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Finite Elements in the Solution of Continuum Field Problems

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ABSTRACT

A finite element functional solution procedure was presented employing variational calculus. The Functionals of field continuum were developed on adoption of Euler minimum integral theorem and finite element procedures on Laplace model. The elements functionals minimization resulted to series of partial differential equations describing the variation of the function of interest at various discrete nodal points. The assembly of the partial differential equations gave a unifying algebraic system of equation was solved for the unique solutions of the function. To simulate the finite element model, boundary conditions of temperature field was assumed. The solution and post processing of FEM of this study showed that once the stiffness matrix of a continuum is established and the boundary conditions specified the continuum is solved uniquely. Regression method was used to establish the error associated with FEM results and to establish a simple prediction model for environmental temperatures. The procedure of this study presented the basis for insulation design for solid, hollow or shell pipes in fluid transport design in oil and gas transport system. The finite element method evaluated the temperature distribution of the region to serve as a guide in quantifying quantity of heat to the environment from the transit fluid. The error of FEM prediction was estimated at 0.006 and the coefficient of determination for goodness of regression fit is estimated as 0.99999. This study also presents an approximate procedure for processing polar systems as rectangular systems by using the circumference of the circular section as one dimensional independent variable and the difference between the inner and outer radius (thickness) as the second independent variable.

Keywords: *Calculus of variation, functional, finite element, continuum, field problems, boundary conditions.*

1. INTRODUCTION

The essence of this work is to use a simple bounded region to present a procedure to solve functional problems in fluid transport piping system. A finite element functional solution procedure was presented using variational calculus. Mechanics field problems of heat and mass transfer and fluid flow problems in oil and gas industries is intended to be solved following the formulations of this article. Fluid mechanics problems can be studied by isolating a discrete domain with boundary values such as fluids within shelled pipes in which boundary conditions are known. The functional approach expects the flow function to be approximate to known field phenomenon, such as wave phenomenon, , diffusion phenomenon and potential phenomenon, though, some complex phenomena of engineering sciences may be a combination of the above phenomena [1].

An immense number of analytical solutions for conduction heat-transfer problems has been accumulated in the literature over the years past. Even so, in many practical situations the geometry or boundary conditions are such that an analytical solution has not been obtained at all, or if the solution has been developed, it involves such a complex series solution that numerical evaluation becomes exceedingly difficult [2]. However, presently the most fruitful approach to the problem is one based on finite different techniques, the basic principles of which are outlined in [3, 4, 5, and 6]. But because of the short falls of finite difference method the major objective of this article is to present the finite element method, as the most elegant approach of solving function such as the temperature distribution in a medium. Finite element approach is suited to solving partial differential equation so that this work used the finite element approach to develop the partial differential equations of the function at the discrete nodes, and as such the values of the function at the boundary nodes must be known, may be from the field data. In this study linear triangular element is used to discretize the field domain or the mathematical model representing the function, and the following the variational principle and Euler minimum integral theorem to arrive at the elements equations. The elements equations are then added in a finite element format called assembly and the system of the equations solved after the application of boundary conditions to obtain the value of the function that minimizes the functional at the stationary points. The procedure is analogous to Raleigh – Ritz method for stationary total potential applied to elastic problems [7].

Above all, Most Quasi – harmonic field phenomenon are represented by either the partial derivatives of the function or by the well known Laplace and Poisson’s equation [8]. The velocities, temperatures and densities of fluids in oil and gas pipelines are expected to be known at various sections of transport line as the mentioned variables determine efficiency of the transportation unit.

Also working in polar coordinates for circular or polar systems in engineering and science often posses some difficulties that this study presents an approximate procedure for processing polar systems as rectangular systems by using the circumference of the circular section as one dimensional independent variable and the difference between the inner and outer radius (thickness) as the second independent variable. The function of interest as the dependent variable is then analysed by solving an approximate function through FEM approaches, which in this study is the Laplace steady state 2-D heat equation in cartesian coordinate. The geometry of field quantities or continuum may be a problem to close form solution of field functions encountered in engineering and science that appropriate algorithm becomes necessary to obtain optimum solution, it is then necessary to employ calculus of variation principles and FEM to obtain optimum continuum field functions whose boundary conditions are specified.

The methodology of this work is also supported by the fact that in calculus of variations, instead of attempting to locate points that extremize function of one or more variables that extremize quantities called functional, functions of functions that extremize the functional are found [9]. Also in the finite element process an approximate solution is sought to the problem of minimizing a functional.

2. THEORETICAL BACKGROUND: BASIC FINITE ELEMENT FUNCTIONS

Figure 1 shows the typical triangular element considered, with nodes I,j,m numbered in anticlockwise order.

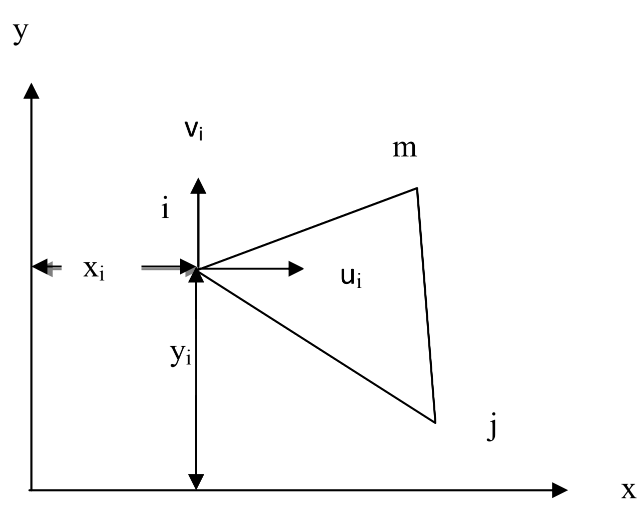

Figure 1. An Element of Continuum for deriving Element Equation.

2.1 Approximation Function Determination

Linear Polynomials of the form

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{1}$$

$$v = \alpha_3 + \alpha_4 x + \alpha_5 y \tag{2}$$

are usually chosen for horizontal and vertical responses or degrees of freedom(DOF).
By passing these polynomials in turn through nodes i,j,m ,the system of the function is obtained for the element following the method of [7] as

$$u_i = \alpha_0 + \alpha_1 x_i + \alpha_2 y_i \tag{3}$$

$$u_j = \alpha_0 + \alpha_1 x_j + \alpha_2 y_j \tag{4}$$

$$u_m = \alpha_0 + \alpha_1 x_m + \alpha_2 y_m \tag{5}$$

2.2 Shape Function and Interpolation Function Definition

By employing Crammer's rule for the polynomial coefficients, α_0, α_1 α_2, also called the shape constants, the interpolation function is established as

$$\mathrm{u} = \frac{1}{2\Delta}\{(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j + (a_m + b_m x + c_m y)u_m\} \tag{6}$$

where

$$a_i = x_j y_m - x_m y_j \tag{7}$$

$$b_i = y_j - y_m = y_{jm} \tag{8}$$

$$c_i = x_m - x_j = x_{mj} \tag{9}$$

and

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = 2(\text{area of triangle, ijm}) \tag{10}$$

So that the shape functions are expressed at the nodes as follows

$$N_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y) \tag{11}$$

$$N_j = \frac{1}{2\Delta}(a_j + b_j x + c_j y) \tag{12}$$

$$N_m = \frac{1}{2\Delta}(a_m + b_m x + c_m y) \tag{13}$$

and interpolation function expressed as

$$u = N_i u_i + N_j u_j + N_m u_m \quad (14)$$

2.3 Basic Field Equations and Approximating Functionals

[8] Presented the general equation governing quasi-harmonic field functions as

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right) + \left(k_y \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial u}{\partial z}\right) + Q = 0 \quad (15)$$

While the theorem of Euler presented by [10] states that if the integral

$$I(u) = \iiint f\left(x, y, z, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) dxdydz \quad (16)$$

is to be minimized, then the necessary and sufficient condition for this minimum to be reached is that the unknown function u (x, y, z) should satisfy the following differential equation

