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The Commutants of the Dunkl Operators on $\mathcal{E}\left(\mathbb{R}^d\right)^*$

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ABSTRACT

We consider the harmonic analysis associated with the Dunkl operators on $\mathbb{R}^d$. We study the Dunkl mean-periodic functions on the space $\mathcal{E}\left(\mathbb{R}^d\right)$ (the space of C^∞-functions). We characterize also the continuous linear mappings-from $\mathcal{E}\left(\mathbb{R}^d\right)$ into itself which commute with the Dunkl operators.

Keywords: Dunkl Operators on $\mathbb{R}^d$; C^∞-Functions on $\mathbb{R}^d$; Dunkl Intertwining Operator; Mean-Periodic Functions; Continuous Linear Mappings

1. Introduction

The Dunkl operators $\mathcal{D}_j$; $j=1,\cdots,d$, on $\mathbb{R}^d$, are differential-difference operators associated with a positive root system $\mathfrak{R}_+$ and a non negative multiplicity function *k*, introduced by Dunkl in [1]. These operators extend the usual partial derivatives and lead to a generalizations of various analytic structure, like the exponential function, the Fourier transform, the translation operators and the convolution product [2-4]. Dunkl proved in [2] that there exists a unique isomorphism V_k from the space of homogeneous polynomials P_n on $\mathbb{R}^d$ of degree *n* onto itself satisfying the transmutation relations:

$$V_k(1)=1,\quad \mathcal{D}_jV_k=V_k\frac{\partial}{\partial x_j};\quad j=1,2,\cdots,d.$$

This operator is called the Dunkl intertwining operator. It has been extended to a topological automorphism of $\mathcal{E}\left(\mathbb{R}^d\right)$ (the space of C^∞-functions on $\mathbb{R}^d$) (see [5]). The operator V_k has the integral representation (see [6]):

$$V_k(f)(x)=\int_{\mathbb{R}^d}f(y)\,\mathrm{d}\Gamma_x(y);\quad f\in\mathcal{E}\left(\mathbb{R}^d\right),\ x\in\mathbb{R}^d,\tag{1}$$

where Γ_x is a probability measure on $\mathbb{R}^d$, such that

$$supp(\Gamma_x)\subset\left\{y\in\mathbb{R}^d:\|y\|\le\|x\|\right\}.$$

The dual intertwining operator ${}^tV_k$ of V_k defined on $\mathcal{E}'\left(\mathbb{R}^d\right)$ (the dual space of $\mathcal{E}\left(\mathbb{R}^d\right)$), by

$$\left\langle {}^tV_k(T),g\right\rangle:=\left\langle T,V_k(g)\right\rangle;\quad T\in\mathcal{E}'\left(\mathbb{R}^d\right)\text{ and }g\in\mathcal{E}\left(\mathbb{R}^d\right).$$

We use the Dunkl intertwining operator V_k and its dual ${}^tV_k$ to study the harmonic analysis associated with the Dunkl operators (Dunkl translation operators, Dunkl convolution, Dunkl transform, Paley-Wiener theorem, etc.). As applications of this theory we study the mean-periodic functions on the space $\mathcal{E}\left(\mathbb{R}^d\right)$ in the Dunkl setting. We characterize also the continuous linear mappings from $\mathcal{E}\left(\mathbb{R}^d\right)$ into itself which commute with the Dunkl operators.

The contents of this paper are as follows. In the second section we recall some results about the Dunkl operators. In particular, we give some properties of the operators V_k and ${}^tV_k$. Next, we define the Dunkl translation operators τ_x, $x\in\mathbb{R}^d$ and the Dunkl convolution product $*_k$ by

$$\tau_xf(y):=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V_k^{-1}(f)(z+t)\,\mathrm{d}\Gamma_x(z)\,\mathrm{d}\Gamma_y(t),\quad f\in\mathcal{E}\left(\mathbb{R}^d\right),\ y\in\mathbb{R}^d,$$

and

$$T*_kf(x):=\left\langle T_y,\tau_xf(-y)\right\rangle,\quad T\in\mathcal{E}'\left(\mathbb{R}^d\right),\ f\in\mathcal{E}\left(\mathbb{R}^d\right).$$

