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Fourier-Bessel Expansions with Arbitrary Radial Boundaries

Muhammad A. Mushref
P. O. Box 9772*, Jeddah, Saudi Arabia*
E-mail: *mmushref@yahoo.co.uk*
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Abstract

Series expansion of single variable functions is represented in Fourier-Bessel form with unknown coefficients. The proposed series expansions are derived for arbitrary radial boundaries in problems of circular domain. Zeros of the generated transcendental equation and the relationship of orthogonality are employed to find the unknown coefficients. Several numerical and graphical examples are explained and discussed.

Keywords: Fourier-Bessel Analysis, Boundary Value Problems, Orthogonality of Bessel Functions

1. Introduction

Several boundary value problems in the applied sciences are frequently solved by expansions in cylindrical harmonics with infinite terms. Problems of circular domain with rounded surfaces often generate infinite series of Bessel functions of the first and second types with unknown coefficients. In this case, the intention is to find the series coefficients which should satisfy the boundary conditions.

The subject of Fourier-Bessel series expansions was investigated and examined in many texts [1-10]. Nearly all of them has derived cylindrical harmonics expansions in $J_0(r)$ for the interval [0, a] only, where $J_0(r)$ is the Bessel function of the first kind with order zero and argument r [8]. The existence of the origin point excludes $Y_0(r)$, Bessel function of the second kind with order zero and argument r, because it goes to negative infinity as r approaches zero [9]. Both $J_0(r)$ and $Y_0(r)$ are shown plotted in **Figure 1.**

In many other problems in the applied sciences, the interval of expansion is found to be [a, b] such that $a, b \in \mathbf{R}$. An example of this could be a hollow cylinder in heat conduction problems or a circular band in vibrations analysis solved in the cylindrical coordinate system. In this case, cylindrical harmonics expansions in both $J_0(r)$ and $Y_0(r)$ are necessary.

In this paper, the derivation of cylindrical harmonics expansion of a single variable function in [a, b] in both $J_0(r)$ and $Y_0(r)$ is solved. In accordance with the boundaries at $r = a$ and $r = b$, zeros of the obtained transcendental equation are first calculated. As shown in **Figure 2**, the solution region is for $a \le r \le b$ where the desired series expansions are forced to be zero at $r = a$ and $r = b$ respectively. Unknown coefficients are then found and the complete series expansion can be achieved.

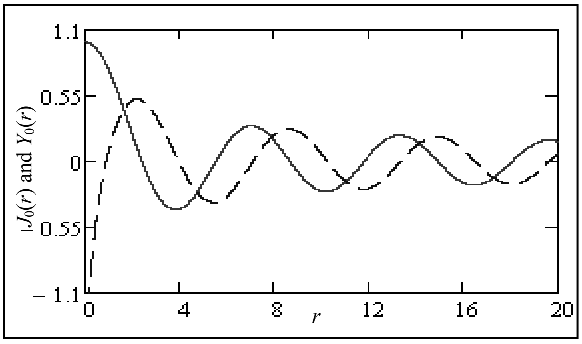

Figure 1. Equation (6), —— $J_0(r)$, – – $Y_0(r)$.

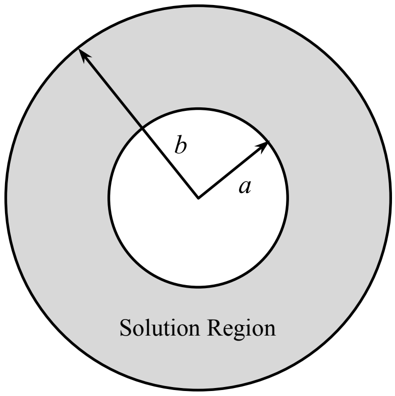

Figure 2. The solution region in radial boundaries.

2. Formulation and Solution

The Bessel differential equation of order zero is well known as [1, 4]:

$$r\frac{d^2}{dr^2}f(r)+\frac{d}{dr}f(r)+\alpha^2 rf(r)=0 \quad (1)$$

$\forall\ \alpha$ and $r \in \mathbf{R}$ and $a \le r \le b$.