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial(\partial u/\partial x)}\right] + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial(\partial u/\partial y)}\right] + \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial(\partial u/\partial z)}\right] - \frac{\partial f}{\partial u} = 0 \quad (17)$$

within the same region, provided u satisfies the same boundary conditions in both cases, where

u = unknown function assumed to be single valued within the region

k_x, k_y, k_z, Q = specified functions of x, y, z

x, y, z = space variables

The equivalent formulation to that of equation (3) is the requirement that the volume integral given below and taken over the whole region, should be,

$$\chi = \iiint \left\{\frac{1}{2}\left[k_x\left(\frac{\partial u}{\partial x}\right)^2 + k_y\left(\frac{\partial u}{\partial y}\right)^2 + k_z\left(\frac{\partial u}{\partial z}\right)^2\right] - Qu\right\} dxdydz \quad (18)$$

3. FORMATION OF FUNCTIONAL OF TWO DIMENSIONAL FIELD FUNCTIONS

The functions that can be approximated by two dimensional Laplace equations such as the two dimensional steady state heat transfer problem with constant thermal conductivity are said to be governed by the equation [2]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (19)$$

The functional of this model is established considering a 2-D case of (15) expressed as

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right)+\left(k_y \frac{\partial u}{\partial y}\right) + Q = 0 \quad (20)$$

So that by (18) the equivalent functional to be minimized becomes

$$\chi = \iiint \left\{\frac{1}{2}\left[k_x \left(\frac{\partial u}{\partial x}\right)^2 + k_y \left(\frac{\partial u}{\partial y}\right)^2\right] - Qu\right\} \mathrm{dxdy} \quad (21)$$

The functional equation (21) is transformed to a more useful form by using the interpolation functions of elements to obtain each element functional.

3.1 Derivation and Description of Element Characteristics and Functional

This process is accomplished by evaluating the contributions to each differential, such as $\frac{\partial \chi}{\partial u_i}$ from a typical element then adding such contribution and equating to zero. Only the elements adjacent to node i will contribute to $\frac{\partial \chi}{\partial u_i}$.

3.2 Element Characteristics and Functional Minimization Scheme

The contribution of elements to the differential $\frac{\partial \chi}{\partial u_i}$ is evaluated as follows:
If the value of functional associated with the element is expressed from (21) as

$$\chi^e = \iiint \left\{\frac{1}{2}\left[k_x \left(\frac{\partial u}{\partial x}\right)^2 + k_y \left(\frac{\partial u}{\partial y}\right)^2\right] - Qu\right\} \mathrm{dxdy} \quad (22)$$

Then by partially differentiating χ^e,

$$\frac{\partial \chi^e}{\partial u_i} = \iint \left\{k_x \frac{\partial u}{\partial x}\frac{\partial}{\partial u_i}\left(\frac{\partial u}{\partial x}\right) + k_y \frac{\partial u}{\partial y}\frac{\partial}{\partial u_i}\left(\frac{\partial u}{\partial y}\right) - Q\frac{\partial u}{\partial u_i}\right\} \mathrm{dxdy} \quad (23)$$

With the interpolation function, u_i defined as

$$u = (N_i, N_j, N_m)\{u\}^e \quad (24)$$

and the shape function defined as

$$N_i = \frac{1}{2\Delta}\{(a_i + b_i x + c_i y)\} \quad (25)$$

(23) becomes

$$\frac{\partial \chi^e}{\partial u_i} = \frac{1}{(2\Delta)^2}\iint \left(k_x\left[b_i, b_j, b_m\right]\{u\}^e b_i + k_y\left[c_i c_j c_m\right]\{u\}^e c_i\right) \mathrm{dxdy} - \frac{1}{2\Delta}\iint Q(a_i + b_i x + c_i y)\,\mathrm{dxdy} \qquad (26)$$

An element contributes three differentials associated to its nodes. The contributions of the three nodes are listed

$$\left\{\frac{\partial \chi}{\partial u}\right\}^e = \begin{Bmatrix} \frac{\partial \chi^e}{\partial u_i} \\ \frac{\partial \chi^e}{\partial u_j} \\ \frac{\partial \chi^e}{\partial u_m} \end{Bmatrix} \qquad (27)$$

Equation (27) can be written in the form

$$\left\{\frac{\partial \chi}{\partial u}\right\}^e = [h]\{u\}^e + \{F\}^e \qquad (28)$$

The matrix $[h]$ is easily written if k_x, k_x are taken as constant within the element, also by nothing that over the area of the element

$$\iint \mathrm{dxdy} = \Delta \qquad (29)$$

$$[h] = \frac{k_x}{4\Delta}\begin{bmatrix} b_i b_i & b_j b_i & b_m b_i \\ b_i b_j & b_j b_j & b_m b_j \\ b_i b_m & b_j b_m & b_m b_m \end{bmatrix} + \frac{k_y}{4\Delta}\begin{bmatrix} c_i b_i & c_j c_i & c_m c_i \\ c_i c_j & c_j c_j & c_m c_j \\ c_i c_m & c_j c_m & c_m c_m \end{bmatrix} \qquad (30)$$

The vector $\{F\}^e$ can be found as follows: If Q is assumed constant within an element then the integral

$$F_i = -\frac{Q}{2\Delta}\iint Q(a_i + b_i x + c_i y)\,\mathrm{dxdy} \qquad (31)$$

can be evaluated

$$F_i = -Q.\frac{(a_i + b_i \bar{x} + c_i \bar{y})}{2} \qquad (32)$$

Where $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid ie

$$\bar{x} = \frac{(x_i + x_j + x_m)}{3}, \quad \bar{y} = \frac{(y_i + y_j + y_m)}{3} \qquad (33)$$

so the integral expressed by (31) becomes

$$F_i = \frac{1}{3}\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = \frac{1}{3}A \qquad (34)$$

3.3 Assembly Procedure for Functional Minimization Scheme

The final equations of the minimization procedure require the assembly of all the differentials of χ and equating of these to zero, expressed typically as

$$\frac{\partial \chi}{\partial u_i} = \Sigma \frac{\partial \chi^e}{\partial u_i} = 0 \qquad (35)$$

Equation (35) says that the functionals of all elements are established and partially differentiated with respect to the associated nodal degrees of freedoms [11].

4. METHODOLOGY

4.1 Finite Element Modeling of 2-D Steady State Regional Function Distribution

Working in polar coordinates for circular or polar systems in engineering and science often posses some difficulties that this study presents an approximate procedure for processing polar systems as rectangular systems by using the circumference of the circular section as one dimensional independent variable and the difference between the inner and outer radius (thickness) as the second independent variable. The function of interest as the dependent variable is then analysed by solving an approximate function through FEM approaches, which in this study is the Laplace steady state 2-D heat equation in cartesian coordinate. The geometry of field quantities or continuum may be a problem to close form solution of field functions encountered in engineering and science that appropriate algorithm becomes necessary to obtain optimum solution, it is then necessary to employ calculus of variation principles and FEM to obtain optimum continuum field functions whose boundary conditions are specified.

In Figure 2, the vertical dimension represents the thickness of a circular pipe or thickness of insulation while the horizontal dimension represents the circumference of circular section of pipe or cylinder. The continuum domain is discretized as shown in Figure 2.