In Section 3, we study the mean-periodic functions associated to the Dunkl operators on $\mathcal{E}\left(\mathbb{R}^d\right)$. We prove that every continuous linear mapping $\mathcal{X}$ from $\mathcal{E}\left(\mathbb{R}^d\right)$

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into itself such that $\mathcal{D}_j\mathcal{X}=\mathcal{X}\mathcal{D}_j$, $j=1,\cdots,d$, has the form

$$\mathcal{X}f(x)=T*_k f(x),\quad T\in\mathcal{E}'(\mathbb{R}^d).$$

In the one-dimensional case (d = 1), the Dunkl convolution operators and the Dunkl mean-periodic functions are studied in [7-9], on the space of entire functions on $\mathbb{C}$.

2. The Dunkl Harmonic Analysis on $\mathbb{R}^d$

We consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,.\rangle$ and norm $\|y\|:=\sqrt{\langle y,y\rangle}$.

For $\alpha\in\mathbb{R}^d\setminus\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha\subset\mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y:=y-\frac{2\langle\alpha,y\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $\Re\subset\mathbb{R}^d\setminus\{0\}$ is called a root system, if $\Re\cap\mathbb{R}.\alpha=\{-\alpha,\alpha\}$ and $\sigma_\alpha\Re=\Re$ for all $\alpha\in\Re$. We assume that it is normalized by $\|\alpha\|^2=2$ for all $\alpha\in\Re$.

For a root system $\Re$, the reflections σ_α, $\alpha\in\Re$ generate a finite group $G\subset O(d)$, the reflection group associated with $\Re$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta\in\mathbb{R}^d\setminus\bigcup_{\alpha\in\Re}H_\alpha$, we fix the positive subsystem:

$$\Re_+:=\{\alpha\in\Re:\langle\alpha,\beta\rangle>0\}.$$

Then for each $\alpha\in\Re$ either $\alpha\in\Re_+$ or $-\alpha\in\Re_+$.

Let $k:\Re\to\mathbb{C}$ be a multiplicity function on $\Re$ (*i.e.* a function which is constant on the orbits under the action of G). For abbreviation, we introduce the index:

$$\gamma=\gamma(k):=\sum_{\alpha\in\Re_+}k(\alpha).$$

Moreover, let w_k denotes the weight function:

$$w_k(y):=\prod_{\alpha\in\Re_+}|\langle\alpha,y\rangle|^{2k(\alpha)},\quad y\in\mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

The Dunkl operators $\mathcal{D}_j$; $j=1,\cdots,d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(y):=\frac{\partial}{\partial y_j}f(y)+\sum_{\alpha\in\Re_+}k(\alpha)\alpha_j\frac{f(y)-f(\sigma_\alpha y)}{\langle\alpha,y\rangle}.$$

For $y\in\mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x)=y_j u(x,y)$; $j=1,\cdots,d$, with $u(0,y)=1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [2,3]. This kernel has the Laplace-type representation [6]:

$$E_k(x,z)=\int_{\mathbb{R}^d}e^{\langle y,z\rangle}\mathrm{d}\Gamma_x(y);\quad x\in\mathbb{R}^d,\ z\in\mathbb{C}^d,\tag{2}$$

where $\langle y,z\rangle:=\sum_{i=1}^d y_i z_i$ and Γ_x is the measure on $\mathbb{R}^d$ given by (1).

We denote by $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$, and by $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$ of compact support.

Theorem 1. (See [5], Theorem 6.3). *The Dunkl intertwining operator V_k defined by*

$$V_k(f)(x)=\int_{\mathbb{R}^d}f(y)\mathrm{d}\Gamma_x(y);$$
$$f\in\mathcal{E}(\mathbb{R}^d),\ x\in\mathbb{R}^d,$$

is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself, and satisfies:

$$\begin{gathered}\mathcal{D}_j(V_k(f))=V_k\left(\frac{\partial}{\partial x_j}f\right);\\ f\in\mathcal{E}(\mathbb{R}^d)\text{ and } j=1,\cdots,d,\\ V_k(f)(0)=f(0).\end{gathered}\tag{3}$$

From Theorem 1, we deduce also the following results.

