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A Note on Directed 5-Cycles in Digraphs*

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ABSTRACT

In this note, it is proved that if $\alpha \geq 0.24817$, then any digraph on *n* vertices with minimum outdegree at least αn contains a directed cycle of length at most 5.

Keywords: Digraph; Directed Cycle

1. Introduction

Let $G = (V, E)$ be a digragh without loops or parallel edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the arc-set. In 1978, Caccetta and Häggkvist [1] made the following conjecture:

Conjecture 1.1 *Any digraph on n vertices with minimum outdegree at least r contains a directed cycle of length at most* $\lceil n/r \rceil$.

Trivially, this conjecture is true for $r = 1$, and it has been proved for $r = 2$ by Caccetta and Häggkvist [1], $r = 3$ by Hamildoune [2], $r = 4$ and $r = 5$ by Hoáng and Reed [3], $r < \sqrt{n/2}$ by Shen [4]. While the general conjecture is still open, some weaker statements have been obtained. A summary of results and problems related to the Caccetta-Häggkvist conjecture sees Sullivan [5].

For the conjecture, the case $r = n/2$ is trivial, the case $r = n/3$ has received much attention, but this special case is still open. To prove the conjecture, one may seek as small a constant α as possible such that any digraph on *n* vertices with minimum outdegree at least αn contains a directed triangle. The conjecture is that $\alpha = 1/3$. Caccetta and Häggkvist [1] obtained
$\alpha \leq \left(3-\sqrt{5}\right)/2 \approx 0.3819$, Bondy [6] showed
$\alpha \leq \left(2\sqrt{6}-3\right)/5 \approx 0.3797$, Shen [7] gave
$\alpha \leq 3-\sqrt{7} \approx 0.3542$, Hamburger, Haxell, and Kostochka [8] improved it to 0.35312. Hladký, Král' and Norin [9] further improved this bound to 0.3465. Namely, any digraph on *n* vertices with minimum out-degree at least 0.3465*n* contains a directed triangle. Very recently, Lichiardopol [10] showed that for $\beta \geq 0.343545$, any digraph on *n* vertices with both minimum out-degree and minimum in-degree at least βn contains a cycle of length at most 3.

In this note, we consider the minimum constant α such that any digraph on *n* vertices with minimum outdegree at least αn contains a directed cycle of length at most 5. The conjecture is that $\alpha = 1/5$. By refining the combinatorial techniques in [6,7,11], we prove the following result.

Theorem 1.2 *If* $\alpha \geq 0.24817$, *then any digraph on n vertices with minimum outdegree at least* αn *contains a directed cycle of length at most* 5.

2. Proof of Theorem 1.2

We prove Theorem 1.2 by induction on *n*. The theorem holds for $n \leq 5$ clearly. Now assume that the theorem holds for all digraphs with fewer than *n* vertices for $n \geq 5$. Let *G* be a digraph on *n* vertices with minimum outdegree at least αn. Suppose *G* contains no directed cycles with length at most 5. We can, without loss of generality, suppose that *G* is *r*-outregular, where $r = \lceil \alpha n \rceil$, that is, every vertex is of the outdegree *r* in *G*. We will try to deduce a contradiction. First we present some notations following [7].

For any $v \in V(G)$, let

$$N^+(v) = \{u \in V(G) : (v,u) \in E(G)\},$$

and $\deg^+(v) = |N^+(v)|$, the outdegree of v;

$$N^-(v) = \{u \in V(G) : (u,v) \in E(G)\},$$

and $\deg^-(v) = |N^-(v)|$, the indegree of v.

We say $\langle u, v, w \rangle$ a *transitive triangle* if

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$(u,v),(v,w),(u,w) \in E(G)$. The arc (u,v) is called the base of the transitive triangle.

For any $(u,v) \in E(G)$, let

$$P(u,v) = N^{+}(v) \setminus N^{+}(u),$$

and $p(u,v) = \left|N^{+}(v) \setminus N^{+}(u)\right|$, the number of induced 2-path with the first arc (u,v);

$$Q(u,v) = N^{-}(u) \setminus N^{-}(v),$$

and $q(u,v) = \left|N^{-}(u) \setminus N^{-}(v)\right|$, the number of induced 2-path with the last arc (u,v);

$$T(u,v) = N^{+}(u) \cap N^{+}(v),$$

and $t(u,v) = \left|N^{+}(u) \cap N^{+}(v)\right|$, the number of transitive triangles with base (u,v).