The general solution to Equation (1) for real values of α is known to be [2, 3]:

$$f(r)=\sum_{n=0}^{\infty} A_n J_0(\alpha r)+B_n Y_0(\alpha r) \quad (2)$$

As in Equation (1), the assumed boundary conditions at $r = a$ and $r = b$ are of Dirichlet type as $f(a) = 0$ and $f(b) = 0$ respectively. Both A_n and B_n are then related as:

$$A_n=-\frac{Y_0(\alpha a)}{J_0(\alpha a)}B_n \quad (3)$$

$$A_n=-\frac{Y_0(\alpha b)}{J_0(\alpha b)}B_n \quad (4)$$

Going after the elimination method, the transcendental equation can be obtained as:

$$J_0(\alpha a)Y_0(\alpha b)-J_0(\alpha b)Y_0(\alpha a)=0 \quad (5)$$

In order for Equation (5) to be satisfied, there exist many zeros or values of α to be calculated. Thus, in all former and coming equations α can be replaced by α_n which are the zeros obtained from the transcendental equation $\forall\ n \in \mathbf{I}$. That is:

$$J_0(\alpha_n a)Y_0(\alpha_n b)-J_0(\alpha_n b)Y_0(\alpha_n a)=0 \quad (6)$$

The orthogonality feature of Bessel functions can be applied to Equation (2) by multiplying both sides by $r\left[A_m J_0(\alpha_m r)+B_m Y_0(\alpha_m r)\right]$ and integrating it over all possible values of r from a to b as:

$$\sum_{n=0}^{\infty}\int_a^b rC_n(r)C_m(r)dr=\int_a^b rC_m(r)f(r)dr \quad (7)$$

where,

$$C_m(r)=A_m J_0(\alpha_m r)+B_m Y_0(\alpha_m r) \quad (8)$$

$$C_n(r)=A_n J_0(\alpha_n r)+B_n Y_0(\alpha_n r) \quad (9)$$

The terms under the summation in the left side of Equation (7) are zeros for all values of $m \neq n$ [5, 6, 7]. Hence, Equation (7) can be simplified to:

$$\int_a^b rC_n(r)\{C_n(r)-f(r)\}dr=0 \quad (10)$$

Either Equation (3) or (4) can help. Using Equation (3) we can obtain the B_n coefficients as:

$$B_n=\frac{\int_a^b rS_0(\alpha_n r)f(r)dr}{\int_a^b r\left[S_0(\alpha_n r)\right]^2 dr} \quad (11)$$

where, $S_0(\alpha_n r)$ is given by:

$$S_0(\alpha_n r)=Y_0(\alpha_n r)-\frac{Y_0(\alpha_n a)}{J_0(\alpha_n a)}J_0(\alpha_n r) \quad (12)$$

By Equation (3) or (4), the A_n coefficients can also be found. Once the coefficients A_n and B_n are calculated, the function $f(r)$ can be expanded as in Equation (2).

3. Numerical Examples

The transcendental expression in Equation (6) shows a gradual decay as α increases which mean small magnitudes between high zeros. This leads to the convergence of the series in Equation (2) above as n increases. As a consequence, a finite number of terms in Equation (2) can be sufficient for numerical approximations.

The zeros are first evaluated using the transcendental cross product Bessel functions equation for the interval [a, b]. A graph of Equation (6) is shown in **Figure 3** for the solution regions [0.65, 2.5] and [0.65, 5]. **Table 1** shows the first 50 zeros of Equation (6) for a = 0.65 and b = 2.5. Zeros obtained from the transcendental equation changes according to the values of a and b assumed for the solution region. The data presented in **Table 1** indicates that the calculated zeros are not periodic and should be calculated using a proper numerical technique.

