in\mathbb{N}.$$

If q is not subordinate to p, then there exist points

$$z_0=r_0e^{i\theta}\in\mathbb{U}\text{ and }\varsigma\in\partial\mathbb{U}-E(f),$$

and there exists a number $m\geq n$ for which

$$q(|z|<r_0)\subset p(\mathbb{U}),\quad q(z_0)=p(\varsigma)$$

and

$$z_0q'(z_0)=m\varsigma p'(\varsigma).$$

Theorem 3.2. Let $-1<\alpha<1$. If a function f belongs to the class A and

$$f\ \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{2|\alpha|}{3-|\alpha|}\text{ for }z\in\mathbb{U},$$

then $f\in SQ(\alpha)$.

Proof. Suppose that $\alpha\neq 0$ and $f\notin SQ(\alpha)$ or equivalently,

$$\sqrt{f'(z)}\not\prec q_\alpha(z).$$

Then by Lemma 3.1, there exist $z_0\in\mathbb{U}$ and $\varsigma\in\partial\mathbb{U}$, $\varsigma\neq 1$ and $m>1$ such that

$$\sqrt{f'(z_0)}=q_\alpha(\varsigma)$$

and

$$z\left(\sqrt{f'(z)}\right)'\bigg|_{z=z_0}=m\varsigma q_\alpha'(\varsigma).$$

Since

$$\mathrm{Re}\left\{\frac{2\varsigma-(\alpha+3)/\alpha}{\varsigma-1}\right\}=1+\frac{3+\alpha}{2\alpha},$$

For $|\varsigma|=1(\varsigma\neq 1)$,

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}\\
&=\mathrm{Re}\left\{1+\frac{2m\varsigma q_\alpha'(\varsigma)}{q_\alpha(\varsigma)}\right\}\\
&=1+2m\,\mathrm{Re}\left\{-\frac{3(\alpha+3)/\alpha}{(\varsigma-1)(\alpha\varsigma+3)}-\frac{2\varsigma-3(\alpha+3)/\alpha}{\varsigma-1}\right\}\\
&=1+\frac{2m(\alpha+3)}{\alpha}\mathrm{Re}\left\{\frac{-3}{(\varsigma-1)(\alpha\varsigma+3)}\right\}\\
&\quad-2m\,\mathrm{Re}\left\{\frac{2\varsigma-3(\alpha+3)/\alpha}{\varsigma-1}\right\}\\
&=1+\frac{2m(\alpha+3)}{\alpha}\mathrm{Re}\left\{\frac{-3}{(\varsigma-1)(\alpha\varsigma+3)}\right\}\\
&\quad-2m\,\mathrm{Re}\left\{1+\frac{3+\alpha}{2\alpha}\right\}.
\end{aligned}$$

In case $-1<\alpha<0$, since the inequality (3) induces the following inequality:

$$\frac{9(1+\alpha)}{2(3+\alpha)^2}<\mathrm{Re}\left\{\frac{-3}{(\varsigma-1)(\alpha\varsigma+3)}\right\}\leq\frac{3}{2(3-\alpha)}, \tag{8}$$

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}\\
&\leq 1+\frac{2m(\alpha+3)}{\alpha}\frac{9(1+\alpha)}{2(3+\alpha)^2}-2m\left(1+\frac{3+\alpha}{2\alpha}\right)\\
&=1+2m\left(\frac{9(1+\alpha)}{2\alpha(3+\alpha)}-1-\frac{3+\alpha}{2\alpha}\right)\\
&\leq 1-\frac{3(1+\alpha)}{3+\alpha}=\frac{2|\alpha|}{3-|\alpha|},
\end{aligned}$$

which is a contradiction to the hypothesis. In case $0<\alpha<1$, using the inequality (8) again,

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} \\
&\le 1+\frac{2m(\alpha+3)}{\alpha}\frac{3}{2(3-\alpha)}-2m\left(1+\frac{3+\alpha}{2\alpha}\right) \\
&=1+2m\left(\frac{3(3+\alpha)}{2\alpha(3-\alpha)}-1-\frac{3+\alpha}{2\alpha}\right) \\
&=1+\frac{3m(\alpha-1)}{3-\alpha}\le 1+\frac{3(\alpha-1)}{3-\alpha}=\frac{2|\alpha|}{3-|\alpha|},
\end{aligned}$$

which is a contradiction to the hypothesis, hence $\sqrt{f'(z)}\prec q_\alpha(z)$, and $f\in SQ(\alpha)$.

If we put $\alpha=1/2$ in Theorem 3.2, we can get next Corollary.

Corollary 3.1. For $f\in A$, the following implication holds:

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{2}{5}\Rightarrow f\in SQ(1/2)$$
$$\Rightarrow \mathrm{Re}\left\{\sqrt{f'(z)}\right\}>\frac{27}{49}.$$

Theorem 3.3. Let $\alpha\in(-1,1)$ and let $f\in SQ(\alpha)$. Then f is convex for $|z|<1/3$, if $\alpha=0$, and

$$|z|<\frac{-C+\sqrt{C^2+36|\alpha|}}{6|\alpha|},$$

where $C=2(3-\alpha)+|\alpha|+3$, if $\alpha\neq 0$.