Lemma 2.1 *For any* $(u,v) \in E(G)$,

$$\begin{aligned} n &> r + (1-\alpha)r + (1-\alpha)^2 r + (1-\alpha)^3 t(u,v) \\ &\quad + \deg^{-}(v) + q(u,v). \end{aligned} \tag{1}$$

Proof: To prove this inequality, we consider two cases according to $t(u,v) = 0$ or $t(u,v) > 0$.

If $t(u,v) = 0$, then substituting it into (1) yields

$$n > r + (1-\alpha)r + (1-\alpha)^2 r + \deg^{-}(v) + q(u,v). \tag{2}$$

There exists some $w \in N^{+}(v)$ with outdegree less than αr in the subdigraph of G induced by $N^{+}(v)$ (Otherwise, this subdigraph would contain a directed 4-cycle by the induction hypothesis). Thus

$$\left|N^{+}(w) \setminus N^{+}(v)\right| \geq r - \alpha r.$$

Consider the subdigraph of G induced by $N^{+}(v) \cup N^{+}(w)$, by the induction hypothesis, some vertex $x \in N^{+}(v) \cup N^{+}(w)$ has outdegree less than $\alpha\left|N^{+}(v) \cup N^{+}(w)\right|$ in this subdigraph. Thus, the set of outneighbors of x not in $N^{+}(v) \cup N^{+}(w)$ satisfies

$$\begin{aligned} &\left|N^{+}(x) \setminus \left(N^{+}(v) \cup N^{+}(w)\right)\right| \\ &\geq r - \alpha\left|N^{+}(v) \cup N^{+}(w)\right| \\ &= r - \alpha\left(\left|N^{+}(v)\right| + \left|N^{+}(w) \setminus N^{+}(v)\right|\right) \\ &= (1-\alpha)r - \alpha\left|N^{+}(w) \setminus N^{+}(v)\right|, \end{aligned}$$

Since G has no directed 5-cycle, then $N^{+}(v)$, $N^{+}(w) \setminus N^{+}(v)$, $N^{+}(x) \setminus \left(N^{+}(v) \cup N^{+}(w)\right)$, $N^{-}(v)$ and $N^{-}(u) \setminus N^{-}(v)$ are pairwise-disjoint sets with cardinalities r, $\left|N^{+}(w) \setminus N^{+}(v)\right|$, $\left|N^{+}(x) \setminus \left(N^{+}(v) \cup N^{+}(w)\right)\right|$, $\deg^{-}(v)$ and $q(u,v)$, we have that

$$\begin{aligned} n &> r + \left|N^{+}(w) \setminus N^{+}(v)\right| + \left|N^{+}(x) \setminus \left(N^{+}(v) \cup N^{+}(w)\right)\right| \\ &\quad + \deg^{-}(v) + q(u,v) \\ &\geq r + (1-\alpha)r + (1-\alpha)\left|N^{+}(w) \setminus N^{+}(v)\right| \\ &\quad + \deg^{-}(v) + q(u,v) \\ &\geq r + (1-\alpha)r + (1-\alpha)^2 r + \deg^{-}(v) + q(u,v), \end{aligned}$$

Thus, the inequality (1) holds for $t(u,v) = 0$.

We now assume $t(u,v) > 0$. By the induction hypothesis, there is some vertex $w \in N^{+}(u) \cap N^{+}(v)$ that has outdegree less than $\alpha t(u,v)$ in the subdigraph of G induced by $N^{+}(u) \cap N^{+}(v)$, otherwise, this subdigraph would contain a directed 5-cycle. Also, w has not more than $p(u,v)$ outneighbors in the subdigraph of G induced by $N^{+}(v) \setminus N^{+}(u)$. Let $N^{+}(w) \setminus N^{+}(v)$ be the outneighbors of w which is not in $N^{+}(v)$. Noting that $t(u,v) = r - p(u,v)$, we have that

$$\begin{aligned} \left|N^{+}(w) \setminus N^{+}(v)\right| &\geq r - p(u,v) - \alpha t(u,v) \\ &= (1-\alpha) t(u,v). \end{aligned} \tag{3}$$