Theorem 2. *The dual intertwining operator ${}^tV_k$ of V_k defined on $\mathcal{E}'(\mathbb{R}^d)$ by*

$$\begin{gathered}\langle {}^tV_k(T),g\rangle:=\langle T,V_k(g)\rangle;\\ T\in\mathcal{E}'(\mathbb{R}^d)\text{ and } g\in\mathcal{E}(\mathbb{R}^d),\end{gathered}\tag{4}$$

is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself. Its inverse operator $({}^tV_k)^{-1}$ is given by

$$\begin{gathered}\left\langle({}^tV_k)^{-1}(T),g\right\rangle=\langle T,V_k^{-1}(g)\rangle;\\ T\in\mathcal{E}'(\mathbb{R}^d)\text{ and } g\in\mathcal{E}(\mathbb{R}^d).\end{gathered}\tag{5}$$

We denote by $H(\mathbb{R}^d)$ the space of entire functions on $\mathbb{C}^d$ which are rapidly increasing and of exponential type. We have

$$H(\mathbb{C}^d)=\bigcup_{a\ge0}H_a(\mathbb{C}^d),$$

where $H_a(\mathbb{C}^d)$ is the space of entire functions f on $\mathbb{C}^d$ satisfying

$$\exists N\in\mathbb{N},\quad\sup_{\lambda\in\mathbb{C}^d}(1+\|\lambda\|)^{-N}|f(\lambda)|e^{-a\|\mathrm{Im}\,\lambda\|}<\infty,$$

where

$$\|\lambda\|=\sqrt{\lambda_1^2+\cdots+\lambda_d^2},\quad\lambda=(\lambda_1,\cdots,\lambda_d)\in\mathbb{C}^d.$$

We define the Dunkl transform $\mathcal{F}_k$ on $\mathcal{E}'(\mathbb{R}^d)$ by

$$\begin{gathered}\mathcal{F}_k(T)(\lambda):=\langle T,E_k(-i\lambda,.)\rangle;\\ T\in\mathcal{E}'(\mathbb{R}^d)\text{ and }\lambda\in\mathbb{R}^d.\end{gathered}\tag{6}$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform

$\mathcal{F}$ that is given by

$$\mathcal{F}(T)(\lambda) := \left\langle T, e^{-i\langle \lambda,.\rangle} \right\rangle; \quad T \in \mathcal{E}'\left(\mathbb{R}^d\right) \text{ and } \lambda \in \mathbb{R}^d. \tag{7}$$

Proposition 1. *$\mathcal{F}_k$ admits on $\mathcal{E}'\left(\mathbb{R}^d\right)$ the following decomposition*:

$$\mathcal{F}_k(T) = \mathcal{F} \circ {}^t V_k(T), \quad T \in \mathcal{E}'\left(\mathbb{R}^d\right). \tag{8}$$

Proof. In (4), we take $g = e^{-i\langle \lambda,.\rangle}$ and applying relation (2) we obtain

$$\left\langle {}^t V_k(T), e^{-i\langle \lambda,.\rangle} \right\rangle = \left\langle T, E_k(-i\lambda,.) \right\rangle; \quad T \in \mathcal{E}'\left(\mathbb{R}^d\right).$$

Then the result follows from (6) and (7). □

Theorem 3. (Paley-Wiener theorem). *$\mathcal{F}_k$ is a topological isomorphism from $\mathcal{E}'\left(\mathbb{R}^d\right)$ onto $H\left(\mathbb{C}^d\right)$.*

Proof. The result follows from (8), Theorem 2 and Paley-Wiener theorem for the Fourier transform $\mathcal{F}$ (see [10]). □

Definition 1. The Dunkl translation operators (see [4]) are the operators τ_x, $x \in \mathbb{R}^d$, defined on $\mathcal{E}\left(\mathbb{R}^d\right)$, by

$$\tau_x f(y) := V_{k,x} V_{k,y}\left[V_k^{-1}(f)(x+y)\right], \quad y \in \mathbb{R}^d, \tag{9}$$

which can be written as:

$$\tau_x f(y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_k^{-1}(f)(z+t)\,\mathrm{d}\Gamma_x(z)\,\mathrm{d}\Gamma_y(t).$$

We next collect some properties of Dunkl translation operators (see [4]).