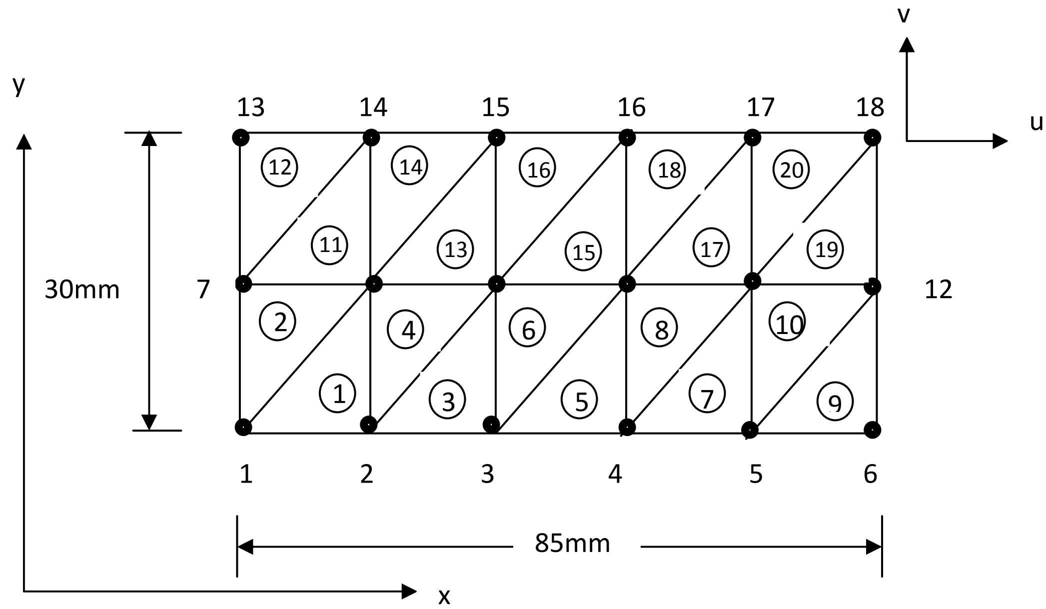

Figure 2. An Idealized Finite Element Model of a Continuum of GRP Composite .

Table 1. Elements Topology Description.

Element number	Active degrees of freedom for assembly	Element coordinates	Element nodes
1	u_1,u_2,u_8,v_1,v_2,v_8	(0,0),(17,0),(17,15)	1,2,8
2	U_1,u_8,u_7,v_1v_8,v_7	(0,0),(17,15),(0,15)	1,8,7
3	u_2,u_3,u_7,v_2,v_3,v_7	(17,0),(34,0),(17,15)	2,3,7
4	u_2,u_7,u_8,v_2,v_7,v_8	(17,0),(34,15),(17,15)	2,7,8
5	$u_3,u_4,u_{10},v_3,v_4,v_{10}$	(34,0),(5,10),(51,15)	3,4,10
6	$u_3,u_{10},u_9,v_3,v_{10},v_9$	(34,0),(51,15),(34,15)	3,10,9
7	$u_4,u_5,u_{11},v_4,v_5,v_{11}$	(51,0),(68,0),(68,15)	4,5,11
8	$u_4,u_{11},u_{10},v_4,v_{111},v_{10}$	(51,0),(68,15),(51,15)	4,11,10
9	$u_5,u_6,u_{12},v_5,v_6,v_{12}$	(68,0),(85,0),(85,15)	5,6,12
10	$u_5,u_{12},u_{11},v_5,v_{12},v_{11}$	(68,0),(85,15),(68,15)	5,12,11
11	$u_7,u_8,u_{14},v_7,v_8,v_{14}$	(0,15),(17,15),(17,30)	7,8,14

12	$u_7,u_{14},u_{13},v_7,v_{14},v_{13}$	(0,15),(17,30),(0,30)	7,14,13
13	$u_8,u_9,u_{15},v_8,v_9,v_{15}$	(17,15),(34,15),(34,30)	8,9,15
14	$u_8,u_{15},u_{14},v_8,v_{15},v_{14}$	(17,15),(34,30),(17,30)	8,15,14
15	$u_9,u_{10},u_{16},v_9,v_{10},v_{16}$	(34,15),(51,15),(51,30)	9,10.16
16	$u_9,u_{16},u_{15},v_9,v_{16},v_{15}$	(34,15),(51,30),(34,30)	9,16,15
17	$u_{10},u_{11},u_{17},v_{10},v_{11},v_{17}$	(51,15),(68,15),(68,30)	10,11,17
18	$u_{10},u_{17},u_{16},v_{10},v_{17},v_{16}$	(51,15),(68,30),(51,30)	10,17,16
19	$u_{11},u_{12},u_{18},v_{11},v_{12},v_{18}$	(68,15),(85,15),(85,30)	11,12,18
20	$u_{11},u_{18},u_{17},v_{11},v_{18},v_{17}$	(68,15),(85,30),(68,30)	11,18,17

4.2 Computation of Elements Partial Differential Equations

By using equation (35) all the systems of partial differential equations are computed for the twenty elements as follows:

4.3 Assumptions

The function is assumed to be Laplace function so that equation (28) reduces to

$$\left\{\frac{\partial \chi}{\partial u}\right\}^e = [h]\{u\}^e \tag{36}$$

and (32) reduces to

$$[h] = \frac{1}{4\Delta}\begin{bmatrix} b_i b_i & b_j b_i & b_m b_i \\ b_i b_j & b_j b_j & b_m b_j \\ b_i b_m & b_j b_m & b_m b_m \end{bmatrix} + \frac{1}{4\Delta}\begin{bmatrix} c_i b_i & c_j c_i & c_m c_i \\ c_i c_j & c_j c_j & c_m c_j \\ c_i c_m & c_j c_m & c_m c_m \end{bmatrix} \tag{37}$$

By computations of (37) and substitution in (36) for all elements respectively the elements PDEs are presented for twenty elements assembly system as follows:

For element 1

$$\begin{Bmatrix} \frac{\partial \chi^1}{\partial u_1} \\ \frac{\partial \chi^1}{\partial u_2} \\ \frac{\partial \chi^1}{\partial u_8} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_8 \end{bmatrix} \tag{38}$$

For element 2

$$\begin{Bmatrix} \frac{\partial \chi^2}{\partial u_1} \\ \frac{\partial \chi^2}{\partial u_8} \\ \frac{\partial \chi^2}{\partial u_7} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_1 \\ u_8 \\ u_7 \end{bmatrix} \tag{39}$$

For element 3

$$\begin{Bmatrix} \frac{\partial \chi^3}{\partial u_2} \\ \frac{\partial \chi^3}{\partial u_3} \\ \frac{\partial \chi^3}{\partial u_9} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_9 \end{bmatrix} \tag{40}$$

By application of symmetry the 3by3 matrix in odd numbered elements are equal and similarly for the

For element 4

$$\begin{Bmatrix} \frac{\partial \chi^4}{\partial u_2} \\ \frac{\partial \chi^4}{\partial u_9} \\ \frac{\partial \chi^4}{\partial u_8} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_2 \\ u_9 \\ u_8 \end{bmatrix} \tag{41}$$

For element 5

$$\begin{Bmatrix} \frac{\partial \chi^5}{\partial u_3} \\ \frac{\partial \chi^5}{\partial u_4} \\ \frac{\partial \chi^5}{\partial u_{10}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_{10} \end{bmatrix} \tag{42}$$

For element 6

$$\begin{Bmatrix} \frac{\partial \chi^6}{\partial u_3} \\ \frac{\partial \chi^6}{\partial u_{10}} \\ \frac{\partial \chi^6}{\partial u_9} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_3 \\ u_{10} \\ u_9 \end{bmatrix} \tag{43}$$

For element 7

$$\begin{Bmatrix} \frac{\partial \chi^7}{\partial u_4} \\ \frac{\partial \chi^7}{\partial u_8} \\ \frac{\partial \chi^7}{\partial u_{11}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_4 \\ u_8 \\ u_{11} \end{bmatrix} \tag{44}$$

For element 8

$$\begin{Bmatrix} \frac{\partial \chi^8}{\partial u_4} \\ \frac{\partial \chi^8}{\partial u_{11}} \\ \frac{\partial \chi^8}{\partial u_{10}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_4 \\ u_{11} \\ u_{10} \end{bmatrix} \tag{45}$$

For element 9

$$\begin{Bmatrix} \frac{\partial \chi^9}{\partial u_5} \\ \frac{\partial \chi^9}{\partial u_6} \\ \frac{\partial \chi^9}{\partial u_{12}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_5 \\ u_6 \\ u_{12} \end{bmatrix} \tag{46}$$