Let's assume that the function $f(r)$ to be expanded as in Equation (2) is sin(r) with a radial solution region in [0.65, 2.5]. The coefficients B_n can be evaluated from Equation (11) and the A_n coefficients are then obtained by Equation (3). Both coefficients are shown in **Tables 2** and **3** respectively for n = 0 to 49.

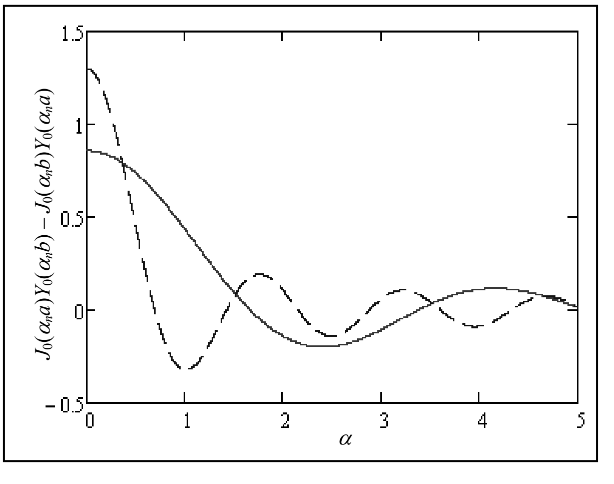

Figure 3. Equation (6), —— [0.65, 2.5], – – [0.65, 5].

Many variations can be noticed for the numerical values of A_n and B_n with a general absolute scale of < 1 except for $B_0 = 2.328$. Some coefficients are in the order of $\times 10^{-3}$ meaning that their associated terms are very small such as B_4 and A_{31} in **Tables 2** and **3** respectively.

The function sin(r) and its approximate expansions are plotted in **Figure 4**. Summation over the first 10 terms produced an acceptable estimation in the interval [0.65, 2.5] with some apparent oscillations around the exact function. An improved approximate expansion is also plotted for n = 0 to 49 with less fluctuations in the same radial domain.

Table 1. First fifty zeros of Equation (6) in [0.65, 2.5].

n	α_n	n	α_n	n	α_n	n	α_n	n	α_n
0	1.663	10	18.676	20	35.659	30	52.642	40	69.624
1	3.376	11	20.374	21	37.358	31	54.34	41	71.322
2	5.08	12	22.073	22	39.056	32	56.038	42	73.02
3	6.782	13	23.771	23	40.754	33	57.736	43	74.718
4	8.4815	14	25.47	24	42.452	34	59.434	44	76.416
5	10.182	15	27.168	25	44.151	35	61.133	45	78.115
6	11.881	16	28.866	26	45.849	36	62.831	46	79.813
7	13.579	17	30.564	27	47.547	37	64.529	47	81.511
8	15.279	18	32.263	28	49.245	38	66.227	48	83.209
9	16.977	19	33.961	29	50.943	39	67.925	49	84.907

Table 2. First fifty B_n for $f(r)$ = sin(r) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	2.328	10	0.154	20	–0.300	30	–0.114	40	0.206
1	–0.101	11	–0.138	21	7.1E-3	31	0.084	41	1.3E-3
2	–0.703	12	0.228	22	0.267	32	–0.123	42	–0.205
3	0.234	13	0.064	23	–0.082	33	–0.047	43	0.057
4	–4.8E-3	14	–0.385	24	–0.030	34	0.250	44	0.042
5	–0.181	15	0.048	25	0.087	35	–0.025	45	–0.068
6	0.455	16	0.231	26	–0.212	36	–0.173	46	0.148
7	0.030	17	–0.110	27	–0.024	37	0.074	47	0.024
8	–0.478	18	0.082	28	0.272	38	–0.036	48	–0.212
9	0.105	19	0.081	29	–0.054	39	–0.061	49	0.037