Proposition 2. *Let $f \in \mathcal{E}\left(\mathbb{R}^d\right)$ and $x, y \in \mathbb{R}^d$. Then*

1) $\tau_0 f = f$, $\tau_x f(y) = \tau_y f(x)$ *and* $\tau_x \circ \tau_y f = \tau_y \circ \tau_x f$.

2) $\mathcal{D}_j(\tau_x f) = \tau_x(\mathcal{D}_j f), \quad j = 1, \cdots, d$.

3) *Product formula*: $\tau_x\left[E_k(\lambda,.)\right](y) = E_k(\lambda, x) E_k(\lambda, y)$, $\lambda \in \mathbb{C}^d$.

4) *The Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, are continuous from $\mathcal{E}\left(\mathbb{R}^d\right)$ onto itself.*

The 4) of Proposition 2 used to investigate the following definition.

Definition 2. Let $T \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $f \in \mathcal{E}\left(\mathbb{R}^d\right)$. The Dunkl convolution product of T and *f*, is the function $T *_k f$ in $\mathcal{E}\left(\mathbb{R}^d\right)$ defined by

$$T *_k f(x) := \left\langle T_y, \tau_x f(-y) \right\rangle, \quad x \in \mathbb{R}^d. \tag{10}$$

We notice that $*_0$ agrees with the convolution $*$ that is given by

$$T * f(x) := \left\langle T_y, f(x-y) \right\rangle; \quad T \in \mathcal{E}'\left(\mathbb{R}^d\right),\ f \in \mathcal{E}\left(\mathbb{R}^d\right). \tag{11}$$

Theorem 4. *Let $T \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $f \in \mathcal{E}\left(\mathbb{R}^d\right)$. Then*

1) $\left({}^t V_k\right)^{-1}(T) *_k V_k(f) = V_k(T * f)$.

2) ${}^t V_k(T) * V_k^{-1}(f) = V_k^{-1}(T *_k f)$.

Proof. Let $T \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $f \in \mathcal{E}\left(\mathbb{R}^d\right)$.

1) From (10) and (5), we have

$$\begin{aligned}&\left({}^t V_k\right)^{-1}(T) *_k V_k(f)(x)\\ &= \left\langle \left({}^t V_k\right)^{-1}(T)_y, \tau_x\left[V_k(f)\right](-y) \right\rangle\\ &= \left\langle T_y, V_{k,y}^{-1} \tau_x\left[V_k(f)\right](-y) \right\rangle.\end{aligned}$$

But from (9), we obtain

$$V_{k,y}^{-1} \tau_x\left[V_k(f)\right](-y) = V_{k,x}(f)(x-y).$$

Thus

$$\begin{aligned}&\left({}^t V_k\right)^{-1}(T) *_k V_k(f)(x)\\ &= \left\langle T_y, V_{k,x}(f)(x-y) \right\rangle = V_{k,x}\left\langle T_y, f(x-y) \right\rangle\\ &= V_k(T * f)(x).\end{aligned}$$

2) From (11) and (4), we have

$$\begin{aligned}&\left({}^t V_k\right)(T) * V_k^{-1}(f)(x)\\ &= \left\langle \left({}^t V_k\right)(T)_y, V_k^{-1}(f)(x-y) \right\rangle\\ &= \left\langle T_y, V_{k,y}\left[V_k^{-1}(f)\right](x-y) \right\rangle.\end{aligned}$$