Proof. Let $f\in SQ(\alpha)$. Then

$$\sqrt{f'(z)}\prec\frac{3}{3+(\alpha-3)z-\alpha z^2}$$

and there exists a Schwarz function $w(z)$ with $w(0)=0$ and $|w(z)|<1$ such that

$$\sqrt{f'(z)}=\frac{3}{3+(\alpha-3)w(z)-\alpha w^2(z)}.$$

Then

$$1+\frac{zf''(z)}{f'(z)}=1+\frac{-2(\alpha-3)zw'(z)+4\alpha zw(z)w'(z)}{3+(\alpha-3)w(z)-\alpha w^2(z)}.$$

Hence

$$\left|\frac{zf''(z)}{f'(z)}\right|=|zw'(z)|\frac{|2(3-\alpha)+4\alpha w(z)|}{|1-w(z)||3+\alpha w(z)|} \tag{9}$$

Using the well-known estimate [7]:

$$|w'(z)|\le\frac{1-|w(z)|^2}{1-|z|^2},$$

We have from (9)

$$\left|\frac{zf''(z)}{f'(z)}\right|\le\frac{|z|(1+|w(z)|)(2(3-\alpha)+4|\alpha||w(z)|)}{(1-|z|^2)(3-|\alpha||w(z)|)}$$

Hence if

$$\frac{|z|(1+|w(z)|)(2(3-\alpha)+4|\alpha||w(z)|)}{(1-|z|^2)(3-|\alpha||w(z)|)}<1, \tag{10}$$

Then f is convex. So it is enough to find the condition of $|z|$ to satisfy the inequality (10). In case $\alpha=0$, then inequality (10) reduces to

$$\frac{2|z|(1+|w(z)|)}{1-|z|^2}<1, \tag{11}$$

And (11) is satisfied for $|z|<1/3$, since $|w(z)|<|z|$. Hence we can conclude that f is convex for $|z|<1/3$, in case $\alpha=0$. Now we suppose that $\alpha\neq 0$ and let $|w(z)|=R$ and $|z|=r$. And let us put

$$\begin{aligned}T(R)&=4|\alpha|rR^2+\left(2(3-\alpha)r+4|\alpha|r+|\alpha|-|\alpha|r^2\right)R\\&\quad+2(3-\alpha)r-3+3r^2.\end{aligned}$$

Now

$$T'(R)=8|\alpha|rR+2(3-\alpha)r+4|\alpha|r+|\alpha|-|\alpha|r^2=0$$

implies

$$R=R_1=-\frac{2(3-\alpha)r+4|\alpha|r+|\alpha|(1-r^2)}{8|\alpha|r}<0.$$

And $T(0)<0$ is equivalent to

$$r<\frac{-(3-\alpha)+\sqrt{\alpha^2-6\alpha+18}}{3}:=r_0.$$

That is, f need not be convex for $r\ge r_0$. And for $r<r_0$, $T(R)=0$ is equivalent to

$$R=R_2=\frac{-B+\sqrt{B^2-16|\alpha|\left(2(3-\alpha)r-3+3r^2\right)}}{8|\alpha|r}.$$

where

$$B=2(3-\alpha)r+4|\alpha|r+|\alpha|-|\alpha|r^2.$$

Put

$$\begin{aligned}P(r)&=64|\alpha|^2r^4+16B|\alpha|r^2\\&\quad+16|\alpha|\left(2(3-\alpha)r-3+3r^2\right).\end{aligned}$$

Then

$$P(0)=-48|\alpha|<0$$

and

$$P(1)=64|\alpha|^2+32|\alpha|(3-\alpha)+64|\alpha|^2+32|\alpha|(3-\alpha)>0.$$

Hence there exists a $r_1\in(0,1)$ such that $P(r_1)=0$ and for $0\le r\le r_1$, $P(r)<0$. Hence for $0\le r\le r_1$, $R_2>r$, $T(R)$ attains its maximum at $R=r$ for $0\le R\le r\le r_1$. Now

$$\begin{aligned}&T(r)<0\\&\Leftrightarrow(1+r)\left(3|\alpha|r^2+\left(2(3-\alpha)+|\alpha|+3\right)r-3\right)<0\\&\Leftrightarrow 3|\alpha|r^2+\left(2(3-\alpha)+|\alpha|+3\right)r-3<0\\&\Leftrightarrow r<\frac{-C+\sqrt{C^2+36|\alpha|}}{6|\alpha|},\end{aligned}$$

where $C=2(3-\alpha)+|\alpha|+3$, which proves Theorem 3.3.

If we put $\alpha=1/2$ in Theorem 3.3, we can get next Corollary.

Corollary 3.2. Let $f\in SQ(1/2)$. Then *f* is convex for

$$|z|<\left(-17+\sqrt{389}\right)/6=0.453847\cdots.$$

4. Acknowledgements

The research was supported by Kyungsung University Research Grants in 2012.

REFERENCES

[1] J. Sokół, "A Certain Class of Starlike Functions," *Computers and Mathematics with Applications*, Vol. 62, No. 2, 2011, pp. 611-619. doi:10.1016/j.camwa.2011.05.041

[2] M. S. Rovertson, "Certain Classes of Starlike Functions," *Michigan Mathematical Journal*, Vol. 76, No. 1, 1954, pp. 755-758.

[3] W. Janowski, "Extremal Problems for a Family of Functions with Positive Real Part and Some Related Families," *Annales Polonici Mathematici*, Vol. 23, 1970, pp. 159-177.

[4] W. Janowski, "Some Extremal Problems for Certain Families of Analytic Functions," *Annales Polonici Mathematici*, Vol. 28, 1973, pp. 297-326.

[5] R. Jurasiska and J. Stankiewics, "Coefficients in Some Classes Defined by Subordination to Multivalent Majorants," *Annales Polonici Mathematici*, Vol. 80, 2003, pp. 163-170.

[6] S. S. Miller and P. T. Mocanu, "Differential Subordinations, Theory and Applications," *Series of Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker Inc., New York, 2000.

[7] P. Duren, "Univalent functions, A Series of Comprehensive Studies in Mathematics," Vol. 259, Springer-Verlag, New York, 1983.