Table 3. First fifty A_n for $f(r)$ = sin(r) in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–0.475	10	0.424	20	0.126	30	–0.242	40	–0.109
1	0.462	11	–0.016	21	–0.102	31	2.1E-3	41	0.074
2	–0.547	12	–0.346	22	0.159	32	0.229	42	–0.099
3	–0.114	13	0.111	23	0.052	33	–0.067	43	–0.044
4	0.675	14	0.021	24	–0.297	34	–0.036	44	0.218
5	–0.092	15	–0.110	25	0.034	35	0.075	45	–0.019
6	–0.338	16	0.279	26	0.193	36	–0.174	46	–0.160
7	0.170	17	0.025	27	–0.087	37	–0.024	47	0.064
8	–0.143	18	–0.333	28	0.054	38	0.236	48	–0.023
9	–0.111	19	0.070	29	0.069	39	–0.044	49	–0.057

Table 4. First fifty B_n for $f(r)$ = cos(r) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	–0.129	10	–0.067	20	0.131	30	0.050	40	–0.090
1	0.338	11	0.571	21	–0.030	31	–0.351	41	–5.5E-3
2	0.286	12	–0.099	22	–0.116	32	0.053	42	0.089
3	–0.919	13	–0.264	23	0.342	33	0.195	43	–0.237
4	2.0E-3	14	0.167	24	0.013	34	–0.109	44	–0.018
5	0.732	15	–0.199	25	–0.364	35	0.105	45	0.281
6	–0.196	16	–0.100	26	0.092	36	0.075	46	–0.064
7	–0.122	17	0.457	27	0.100	37	–0.306	47	–0.100
8	0.207	18	–0.036	28	–0.118	38	0.016	48	0.092
9	–0.433	19	–0.338	29	0.223	39	0.256	49	–0.153

In addition, $f(r)$ = cos(r) is expanded as in Equation (2) and the first fifty coefficients are listed in **Tables 4** and **5** for the B_n and A_n respectively. Similar to the sin(r), the cos(r) coefficients go through several variations with a general absolute scale of < 1 except $A_1 = -1.550$. Also, only four coefficients are in the order of $\times 10^{-3}$ implying that their related terms in the series are extremely small such as B_4 and A_{41} in **Tables 4** and **5** respectively.

Table 5. First fifty A_n for $f(r) = \cos(r)$ in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	0.026	10	–0.184	20	–0.055	30	0.105	40	0.047
1	–1.550	11	0.068	21	0.422	31	–8.9E-3	41	–0.306
2	0.222	12	0.150	22	–0.069	32	–0.100	42	0.043
3	0.448	13	–0.461	23	–0.218	33	0.279	43	0.182
4	–0.287	14	–9.1E-3	24	0.129	34	0.016	44	–0.095
5	0.371	15	0.455	25	–0.139	35	–0.314	45	0.080
6	0.145	16	–0.121	26	–0.084	36	0.076	46	0.070
7	–0.696	17	–0.105	27	0.362	37	0.100	47	–0.268
8	0.062	18	0.145	28	–0.023	38	–0.103	48	0.010
9	0.458	19	–0.289	29	–0.286	39	0.182	49	0.235