But from (9), we obtain

$$V_{k,y}\left[V_k^{-1}(f)\right](x-y) = V_{k,x}^{-1}(\tau_x f)(-y).$$

Thus

$$\begin{aligned}&\left({}^t V_k\right)(T) * V_k^{-1}(f)(x)\\ &= \left\langle T_y, V_{k,x}^{-1}(\tau_x f)(-y) \right\rangle = V_{k,x}^{-1}\left\langle T_y, \tau_x f(-y) \right\rangle\\ &= V_k^{-1}(T *_k f)(x).\end{aligned}$$

Which completes the proof of the theorem. □

Proposition 3. *Let $T \in \mathcal{E}'\left(\mathbb{R}^d\right)$. The mapping $f \to T *_k f$ is continuous from $\mathcal{E}\left(\mathbb{R}^d\right)$ onto itself.*

Proof. Assume that $\{f\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}\left(\mathbb{R}^d\right)$ such that $f_n \to f$ and $T *_k f_n \to g$, as $n \to \infty$, where *f*, *g* being in $\mathcal{E}\left(\mathbb{R}^d\right)$. According to Proposition 2 4), for every $x \in \mathbb{R}^d$, $\tau_x f_n \to \tau_x f$ as $n \to \infty$, in $\mathcal{E}\left(\mathbb{R}^d\right)$. Hence $T *_k f_n(x) \to T *_k f(x)$, as $n \to \infty$, for every $x \in \mathbb{R}^d$. By using the closed graph theorem we conclude that the mapping $f \to T *_k f$ is continuous from $\mathcal{E}\left(\mathbb{R}^d\right)$ into itself. □

The Proposition 3 used to investigate the following definition.

Definition 3. Let $T, S \in \mathcal{E}'\left(\mathbb{R}^d\right)$. The Dunkl convolution product of T and S, is the distribution $T *_k S$ in $\mathcal{E}'\left(\mathbb{R}^d\right)$ defined by

$$\langle T *_k S, f \rangle := \langle T_x, \langle S_y, \tau_x f(y) \rangle \rangle = \langle T, \tilde{S} *_k f \rangle, \quad (12)$$

where $\tilde{S}$ is the distribution in $\mathcal{E}'(\mathbb{R}^d)$ given by

$$\langle \tilde{S}, f \rangle = \langle S, \tilde{f} \rangle, \quad f \in \mathcal{E}(\mathbb{R}^d),$$

with

$$\tilde{f}(x) = f(-x), \quad x \in \mathbb{R}^d.$$

We notice that $*_0$ agrees with the convolution * that is given by

$$\langle T * S, f \rangle := \langle T_x, \langle S_y, f(x+y) \rangle \rangle; \quad T, S \in \mathcal{E}'(\mathbb{R}^d).$$

Proposition 4. *Let* $T, S \in \mathcal{E}'(\mathbb{R}^d)$. *Then*
1) $T *_k S = S *_k T$ *and* $T *_k \delta = T$.
2) $\mathcal{F}_k(T *_k S) = \mathcal{F}_k(T)\mathcal{F}_k(S)$.
3) ${}^tV_k(T *_k S) = {}^tV_k(T) * {}^tV_k(S)$.

Proof. 1) follows from (12).

2) From Proposition 3, the distribution $T *_k S$ belongs to $\mathcal{E}'(\mathbb{R}^d)$, and by (6), we have

$$\mathcal{F}_k(T *_k S)(\lambda) = \langle T *_k S, E_k(-i\lambda, x) \rangle, \quad \lambda \in \mathbb{R}^d.$$

Thus, by (7) and Proposition 2 3), we obtain

$$\begin{aligned}\mathcal{F}_k(T *_k S)(\lambda) &= \langle T_x, \langle S_y, \tau_x E_k(-i\lambda, .)(y) \rangle \rangle \\ &= \mathcal{F}_k(T)(\lambda)\mathcal{F}_k(S)(\lambda).\end{aligned}$$

3) From 2) and (8) we obtain

$$\mathcal{F}\left({}^tV_k(T *_k S)\right) = \mathcal{F}\left({}^tV_k(T) * {}^tV_k(S)\right).$$

Then we deduce the result from the injectivity of the Fourier transform $\mathcal{F}$ on $\mathcal{E}'(\mathbb{R}^d)$. □

3. Commutators and Mean-Periodic Functions

In this section, we use Theorem 4 to study the Dunkl mean-periodic functions on $\mathcal{E}(\mathbb{R}^d)$, and to give a characterization of the continuous linear mappings $\mathcal{X}$ from $\mathcal{E}(\mathbb{R}^d)$ into itself which commute with the Dunkl operators $\mathcal{D}_j$; $j = 1, \cdots, d$.