For element 10

$$\begin{Bmatrix} \frac{\partial \chi^{10}}{\partial u_5} \\ \frac{\partial \chi^{10}}{\partial u_{12}} \\ \frac{\partial \chi^{10}}{\partial u_{11}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_5 \\ u_{12} \\ u_{11} \end{bmatrix} \tag{47}$$

For element 11

$$\begin{Bmatrix} \frac{\partial \chi^{11}}{\partial u_7} \\ \frac{\partial \chi^{11}}{\partial u_8} \\ \frac{\partial \chi^{11}}{\partial u_{14}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_{14} \end{bmatrix} \tag{48}$$

For element 12

$$\begin{Bmatrix} \frac{\partial \chi^{12}}{\partial u_7} \\ \frac{\partial \chi^{12}}{\partial u_{14}} \\ \frac{\partial \chi^{12}}{\partial u_{13}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & 0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_7 \\ u_{14} \\ u_{13} \end{bmatrix} \tag{49}$$

For element 13

$$\begin{Bmatrix} \frac{\partial \chi^{13}}{\partial u_8} \\ \frac{\partial \chi^{13}}{\partial u_9} \\ \frac{\partial \chi^{13}}{\partial u_{15}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_8 \\ u_9 \\ u_{15} \end{bmatrix} \tag{50}$$

For element 14

$$\begin{Bmatrix} \frac{\partial \chi^{14}}{\partial u_8} \\ \frac{\partial \chi^{14}}{\partial u_{15}} \\ \frac{\partial \chi^{14}}{\partial u_{14}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_8 \\ u_{15} \\ u_{14} \end{bmatrix} \tag{51}$$

For element 15

$$\begin{Bmatrix} \frac{\partial \chi^{15}}{\partial u_9} \\ \frac{\partial \chi^{15}}{\partial u_{10}} \\ \frac{\partial \chi^{15}}{\partial u_{16}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_9 \\ u_{10} \\ u_{16} \end{bmatrix} \tag{52}$$

For element 16

$$\begin{Bmatrix} \frac{\partial \chi^{16}}{\partial u_9} \\ \frac{\partial \chi^{16}}{\partial u_{16}} \\ \frac{\partial \chi^{16}}{\partial u_{15}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_9 \\ u_{16} \\ u_{15} \end{bmatrix} \tag{53}$$

For element 17

$$\begin{Bmatrix} \frac{\partial \chi^{17}}{\partial u_{10}} \\ \frac{\partial \chi^{17}}{\partial u_{11}} \\ \frac{\partial \chi^{17}}{\partial u_{17}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{11} \\ u_{17} \end{bmatrix} \tag{54}$$

For element 18

$$\begin{Bmatrix} \frac{\partial \chi^{18}}{\partial u_{10}} \\ \frac{\partial \chi^{18}}{\partial u_{17}} \\ \frac{\partial \chi^{18}}{\partial u_{16}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{17} \\ u_{16} \end{bmatrix} \tag{55}$$

For element 19

$$\begin{Bmatrix} \frac{\partial \chi^{19}}{\partial u_{11}} \\ \frac{\partial \chi^{19}}{\partial u_{12}} \\ \frac{\partial \chi^{19}}{\partial u_{18}} \end{Bmatrix} = \begin{bmatrix} 0.441 & -0.441 & 0.000 \\ -0.441 & 1.008 & -0.567 \\ 0.000 & -0.567 & 0.567 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{18} \end{bmatrix} \quad (56)$$

For element 20

$$\begin{Bmatrix} \frac{\partial \chi^{20}}{\partial u_{11}} \\ \frac{\partial \chi^{20}}{\partial u_{18}} \\ \frac{\partial \chi^{20}}{\partial u_{17}} \end{Bmatrix} = \begin{bmatrix} 0.567 & 0.000 & -0.567 \\ 0.000 & 0.441 & -0.441 \\ -0.567 & -0.441 & 1.008 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{18} \\ u_{17} \end{bmatrix} \quad (57)$$

4.4 Assembly of Elements Equations

All the partial equations are added with reference to the degree of freedom of elements so that for
f = 1, 2, 3, .n nodes, eighteen equations from eighteen nodal freedoms can be expressed as

For f =1

$$\frac{\partial \chi}{\partial u_1} = \Sigma \frac{\partial \chi^e}{\partial u_1} = \frac{\partial \chi^1}{\partial u_1} + \frac{\partial \chi^2}{\partial u_1} = 0$$

so that by addition of all contributions of node 1 to the functional minimization

$$\frac{\partial \chi}{\partial u_1} = 1.008u_1 - 0.441u_2 - 0.567u_7 = 0 \quad (58)$$

For f =2

$$\frac{\partial \chi}{\partial u_2} = \Sigma \frac{\partial \chi^e}{\partial u_2} = \frac{\partial \chi^2}{\partial u_2} + \frac{\partial \chi^3}{\partial u_2} + \frac{\partial \chi^4}{\partial u_2} = 0$$

so that by addition of all contributions of node 2 to the functional minimization

$$\frac{\partial \chi}{\partial u_2} = -0.441u_1 + 2.016u_2 - 0.441u_3 - 1.134u_8 = 0 \quad (59)$$

For f =3

$$\frac{\partial \chi}{\partial u_3} = \Sigma \frac{\partial \chi^e}{\partial u_3} = \frac{\partial \chi^4}{\partial u_3} + \frac{\partial \chi^5}{\partial u_3} + \frac{\partial \chi^6}{\partial u_3} = 0$$

so that by addition of all contributions of node 3 to the functional minimization

$$\frac{\partial \chi}{\partial u_3} = -0.441u_2 + 2.016u_3 - 0.441u_4 - 1.134u_9 = 0 \quad (60)$$

For f = 4

$$\frac{\partial\chi}{\partial u_4}=\Sigma\frac{\partial\chi^e}{\partial u_4}=\frac{\partial\chi^5}{\partial u_4}+\frac{\partial\chi^7}{\partial u_4}+\frac{\partial\chi^8}{\partial u_4}=0$$

so that by addition of all contributions of node 4 to the functional minimization

$$\frac{\partial\chi}{\partial u_4}=-0.441u_3+2.016u_4-0.441u_5-1.134u_{10}=0 \tag{61}$$

For f = 5

$$\frac{\partial\chi}{\partial u_5}=\Sigma\frac{\partial\chi^e}{\partial u_5}=\frac{\partial\chi^7}{\partial u_5}+\frac{\partial\chi^9}{\partial u_5}+\frac{\partial\chi^{10}}{\partial u_5}=0$$

so that by addition of all contributions of node 5 to the functional minimization

$$\frac{\partial\chi}{\partial u_5}=-0.441u_4+2.016u_5-0.441u_6-1.134u_{11}=0 \tag{62}$$

For f = 6

$$\frac{\partial\chi}{\partial u_6}=\Sigma\frac{\partial\chi^e}{\partial u_6}=\frac{\partial\chi^9}{\partial u_6}=0$$

so that by addition of all contributions of node 6 to the functional minimization

$$\frac{\partial\chi}{\partial u_6}=-0.441u_5+1.008u_6-0.567u_{12}=0 \tag{63}$$

For f = 7

$$\frac{\partial\chi}{\partial u_7}=\Sigma\frac{\partial\chi^e}{\partial u_7}=\frac{\partial\chi^2}{\partial u_7}+\frac{\partial\chi^{11}}{\partial u_7}+\frac{\partial\chi^{12}}{\partial u_7}=0$$

so that by addition of all contributions of node 7 to the functional minimization

$$\frac{\partial\chi}{\partial u_7}=-0.567u_1+2.016u_7-0.882u_8-0.567u_{13}=0 \tag{64}$$

For f = 8

$$\frac{\partial\chi}{\partial u_8}=\Sigma\frac{\partial\chi^e}{\partial u_8}=\frac{\partial\chi^1}{\partial u_8}+\frac{\partial\chi^3}{\partial u_8}+\frac{\partial\chi^4}{\partial u_8}+\frac{\partial\chi^{11}}{\partial u_8}+\frac{\partial\chi^{13}}{\partial u_8}+\frac{\partial\chi^{14}}{\partial u_8}=0$$

so that by addition of all contributions of node 8 to the functional minimization

$$\frac{\partial\chi}{\partial u_8}=-1.134u_2-2.016u_5-0.882u_7+4.032u_8-0.882u_9-1.134u_{14}=0 \tag{65}$$