Because G has no directed triangle, all outneighbors of w are neither in $N^{+}(v)$ nor in $N^{-}(u) \setminus N^{-}(v)$. Consider the subdigraph of G induced by $N^{+}(v) \cup N^{+}(w)$, by the induction hypothesis, there is some vertex $x \in N^{+}(v) \cup N^{+}(w)$ that has outdegree less than $\alpha\left|N^{+}(v) \cup N^{+}(w)\right|$ in this subdigraph. Thus, the set of outneighbors of x not in $N^{+}(v) \cup N^{+}(w)$ satisfies

$$\begin{aligned} &\left|N^{+}(x) \setminus \left(N^{+}(v) \cup N^{+}(w)\right)\right| \\ &\geq r - \alpha\left|N^{+}(v) \cup N^{+}(w)\right| \\ &= r - \alpha\left(\left|N^{+}(v)\right| + \left|N^{+}(w) \setminus N^{+}(v)\right|\right) \\ &= (1-\alpha)r - \alpha\left|N^{+}(w) \setminus N^{+}(v)\right|, \end{aligned} \tag{4}$$

Since G has no directed 4-cycle, all outneighbors of w are neither in $N^{-}(v)$ nor in $N^{-}(u) \setminus N^{-}(v)$. Consider the subdigraph of G induced by

$$N^{+}(v) \cup N^{+}(w) \cup N^{+}(x),$$

by the induction hypothesis, there is some vertex

$$y \in N^{+}(v) \cup N^{+}(w) \cup N^{+}(x)$$

that has outdegree less than

$$\alpha\left|N^{+}(v) \cup N^{+}(w) \cup N^{+}(x)\right|$$

in this subdigraph. Thus, the set of outneighbors of y not in $N^{+}(v) \cup N^{+}(w) \cup N^{+}(x)$ satisfies

$$\begin{aligned}
&\left|N^+(y)\setminus\left(N^+(v)\cup N^+(w)\cup N^+(x)\right)\right| \\
&\ge r-\alpha\left|N^+(v)\cup N^+(w)\cup N^+(x)\right| \\
&= r-\alpha\left(\left|N^+(v)\cup N^+(w)\right|\right. \\
&\quad \left.+\left|N^+(x)\setminus N^+(v)\cup N^+(w)\right|\right) \\
&= (1-\alpha)r-\alpha\left|N^+(w)\setminus N^+(v)\right| \\
&\quad -\alpha\left|N^+(x)\setminus N^+(v)\cup N^+(w)\right|,
\end{aligned} \tag{5}$$

Because G has no directed cycle of length at most 5, then $N^+(v)$, $N^+(w)\setminus N^+(v)$,

$$N^+(x)\setminus\left(N^+(v)\cup N^+(w)\right),$$

$$N^+(y)\setminus(N^+(v)\cup N^+(w)\cup N^+(x),$$

$N^-(v)$ and $N^-(u)\setminus N^-(v)$ are pairwise-disjoint sets of cardinalities r, $\left|N^+(w)\setminus N^+(v)\right|$,

$$\left|N^+(x)\setminus\left(N^+(v)\cup N^+(w)\right)\right|,$$

$$\left|N^+(y)\setminus\left(N^+(v)\cup N^+(w)\cup N^+(x)\right)\right|,$$

$\deg^-(v)$ and $q(u,v)$, we have that

$$\begin{aligned}
n &> r+\left|N^+(w)\setminus N^+(v)\right| \\
&\quad +\left|N^+(x)\setminus\left(N^+(v)\cup N^+(w)\right)\right| \\
&\quad +\left|N^+(y)\setminus\left(N^+(v)\cup N^+(w)\cup N^+(x)\right)\right| \\
&\quad +\deg^-(v)+q(u,v)
\end{aligned}$$