3.1. Mean-Periodic Functions

Definition 4. A function f in $\mathcal{E}(\mathbb{R}^d)$ is said mean-periodic, if there exists $T \in \mathcal{E}'(\mathbb{R}^d)$ and $T \neq 0$, such that

$$T *_k f(x) = 0, \text{ for all } x \in \mathbb{R}^d.$$

For example, let $x_0 \in \mathbb{R}^d \setminus \{0\}$. The function f in $\mathcal{E}(\mathbb{R}^d)$ satisfying

$$\tau_x f(-x_0) = 0,$$

is mean-periodic, because we have

$$\tau_x f(-x_0) = \delta_{x_0} *_k f(x),$$

δ_{x_0} being the Dirac measure at x_0.

We now characterize the Dunkl mean-periodic functions on $\mathcal{E}(\mathbb{R}^d)$.

Theorem 5. *A function f is mean-periodic function if and only if the function* $V_k^{-1}(f)$ *is a classical mean-periodic function.*

Proof. Let f be a mean-periodic function, then there exists $T \in \mathcal{E}'(\mathbb{R}^d)$ and $T \neq 0$, such that

$$T *_k f = 0.$$

Applying V_k^{-1} to this equation, then Theorem 4 2) implies that

$${}^tV_k(T) * V_k^{-1}(f) = 0.$$

From Theorem 2, ${}^tV_k(T) \neq 0$, thus $V_k^{-1}(f)$ is a classical mean-periodic function.

Conversely, if $V_k^{-1}(f)$ is a classical mean-periodic function, there exists $T \in \mathcal{E}'(\mathbb{R}^d)$ and $T \neq 0$, such that

$$T * V_k^{-1}(f) = 0.$$

Applying V_k to this equation, then Theorem 4 1) implies that

$$\left({}^tV_k\right)^{-1}(T) *_k f = 0.$$

From Theorem 2, $\left({}^tV_k\right)^{-1}(T) \neq 0$, thus f is a mean-periodic function. □

Remark 1. Let $\lambda \in \mathbb{R}^d$ and $\nu \in \mathbb{N}^d$. From [11] the functions

$$F_{\lambda,\nu}(x) = i^{|\nu|} x^\nu e^{i\langle \lambda, x \rangle}, \quad x \in \mathbb{R}^d,$$

are classical mean-periodic functions. Then from Theorem 5, the functions

$$E_{k,\lambda,\nu}(x) = i^{|\nu|} V_k(F_{\lambda,\nu})(x) = D_\lambda^\nu \left[E_k(i\lambda, x)\right], \quad x \in \mathbb{R}^d,$$

are mean-periodic functions.

3.2. Commutator of Dunkl Operators

In this section, we give a characterization of the contenuous linear mappings $\mathcal{X}$ from $\mathcal{E}(\mathbb{R}^d)$ into itself which commute with the Dunkl operators $\mathcal{D}_j$; $j = 1, \cdots, d$.

Lemma 1. *Let* $\mathcal{A}$ *be a continuous linear mapping from* $\mathcal{E}(\mathbb{R}^d)$ *into itself, such that* $\frac{\partial}{\partial x_j}\mathcal{A} = \mathcal{A}\frac{\partial}{\partial x_j}$, $j = 1, \cdots, d$, *on* $\mathcal{E}(\mathbb{R}^d)$, *then* $\mathcal{A}$ *has the form*