For f = 9

$$\frac{\partial\chi}{\partial u_9}=\Sigma\frac{\partial\chi^e}{\partial u_9}=\frac{\partial\chi^3}{\partial u_9}+\frac{\partial\chi^4}{\partial u_9}+\frac{\partial\chi^6}{\partial u_9}+\frac{\partial\chi^{13}}{\partial u_9}+\frac{\partial\chi^{15}}{\partial u_9}+\frac{\partial\chi^{16}}{\partial u_9}=0$$

so that by addition of all contributions of node 9 to the functional minimization

$$\frac{\partial\chi}{\partial u_9}=-1.134u_3-0.882u_8+4.032u_9-0.882u_{10}-1.134u_{15}=0 \tag{66}$$

For f = 10

$$\frac{\partial\chi}{\partial u_{10}}=\Sigma\frac{\partial\chi^e}{\partial u_{10}}=\frac{\partial\chi^5}{\partial u_{10}}+\frac{\partial\chi^4}{\partial u_{10}}+\frac{\partial\chi^6}{\partial u_{10}}+\frac{\partial\chi^{15}}{\partial u_{10}}+\frac{\partial\chi^{16}}{\partial u_{10}}=0$$

so that by addition of all contributions of node 10 to the functional minimization

$$\frac{\partial \chi}{\partial u_5} = -1.134u_4 - 0.882u_9 + 4.032u_{10} - 0.882u_{11} - 1.134u_{16} = 0 \quad (67)$$

For f = 11

$$\frac{\partial \chi}{\partial u_{11}} = \sum \frac{\partial \chi^e}{\partial u_{11}} = \frac{\partial \chi^5}{\partial u_{11}} + \frac{\partial \chi^6}{\partial u_{11}} + \frac{\partial \chi^{10}}{\partial u_{11}} + \frac{\partial \chi^{12}}{\partial u_{11}} + \frac{\partial \chi^{17}}{\partial u_{11}} + \frac{\partial \chi^{20}}{\partial u_{11}} = 0$$

so that by addition of all contributions of node11 to the functional minimization

$$\frac{\partial \chi}{\partial u_{11}} = -1.134u_5 - 0.882u_{10} + 4.032u_{11} - 0.882u_{12} - 1.134u_{17} = 0 \quad (68)$$

For f = 12

$$\frac{\partial \chi}{\partial u_{12}} = \sum \frac{\partial \chi^e}{\partial u_{12}} = \frac{\partial \chi^7}{\partial u_{12}} + \frac{\partial \chi^{11}}{\partial u_{12}} + \frac{\partial \chi^{12}}{\partial u_{12}} = 0$$

so that by addition of all contributions of node 12 to the functional minimization

$$\frac{\partial \chi}{\partial u_{12}} = -0.567u_6 - 0.882u_{11} + 2.016u_{12} - 0.567u_{13} = 0 \quad (69)$$

For f = 13

$$\frac{\partial \chi}{\partial u_{13}} = \sum \frac{\partial \chi^e}{\partial u_{13}} = \frac{\partial \chi^{12}}{\partial u_{13}} = 0$$

so that by addition of all contributions of node 13 to the functional minimization

$$\frac{\partial \chi}{\partial u_{13}} = -0.567u_7 + 1.008u_{13} - 0.441u_{14} = 0 \quad (70)$$

For f = 14

$$\frac{\partial \chi}{\partial u_{14}} = \sum \frac{\partial \chi^e}{\partial u_{14}} = \frac{\partial \chi^7}{\partial u_{14}} + \frac{\partial \chi^{11}}{\partial u_{14}} + \frac{\partial \chi^{12}}{\partial u_{14}} = 0$$

so that by addition of all contributions of node 14 to the functional minimization

$$\frac{\partial \chi}{\partial u_{14}} = -1.134u_8 - 0.441u_{13} + 2.016u_{14} - 0.441u_{15} = 0 \quad (71)$$

For f = 15

$$\frac{\partial \chi}{\partial u_{15}} = \sum \frac{\partial \chi^e}{\partial u_{15}} = \frac{\partial \chi^{13}}{\partial u_{15}} + \frac{\partial \chi^{14}}{\partial u_{15}} + \frac{\partial \chi^{16}}{\partial u_{15}} = 0$$

so that by addition of all contributions of node 15 to the functional minimization

$$\frac{\partial \chi}{\partial u_{15}} = -1.134u_9 - 0.441u_{14} + 2.016u_{15} - 0.441u_{16} = 0 \quad (72)$$

For f = 16

$$\frac{\partial \chi}{\partial u_{16}} = \sum \frac{\partial \chi^e}{\partial u_{16}} = \frac{\partial \chi^{13}}{\partial u_{16}} + \frac{\partial \chi^{14}}{\partial u_{16}} + \frac{\partial \chi^{16}}{\partial u_{16}} = 0$$

so that by addition of all contributions of node 16 to the functional minimization

$$\frac{\partial \chi}{\partial u_{16}} = -1.134u_{10} - 0.441u_{15} + 2.016u_{16} - 0.441u_{17} = 0 \quad (73)$$

For f = 17

$$\frac{\partial \chi}{\partial u_{17}} = \Sigma \frac{\partial \chi^{e}}{\partial u_{17}} = \frac{\partial \chi^{17}}{\partial u_{17}} + \frac{\partial \chi^{18}}{\partial u_{17}} + \frac{\partial \chi^{19}}{\partial u_{17}} = 0$$

so that by addition of all contributions of node17 to the functional minimization

$$\frac{\partial \chi}{\partial u_{17}} = -1.134u_{11} - 0.441u_{16} + 2.016u_{17} - 0.441u_{18} = 0 \quad (74)$$

For f = 18

$$\frac{\partial \chi}{\partial u_{18}} = \Sigma \frac{\partial \chi^{e}}{\partial u_{18}} = \frac{\partial \chi^{19}}{\partial u_{18}} + \frac{\partial \chi^{20}}{\partial u_{18}} = 0$$

so that by addition of all contributions of node 18 to the functional minimization

$$\frac{\partial \chi}{\partial u_{18}} = -0.567u_{12} - 0.441u_{17} + 1.008u_{18} = 0 \quad (75)$$

4.5 Boundary Conditions and Solutions of FEM

The above system formed by (58-75) is expressed in FEM matrix format as

$$[K][u] = [f] \quad (76)$$

Where

$[K]$ = the vector of the coefficient matrix called the stiffness matrix in FEM terminology
$[u]$ = the column vector regarded as the nodal degree of freedom (DOF)
$[f]$ = external influence column vector

Having established the boundary values of a field function and also the stiffness of the field, the model representing the field response can be solved uniquely.

4.6 Statement of the Field Problem

In this study the major objective is to set up FEM that solves continuum functions. To demonstrate the application of this model, thermal field region is solved. The model of (76) is then solved applying the specified boundary conditions of Figure 2:

4.7 Boundary Condition of Hot and Cold Surfaces

This may represent the case of a hot fluid in a hollow pipe with outside cold fluid or the case of shell pipe or the case of insulated heat transfer surfaces.