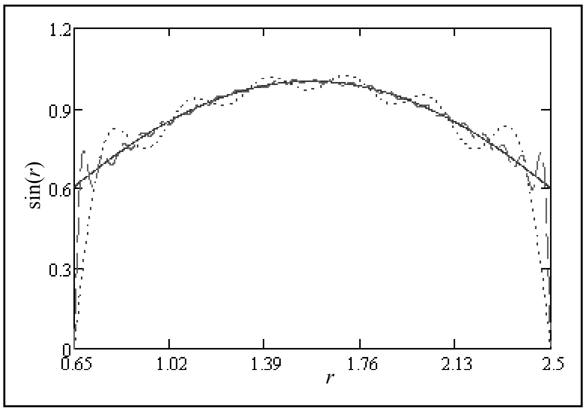

Figure 4. —— sin(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

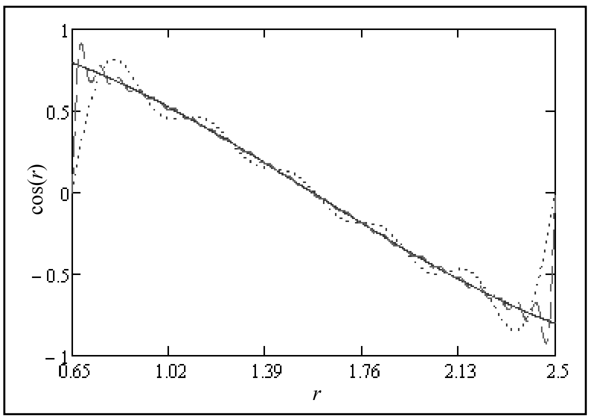

Figure 5. —— cos(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

The function cos(r) and its estimated expansions are shown plotted in **Figure 5**. Finite summation over the first 10 terms generated a satisfactory estimation in the interval [0.65, 2.5] with several obvious oscillations close to the exact function. A better approximate expansion is also plotted for n = 0 to 49 with less fluctuations in the same solution region.

The calculated coefficients for the function e^r are also shown in **Tables 6** and **7** for B_n and A_n respectively. Apparently, the coefficients swing around the exact values with an absolute level of > 1 or < 1.

The greatest values in **Tables 6** and **7** are found as B_0 = 13.852 and A_1 = 11.499. In addition, no coefficients are calculated in the order of $\times 10^{-3}$ implying that all coefficients are to be included in the series expansion.

The function exp(r) and its estimated expansions are shown plotted in **Figure 6** in [0.65, 2.5]. A satisfactory estimation of a finite summation over the first 10 terms are generated with several oscillations close to the exact function. A good approximated expansion is also plotted for n = 0 to 49 with fewer variations in the same solution region.

The last numerical example to be discussed is the square function expressed as:

$$f(r) = \begin{cases} 1 & 1.26 \le r \le 1.88 \\ -1 & otherwise \end{cases} \tag{13}$$

The calculated B_n and A_n coefficients for this function are shown in **Tables 8** and **9** respectively. Similar to former expansions, both coefficients vary about the exact values of Equation (13). The B_n coefficients have a general absolute level of < 1 except B_2, B_8, B_{14}, B_{20} and B_{26} that have an absolute scale of > 1. Furthermore, the A_n coefficients show an absolute level of < 1 except the absolute values of A_2, A_{32}, A_{38} and A_{44} that are > 1. Some B_n and A_n coefficients are calculated in the order of $\times 10^{-3}$ like A_0 or

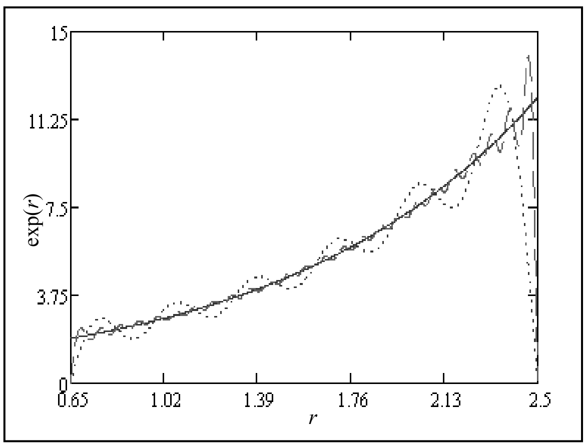

Figure 6. —— exp(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

Table 6. First fifty B_n for $f(r) = \exp(r)$ in [0.65, 2.5].

n	B_n	N	B_n	n	B_n	n	B_n	n	B_n
0	13.852	10	2.217	20	–4.350	30	–1.660	40	2.987
1	–2.506	11	–5.266	21	0.274	31	3.254	41	0.051
2	–9.361	12	3.298	22	3.873	32	–1.780	42	–2.971
3	8.069	13	2.443	23	–3.169	33	–1.813	43	2.202
4	–0.068	14	–5.566	24	–0.433	34	3.618	44	0.602
5	–6.632	15	1.841	25	3.372	35	–0.971	45	–2.611
6	6.506	16	3.343	26	–3.078	36	–2.505	46	2.140
7	1.113	17	–4.227	27	–0.931	37	2.841	47	0.932
8	–6.867	18	1.182	28	3.937	38	–0.525	48	–3.072
9	3.985	19	3.127	29	–2.069	39	–2.371	49	1.419