$$\mathcal{A}f(x) = T_0 * f(x), \quad T_0 \in \mathcal{E}'(\mathbb{R}^d).$$

Proof. For a fixed $x \in \mathbb{R}$, the map $f \to \mathcal{A}f(x)$ is a continuous form on $\mathcal{E}(\mathbb{R}^d)$. So there exists $T_x \in \mathcal{E}'(\mathbb{R}^d)$, such that

$$\mathcal{A}f(x) = \langle T_x, f \rangle, \quad x \in \mathbb{R}^d.$$

Using the fact $\frac{\partial}{\partial x_j}\mathcal{A} = \mathcal{A}\frac{\partial}{\partial x_j}$, $j = 1,\cdots,d$, on $\mathcal{E}(\mathbb{R}^d)$, we deduce

$$\frac{\partial}{\partial x_j}\mathcal{F}(T_x)(\lambda) = -i\lambda_j\mathcal{F}(T_x)(\lambda), \quad j = 1,\cdots,d.$$

Then

$$\mathcal{F}(T_x)(\lambda) = e^{-i\langle\lambda,x\rangle}\mathcal{F}(T_0)(\lambda) = \mathcal{F}(\delta_x)(\lambda)\mathcal{F}(T_0)(\lambda).$$

Thus,

$$T_x = \delta_x * T_0,$$

and

$$\mathcal{A}f(x) = \langle \delta_x * T_0, f\rangle = \langle \delta_x, \langle T_0, f(t-y)\rangle\rangle = \langle T_0, f(x-y)\rangle = T_0 * f(x).$$

□

Lemma 2. *Every continuous linear mapping* $\mathcal{B}$ *from* $\mathcal{E}(\mathbb{R}^d)$ *into itself, such that* $\mathcal{D}_j\mathcal{B} = \mathcal{B}\frac{\partial}{\partial x_j}$, $j = 1,\cdots,d$, has the form

$$\mathcal{B}f(x) = T *_k V_k(f)(x), \quad T \in \mathcal{E}'(\mathbb{R}^d).$$

Proof. Applying V_k^{-1} to the relation $\mathcal{D}_j\mathcal{B} = \mathcal{B}\frac{\partial}{\partial x_j}$, $j = 1,\cdots,d$, and using the fact that $V_k^{-1}\mathcal{D}_j = \frac{\partial}{\partial x_j}V_k^{-1}$, $j = 1,\cdots,d$, we obtain the deduce

$$\frac{\partial}{\partial x_j}V_k^{-1}\mathcal{B} = V_k^{-1}\mathcal{B}\frac{\partial}{\partial x_j}, \quad j = 1,\cdots,d.$$

By applying Lemma 1, we deduce that $V_k^{-1}\mathcal{B} = \mathcal{A}$, and Theorem 4 1) yields

$$\mathcal{B}f(x) = V_k\mathcal{A}f(x) = V_k(T_0 * f)(x) = \left({}^tV_k\right)^{-1}(T_0) *_k V_k(f)(x) = T *_k V_k(f)(x),$$

where $T = \left({}^tV_k\right)^{-1}(T_0)$. □

We now establish the main result of this paragraph.

Theorem 6. *Every the continuous linear mapping* $\mathcal{X}$ *from* $\mathcal{E}(\mathbb{R}^d)$ *into itself, such that* $\mathcal{D}_j\mathcal{X} = \mathcal{X}\mathcal{D}_j$, $j = 1,\cdots,d$, has the form

$$\mathcal{X}f(x) = T *_k f(x), \quad T \in \mathcal{E}'(\mathbb{R}^d).$$

Proof. Using the relation $\mathcal{D}_jV_k = V_k\frac{\partial}{\partial x_j}$, $j = 1,\cdots,d$, and the fact that $\mathcal{D}_j\mathcal{X} = \mathcal{X}\mathcal{D}_j$, $j = 1,\cdots,d$, we obtain

$$\mathcal{D}_j\mathcal{X}V_k = \mathcal{X}\mathcal{D}_jV_k = \mathcal{X}V_k\frac{\partial}{\partial x_j}, \quad j = 1,\cdots,d.$$

By applying Lemma 2, we deduce that $\mathcal{X}V_k = \mathcal{B}$, and hence

$$\mathcal{X}f(x) = \mathcal{B}V_k^{-1}f(x) = T *_k f(x).$$

□

Remark 2. Let $\mathcal{X}$ be continuous linear mapping $\mathcal{X}$ from $\mathcal{E}(\mathbb{R}^d)$ into itself, such that $\mathcal{D}_j\mathcal{X} = \mathcal{X}\mathcal{D}_j$, $j = 1,\cdots,d$.