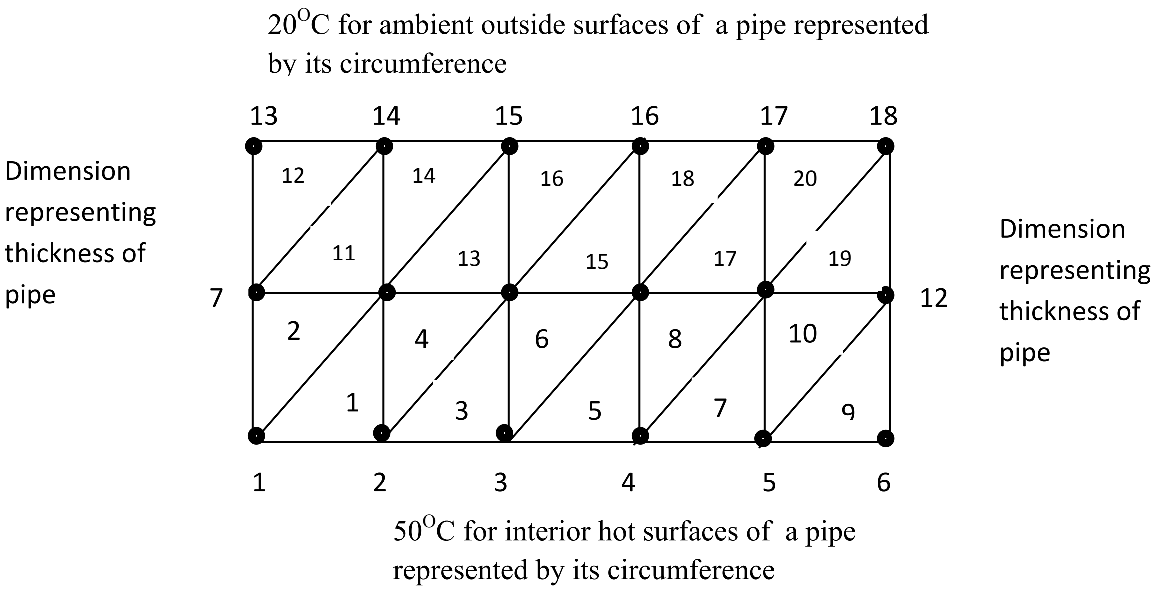

Figure 3. Schematic Representation of hot and ambient surfaces.

By employing (58-75) as

$$-0.567u_1 + 2.016u_7 - 0.882u_8 - 0.567u_{13} = 0 \quad (77)$$

$$-1.134u_2 - 2.016u_3 - 0.882u_7 + 4.032u_8 - 0.882u_9 - 1.134u_{14} = 0 \quad (78)$$

$$-1.134u_3 - 0.882u_8 + 4.032u_9 - 0.882u_{10} - 1.134u_{15} = 0 \quad (79)$$

$$-1.134u_4 - 0.882u_9 + 4.032u_{10} - 0.882u_{11} - 1.134u_{16} = 0 \quad (80)$$

$$-1.134u_5 - 0.882u_{10} + 4.032u_{11} - 0.882u_{12} - 1.134u_{17} = 0 \quad (81)$$

$$-0.567u_6 - 0.882u_{11} + 2.016u_{12} - 0.567u_{18} = 0 \quad (82)$$

and using the boundary conditions as:

For

$u_{13} = u_{14} = u_{15} = u_{16} = u_{17} = u_{18} = 20$, $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 50$,

the system formed by equation (77-82) reduces to

$$2.016u_7 - 0.882u_8 = 39.69 \quad (83)$$

$$-0.882u_7 + 4.032u_8 - 0.882u_9 = 180.18 \quad (84)$$

$$-0.882u_8 + 4.032u_9 - 0.882u_{10} = 79.38 \quad (85)$$

$$-0.882u_9 + 4.032u_{10} - 0.882u_{11} = 79.38 \quad (86)$$

$$-0.882u_{10} + 4.032u_{11} - 0.882u_{12} = 79.38 \quad (87)$$

$$-0.882u_{11} + 2.016u_{12} = 39.69 \quad (88)$$

and in matrix form

$$\begin{bmatrix} 2.016 & -0.882 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.882 & 4.032 & -0.882 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.882 & 4.032 & -0.882 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.882 & 4.032 & -0.882 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.882 & 4.032 & -0.882 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.882 & 2.016 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 39.69 \\ 160.18 \\ 79.38 \\ 79.38 \\ 79.38 \\ 39.69 \end{bmatrix} \quad (89)$$

(89) is solved by LU-decomposition to obtain

$u_7 = 47.8089, u_8 = 64.2775, u_9 = 41.7453, u_{10} = 36.55796, u_{11} = 35.3769, u_{12} = 35.1649$

For

$u_{13} = u_{14} = u_{15} = u_{16} = u_{17} = u_{18} = 30,\ u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 50,$

the system formed by equation (77-82) reduces to

$$\begin{bmatrix} 2.016 & -0.882 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.882 & 4.032 & -0.882 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.882 & 4.032 & -0.882 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.882 & 4.032 & -0.882 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.882 & 4.032 & -0.882 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.882 & 2.016 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 45.36 \\ 191.52 \\ 90.72 \\ 90.72 \\ 90.72 \\ 45.36 \end{bmatrix} \quad (90)$$

and

$u_7 = 52.8089, u_8 = 69.27747, u_9 = 46.74525, u_{10} = 41.55796, u_{11} = 40.37687, u_{12} = 40.16488$

For

$u_{13} = u_{14} = u_{15} = u_{16} = u_{17} = u_{18} = 40,\ u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 50,$

the system formed by equation (77-82) reduces to

$$\begin{bmatrix} 2.016 & -0.882 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.882 & 4.032 & -0.882 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.882 & 4.032 & -0.882 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.882 & 4.032 & -0.882 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.882 & 4.032 & -0.882 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.882 & 2.016 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 51.03 \\ 202.86 \\ 102.06 \\ 102.06 \\ 102.06 \\ 51.03 \end{bmatrix} \quad (91)$$

and

$u_7 = 57.80887, u_8 = 74.27747, u_9 = 51.74525, u_{10} = 46.55796, u_{11} = 45.37687, u_{12} = 45.16488$

For

$u_{13} = u_{14} = u_{15} = u_{16} = u_{17} = u_{18} = 50,\ u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 50,$

the system formed by equation (77-82) reduces to

$$\begin{bmatrix} 2.016 & -0.882 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.882 & 4.032 & -0.882 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.882 & 4.032 & -0.882 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.882 & 4.032 & -0.882 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.882 & 4.032 & -0.882 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.882 & 2.016 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 56.7 \\ 214.2 \\ 113.4 \\ 113.4 \\ 113.4 \\ 56.7 \end{bmatrix} \quad (92)$$

and

$u_7 = 62.80889, u_8 = 79.27747, u_9 = 56.74525, u_{10} = 51.55796, u_{11} = 50.37687, u_{12} = 50.16488$

For

$u_{13} = u_{14} = u_{15} = u_{16} = u_{17} = u_{18} = 60, u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 50,$

the system formed by equation (77-82) reduces to

$$\begin{bmatrix} 2.016 & -0.882 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.882 & 4.032 & -0.882 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.882 & 4.032 & -0.882 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.882 & 4.032 & -0.882 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.882 & 4.032 & -0.882 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.882 & 2.016 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 62.37 \\ 225.54 \\ 124.74 \\ 124.74 \\ 124.74 \\ 62.37 \end{bmatrix} \quad (93)$$

and

$u_7 = 67.80889, u_8 = 84.27747, u_9 = 61.74525, u_{10} = 56.5576, u_{11} = 55.37687, u_{12} = 55.16488$

5. ANALYSIS OF FEM RESULTS

Table 4. Predicted Interior Temperayures Given Boundary Conditions.

Ti(^oC)	Ta(^oC)	u_7(^oC)	u_8(^oC)	u_9(^oC)	u_{10}(^oC)	u_{11}(^oC)	u_{12}(^oC)
50	20	47.8089	64.2775	41.7453	36.55796	35.3769	35.1649
50	30	52.8089	69.27747	46.74525	41.55796	40.37687	40.16488
50	40	57.80887	74.27747	51.74525	46.55796	45.37687	45.16488
50	50	62.80889	79.27747	56.74525	51.55796	50.37687	50.16488
50	60	67.80889	84.27747	61.74525	56.5576	55.37687	55.16488

5.1 Regression Modelling of FEM Result

The correctness of FEM result is established by using multiple linear regression analysis approach with assumed boundary conditions as dependent variable and FEM predicted interior values of the function as independent variables. This analysis approach enabled the estimation of error associated with FEM and establishment of a design friendly model for the prediction of the maximum temperature within the region which was experienced in node8 where the temperature peaked to 84.27747°C. In design practice this point is critical.