Substituting (3), (4) and (5) into this inequalities yields

$$\begin{aligned}
n &> r+\left|N^+(w)\setminus N^+(v)\right| \\
&\quad +\left|N^+(x)\setminus\left(N^+(v)\cup N^+(w)\right)\right| \\
&\quad +\left|N^+(y)\setminus\left(N^+(v)\cup N^+(w)\cup N^+(x)\right)\right| \\
&\quad +\deg^-(v)+q(u,v) \\
&= r+|N^+(w)\setminus N^+(v)|+(1-\alpha)r \\
&\quad -\alpha\left|N^+(w)\setminus N^+(v)\right| \\
&\quad +(1-\alpha)\left|N^+(x)\setminus\left(N^+(v)\cup N^+(w)\right)\right| \\
&\quad +\deg^-(v)+q(u,v) \\
&\ge r+(1-\alpha)r+(1-\alpha)^2 r \\
&\quad +(1-\alpha)^2\left|N^+(w)\setminus N^+(v)\right|+\deg^-(v)+q(u,v) \\
&\ge r+(1-\alpha)r+(1-\alpha)^2 r \\
&\quad +(1-\alpha)^3 t(u,v)+\deg^-(v)+q(u,v)
\end{aligned}$$

as desired, and so the lemma follows.

Connect to Proof of Theorem 1.2

Recalling that $t(u,v)=r-p(u,v)$, we can rewrite the inequality (1) as

$$\begin{aligned}
&\left(3\alpha-3\alpha^2+\alpha^3\right)t(u,v) \\
&>\left(4-3\alpha+\alpha^2\right)r-n+\deg^-(v)+q(u,v)-p(u,v).
\end{aligned} \tag{6}$$

Summing over all $(u,v)\in E(G)$, we have that

$$\sum_{(u,v)\in E(G)} t(u,v)=t, \tag{7}$$

where t is the number of transitive triangles in G, and

$$\sum_{(u,v)\in E(G)}\left(4-3\alpha+\alpha^2\right)r-n=nr\left[\left(4-3\alpha+\alpha^2\right)r-n\right]. \tag{8}$$

By Cauchy's inequality and the first theorem on graph theory (see, for example, Theorem 1.1 in [12]), we have that

$$\begin{aligned}
\sum_{(u,v)\in E(G)}\deg^-(v) &= \sum_{v\in V(G)}\left(\deg^-(v)\right)^2 \\
&\ge \frac{1}{n}\left(\sum_{v\in V(G)}\deg^-(v)\right)^2 = nr^2,
\end{aligned}$$

that is

$$\sum_{(u,v)\in E(G)}\deg^-(v)\ge nr^2. \tag{9}$$

Because $\sum_{(u,v)\in E(G)} p(u,v)$ and $\sum_{(u,v)\in E(G)} q(u,v)$ are both equal to the number of induced directed 2-paths in G, it follows that

$$\sum_{(u,v)\in E(G)} p(u,v)=\sum_{(u,v)\in E(G)} q(u,v). \tag{10}$$

Summing over all $(u,v)\in E(G)$ for the inequality (6) and substituting inequalities (7)-(10) into that inequality yields,

$$\left(3\alpha-3\alpha^2+\alpha^3\right)t>\left(5-3\alpha+\alpha^2\right)nr^2-n^2r. \tag{11}$$

Noting that $t\le n\binom{r}{2}$ (see Shen [7]), we have that

$$\begin{aligned}
t\left(3\alpha-3\alpha^2+\alpha^3\right) &\le n\binom{r}{2}\left(3\alpha-3\alpha^2+\alpha^3\right) \\
&< \frac{nr^2}{2}\left(3\alpha-3\alpha^2+\alpha^3\right).
\end{aligned} \tag{12}$$

Combining (11) with (12) yields

$$\left(5-3\alpha+\alpha^2\right)nr^2-n^2r<\frac{nr^2}{2}\left(3\alpha-3\alpha^2+\alpha^3\right). \tag{13}$$

Dividing both sides of the inequality (13) by $\frac{nr^2}{2}$, and noting that $r = \lceil \alpha n \rceil \geq \alpha n$, we get

$$2\left(5-3\alpha+\alpha^2\right)-\frac{2}{\alpha}<\left(3\alpha-3\alpha^2+\alpha^3\right),$$

that is

$$\alpha^4-5\alpha^3+9\alpha^2-10\alpha+2>0.$$

We obtain that $\alpha < 0.248164$, a contradiction. This completes the proof of the theorem.

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