By virtue of Theorem 6, we can find $T \in \mathcal{E}'(\mathbb{R}^d)$ such that

$$\mathcal{X}f(x) = T *_k f(x) = \langle T_y, \tau_x f(-y)\rangle, \quad f \in \mathcal{E}(\mathbb{R}^d).$$

In particular (by Proposition 2 3)), for every $x \in \mathbb{R}^d$, we have

$$\mathcal{X}E_k(.,z)(x) = \langle T_y, \tau_x E_k(.,z)(-y)\rangle = E_k(x,z)\langle T_y, E_k(-y,z)\rangle.$$

We put $\Psi(z) := \langle T_y, E_k(-y,z)\rangle$, we obtain

$$\mathcal{X}E_k(.,z)(x) = E_k(x,z)\Psi(z), \quad z \in \mathbb{R}^d.$$

Hence, for every $z \in \mathbb{R}^d$, $E_k(.,z)$ is an eigenfunction of $\mathcal{X}$ associated with the eigenvalue $\Psi(z)$. □

REFERENCES

[1] C. F. Dunkl, "Differential-Difference Operators Associated with Reflections Groups," *Transactions of the American Mathematical Society*, Vol. 311, 1989, pp. 167-183. doi:10.1090/S0002-9947-1989-0951883-8

[2] C. F. Dunkl, "Integral Kernels with Reflection Group Invariance," *Canadian Journal of Mathematics*, Vol. 43, No. 6, 1991, pp. 1213-1227. doi:10.4153/CJM-1991-069-8

[3] M. F. E. de Jeu, "The Dunkl Transform," *Inventiones Mathematicae*, Vol. 113, No. 1, 1993, pp. 147-162. doi:10.1007/BF01244305

[4] K. Trimèche, "Paley-Wiener Theorems for the Dunkl Transform and Dunkl Translation Operators," *Integral Transforms and Special Functions*, Vol. 13, No. 1, 2002, pp. 17-38. doi:10.1080/10652460212888

[5] K. Trimèche, "The Dunkl Intertwining Operator on Spaces of Functions and Distributions and Integral Representation of Dual," *Integral Transforms and Special Functions*, Vol. 12, No. 4, 2001, pp. 349-374. doi:10.1080/10652460108819358

[6] M. Rösler, "Positivity of Dunkl's Intertwining Operator," *Duke Mathematical Journal*, Vol. 98, 1999, pp. 445-463.

[7] J. J. Betancor, M. Sifi and K. Trimèche, "Hypercyclic and Chaotic Convolution Operators Associated with the Dunkl Operator on $\mathbb{C}$," *Acta Mathematica Hungarica*, Vol. 106, No. 1-2, 2005, pp. 101-116.

doi:10.1007/s10474-005-0009-1

[8] J. J. Betancor, M. Sifi and K. Trimèche, “Intertwining Operator and the Commutators of the Dunkl Operator on $\mathbb{C}$,” *Mathematical Sciences Research Journal*, Vol. 10, 2006, pp. 66-78.

[9] N. B. Salem and S. Kallel, “Mean-Periodic Functions Associated with the Dunkl Operators,” *Integral Transforms and Special Functions*, Vol. 15, No. 2, 2004, pp. 155-179. doi:10.1080/10652460310001600735

[10] L. Hörmander, “Linear Partial Differential Operators,” Springer-Verlag, Berlin, Gottingen, Heidelberg, 1964.

[11] L. Schwartz, “Théorie Générale des Fonctions Moyennes Périodiques,” *Annals of Mathematics*, Vol. 48, No. 4, 1947, pp. 857-929. doi:10.2307/1969386