Table 9. Data for Regression Model of FEM result.

u_7(°C)	u_8(°C)	u_9(°C)	u_{10}(°C)	u_{11}(°C)	u_{12}(°C)	T(°C)
47.8089	64.2775	41.7453	36.55796	35.3769	35.1649	20
52.8089	69.27747	46.74525	41.55796	40.37687	40.16488	30
57.80887	74.27747	51.74525	46.55796	45.37687	45.16488	40
62.80889	79.27747	56.74525	51.55796	50.37687	50.16488	50
67.80889	84.27747	61.74525	56.5576	55.37687	55.16488	60

The function of six variables for multiple linear regression can be expressed as

$$T = \beta_0 + \beta_1 u_7 + \beta_2 u_8 + \beta_3 u_9 + \beta_4 u_{10} + \beta_5 u_{11} + \beta_6 u_{12} \quad (94)$$

So that by following regression method assuming the linear equation of the form

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + e \quad (95)$$

following least squares approach that minimizes the sum of squares of residuals the normal equation that evaluates the coefficients of (94) are established.
By expressing the sum of squares of residuals as

$$s_r = \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})]^2 \quad (96)$$

By minimizing sum of squares of residuals with respect to the coefficients(estimators), of (96) and equating to zero,

$$\frac{\partial s_r}{\partial \beta_0} = \sum_{i=1}^{n} (-2)[y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (97)$$

$$\frac{\partial s_r}{\partial \beta_1} = \sum_{i=1}^{n} (-2) x_{1i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (98)$$

$$\frac{\partial s_r}{\partial \beta_2} = \sum_{i=1}^{n} (-2) x_{2i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (99)$$

$$\frac{\partial s_r}{\partial \beta_3} = \sum_{i=1}^{n} (-2) x_{3i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (100)$$

$$\frac{\partial s_r}{\partial \beta_4} = \sum_{i=1}^{n} (-2) x_{4i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (101)$$

$$\frac{\partial s_r}{\partial \beta_5} = \sum_{i=1}^{n} (-2) x_{5i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (102)$$

$$\frac{\partial s_r}{\partial \beta_6} = \sum_{i=1}^{n} (-2) x_{6i} [y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{5i} + \beta_6 x_{6i})] = 0 \quad (103)$$

The 7by7 system of equations for solving for the coefficients are expressed from the preceding seven equations as

$$\sum y = \beta_0 n + \beta_1 \sum x_1 + \beta_2 \sum x_2 + \beta_3 \sum x_3 + \beta_4 \sum x_4 + \beta_5 \sum x_5 + \beta_6 \sum x_6 \tag{104}$$

$$\sum x_1 y = \beta_0 \sum x_2 + \beta_1 \sum x_1^2 + \beta_2 \sum x_1 x_2 + \beta_3 \sum x_1 x_3 + \beta_4 \sum x_1 x_4 + \beta_5 \sum x_1 x_5 + \beta_6 \sum x_1 x_6 \tag{105}$$

$$\sum x_2 y = \beta_0 \sum x_1 + \beta_1 \sum x_1 x_2 + \beta_2 \sum x_2^2 + \beta_3 \sum x_2 x_3 + \beta_4 \sum x_2 x_4 + \beta_5 \sum x_2 x_5 + \beta_6 \sum x_2 x_6 \tag{106}$$

$$\sum x_3 y = \beta_0 \sum x_3 + \beta_1 \sum x_1 x_3 + \beta_2 \sum x_2 x_3 + \beta_3 \sum x_3^2 + \beta_4 \sum x_3 x_4 + \beta_5 \sum x_3 x_5 + \beta_6 \sum x_3 x_6 \tag{107}$$

$$\sum x_4 y = \beta_0 \sum x_4 + \beta_1 \sum x_1 x_4 + \beta_2 \sum x_2 x_4 + \beta_3 \sum x_3 x_4 + \beta_4 \sum x_4^2 + \beta_5 \sum x_4 x_5 + \beta_6 \sum x_4 x_6 \tag{108}$$

$$\sum x_5 y = \beta_0 \sum x_5 + \beta_1 \sum x_1 x_5 + \beta_2 \sum x_2 x_5 + \beta_3 \sum x_3 x_5 + \beta_4 \sum x_4 x_5 + \beta_5 \sum x_5^2 + \beta_6 \sum x_5 x_6 \tag{109}$$

$$\sum x_6 y = \beta_0 \sum x_6 + \beta_1 \sum x_1 x_6 + \beta_2 \sum x_2 x_6 + \beta_3 \sum x_3 x_6 + \beta_4 \sum x_4 x_6 + \beta_5 \sum x_5 x_6 + \beta_6 \sum x_6^2 \tag{110}$$

This system of equations is reduced in matrix form as

$$[a][\beta]=[N] \tag{111}$$

So that (111) becomes

$$\begin{bmatrix} 5 & 289.0445 & 371.3874 & 258.7263 & 232.7891 & 226.8844 & 230.8244 \\ 289.0445 & 16959.33852 & 21719.4928 & 15206.6796 & 13707.2746 & 13365.9337 & 13618.7032 \\ 371.3874 & 21719.4928 & 27835.7194 & 19467.5377 & 17540.9822 & 17102.3993 & 17420.0556 \\ 258.7263 & 15206.6796 & 19467.5377 & 27835.7194 & 12295.7274 & 11990.0324 & 12219.0827 \\ 232.7891 & 13707.2746 & 17540.9823 & 12295.7274 & 11088.1440 & 10813.2953 & 11021.6778 \\ 226.8844 & 13365.9337 & 17102.3993 & 11990.0324 & 10813.2953 & 10545.3038 & 10749.0906 \\ 230.8244 & 13618.7032 & 17420.0556 & 12219.0827 & 11021.6778 & 10749.0906 & 10975.9821 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}$$

$$= \begin{bmatrix} 200 \\ 12061.7777 \\ 15355.4952 \\ 10849.051 \\ 9811.556 \\ 9575.3746 \\ 9782.9764 \end{bmatrix} \tag{112}$$

The solution of (112) by LU-decomposition gave

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \begin{bmatrix} -57.05665 \\ 3.416156 \\ -1.248045 \\ -0.000001 \\ -085594 \\ -0.082o83 \\ -0.00034547 \end{bmatrix}$$

So that (95) can be expressed as

$$T = 3.1615x_1 - 1.2480x_2 - 0.000001x_3 - 0.0855x_4$$

$$-0.0820x_8 - 0.0003x_6 - 57.0567 \quad (113)$$

5.2 Analysis of Model

The predicted values of the environmental boundary conditions with (113) are as in Colum 8 of Table 10.

Table 10. Prediction with model.

u_7	u_8	u_9	u_{10}	u_{11}	u_7	T_i	Tp
47.8089	64.2775	41.7453	36.55796	35.3769	35.1649	20	20.00316
52.8089	69.27747	46.74525	41.55796	40.37687	40.16488	30	30.00405
57.80887	74.27747	51.74525	46.55796	45.37687	45.16488	40	40.00479
62.80889	79.27747	56.74525	51.55796	50.37687	50.16488	50	50.006
67.80889	84.27747	61.74525	56.5576	55.37687	55.16488	60	60.00659

Table 11. Computations for Error analysis.

T_{iav} = 40 = average of values on column one of this table				
	T_i	T_p	$(T_i - T_{iav})^2$	$(T_i - T_P)^2$
	20	20.0032	400	9.9856E-06
	30	30.0041	100	1.6402E-05
	40	40.0048	0	2.2944E-05
	50	50.006	100	1.608E-05
	60	60.0066	400	4.3428E-05
sum	**200**	**200.0246**	**1000**	**0.00012884**

5.3 Computations for Fitness of Regression Model

The coefficient of determination, r^2 is expressed as

$$r^2 = \frac{S_t - S_r}{S_t} \quad (114)$$

while the Standard Error of Regression, S_{er} is expressed as

$$S_{er} = \sqrt{\frac{S_r}{n-(m+1)}} \tag{115}$$

So that by (113) and (114) and Table11

$$r^2 = \frac{1000 - 0.00010884}{1000} = 0.9999$$

$$S_{er} = \sqrt{\frac{0.00010884}{5-(1+1)}} = 0.006$$

6. 3-D PRESENTATION OF RESULTS

Table 12. Interior Temperature Distribution at Varyuig Ambient Condition.

n	20 °C	30 °C	40 °C	50 °C	60 °C
1	50	50	50	50	50
2	50	50	50	50	50
3	50	50	50	50	50
4	50	50	50	50	50
5	50	50	50	50	50
6	50	50	50	50	50
7	47.8089	52.8089	57.80887	62.80889	67.80889
8	64.2775	69.27747	74.27747	79.27747	84.27747
9	41.7453	46.74525	45.37687	56.74525	61.74525
10	36.55796	41.55796	45.16488	51.55796	56.5576
11	35.3769	40.37687	35.3768	50.37687	55.37687
12	35.169	40.16488	35.1649	50.16488	55.16488
13	20	30	40	50	60
14	20	30	40	50	60
15	20	30	40	50	60
16	20	30	40	50	60
17	20	30	40	50	60
18	20	30	40	50	60

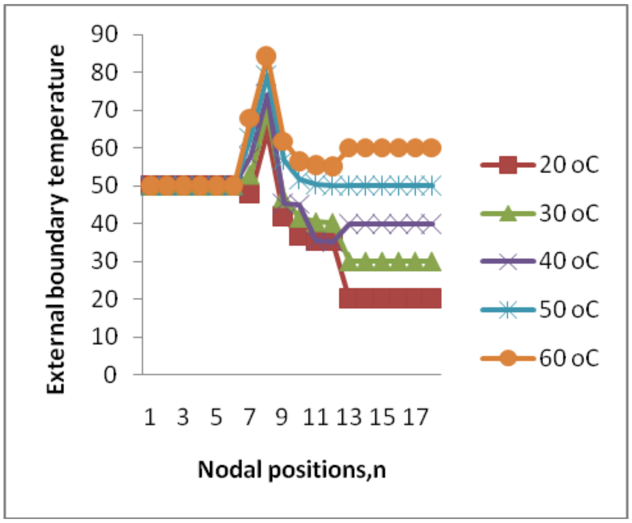

Figure 4. Line plot to depiction of temperature at nodal points using Table 12.

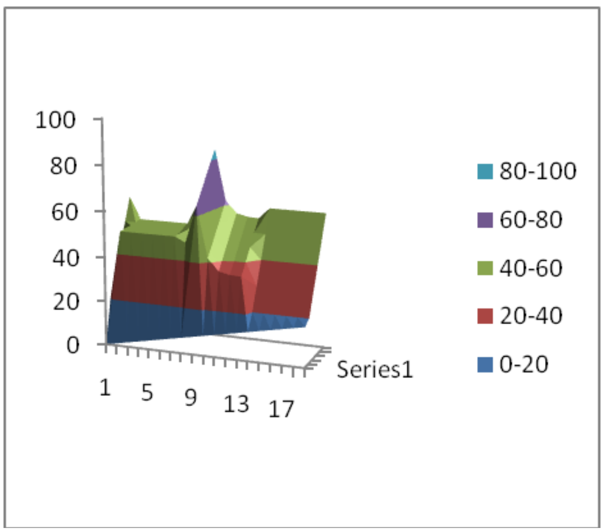

Figure 5. 3-D plot of temperature at nodal points using Table 10.

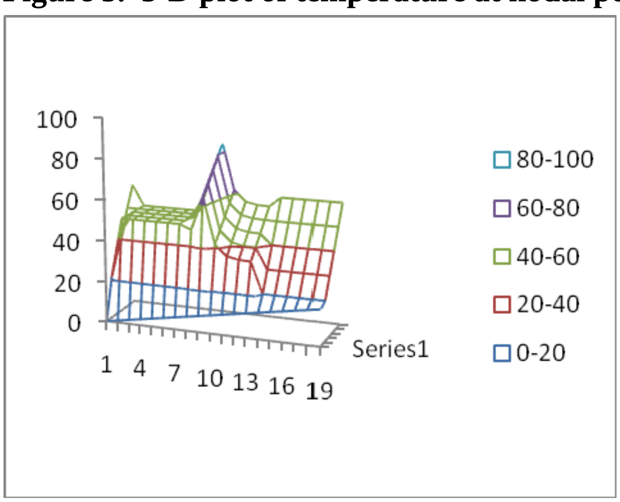

Figure 6. 3-D surface plot of temperature at nodal points using Table 10.

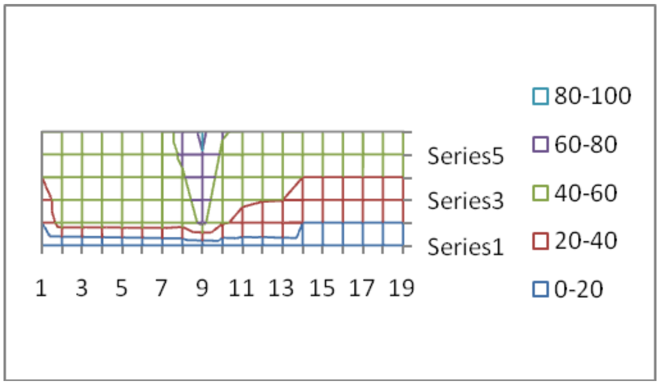

Figure 7. Contour plot of temperature at nodal points using Table 10.

7. DISCUSSION OF RESULTS.

This study however assumed the values of the ambient boundry and source temperatures of the region. The finite element model was used to evaluate the interior temperatures at fixed source temperature of 50^0C on the assumption of ambient temperatures of 20^0C, 30^0C, 40^0C, 50^0C and 60^0C. The finite element results were further used as experimental data in order to establish a predictive model to evaluate the error of FEM and as a model to serve whenever a design for similar abstraction is contemplated. Table10 was used to establish a regression model after a scatter plot that showed that the ambient temperature, T is linearly dependent on the temperature of the interior nodes. This is also confirmed by the goodness of fit described by the correlation coefficient of R and coefficient of determination R^2. The standard error of regression is also estimated through error analysis as 0.006.

The graphics of Figures 4, 6, 5and 7show clearly that a potential flow exists and that the values of function at node 8 is highest and increases as the boundary values increases.

The multiple linear regression fit of this study shows that the FEM approach is appropriate in the solution of functions of Laplace model as the error of the study is only 0.006 and the coefficient of determination is only 0.99999 while the correlation coefficient is 99.99%, showing that the variability of the results is due to linear relationship of the boundary temperatures with the source temperature.

The predictions of this study also show that the material of this region is not expected to be of temperature above 84.27747 ^{o}C and points at first location of failure when temperature is a critical design parameter. The quantity of heat attained by the material of this study can be quantified by the predictions of this study when appropriate relations are applied.

8. CONCLUSSIONS

The procedure of this study presented the basis for insulation design for solid, hollow or shell pipes in fluid transport design in oil and gas transport system. The finite element method evaluated the temperature distribution of the region to serve as a guide to quantify heat to the environment from the transit fluid. This study also presents an approximate procedure for processing polar systems as rectangular systems by using the circumference of the circular section as one dimensional independent variable and the difference between the inner and outer radius (thickness) as the second independent variable.

Above all the multiple linear regression fit of predictions of this study shows that the FEM approach is appropriate in the solution of functions of Laplace model as the error of the study is only 0.006 and the coefficient of determination is only 0.99999 while the correlation coefficient is 0.99999(99.99%), showing that the boundary temperature is linearly related to the interior or the source temperature.

The predictions of this study also show that the material of this region is not expected to be of temperature above 84.27747^{o}C and that the quantity of heat attained by the material of this study can be quantified when appropriate relations are applied.

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