Table 7. First fifty A_n for $f(r) = \exp(r)$ in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–2.828	10	6.108	20	1.821	30	–3.511	40	–1.575
1	11.499	11	–0.625	21	–3.915	31	0.083	41	2.841
2	–7.286	12	–4.992	22	2.304	32	3.317	42	–1.431
3	–3.934	13	4.263	23	2.019	33	–2.586	43	–1.691
4	9.496	14	0.304	24	–4.301	34	–0.526	44	3.168
5	–3.360	15	–4.213	25	1.292	35	2.912	45	–0.747
6	–4.824	16	4.033	26	2.799	36	–2.523	46	–2.318
7	6.376	17	0.969	27	–3.353	37	–0.925	47	2.490
8	–2.059	18	–4.814	28	0.782	38	3.423	48	–0.336
9	–4.216	19	2.675	29	2.650	39	–1.690	49	–2.185

Table 8. First fifty B_n for Equation (13) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	0.026	10	1.64E-3	20	1.527	30	–0.021	40	–0.051
1	0.1	11	0.081	21	–5E-3	31	–0.404	41	–3E-4
2	3.515	12	0.031	22	–0.025	32	0.614	42	–0.037
3	–0.516	13	–0.205	23	0.042	33	–9E-3	43	–0.329
4	–3E-4	14	1.968	24	–5E-3	34	–0.05	44	–0.201
5	0.105	15	–0.053	25	–0.37	35	8E-3	45	–0.07
6	0.048	16	–9.1E-3	26	1.072	36	–0.033	46	–0.048
7	–0.076	17	0.061	27	6E-3	37	–0.388	47	–3E-3
8	2.447	18	0.013	28	–0.037	38	0.179	48	–0.037
9	–0.17	19	–0.305	29	0.025	39	–0.039	49	–0.228

Table 9. First fifty A_n for Equation (13) in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–5E-3	10	4.5E-3	20	–0.639	30	–0.045	40	0.027
1	–0.457	11	9.6E-3	21	0.069	31	–0.01	41	–0.016
2	2.735	12	–0.047	22	–0.015	32	–1.144	42	–0.018
3	0.252	13	–0.375	23	–0.027	33	–0.013	43	0.253
4	0.039	14	–0.108	24	–0.052	34	7.2E-3	44	–1.056
5	0.053	15	0.121	25	–0.142	35	–0.024	45	–0.02
6	–0.035	16	–0.011	26	–0.975	36	–0.033	46	0.052
7	–0.433	17	–0.014	27	0.02	37	0.126	47	–6.8E-3
8	0.734	18	–0.053	28	–7.3E-3	38	–1.168	48	–4E-3
9	0.18	19	–0.261	29	–0.032	39	–0.027	49	0.352

in the order of $\times 10^{-4}$ such as B_{41} indicating that their associated terms in the series are very small.

The function expressed by Equation (13) and its approximate expansions are plotted in **Figure 7**. Summation over the first 10 terms produced an acceptable estimation in the interval [0.65, 2.5] with some noticeable oscillations around the exact function. A better approximate expansion is also plotted for $n = 0$ to 49 with less fluctuations in the same radial domain.

In all graphical plots previously shown, the curves return to zero at the assumed boundaries $a = 0.65$ and $b = 2.5$. In addition, accuracy of the expanded curves may appear better as n increases due to larger number of terms involved in the series and less fluctuations seen around the exact values.

4. Conclusions

Functions were expanded as a Fourier-Bessel series summation in both $J_0(r)$ and $Y_0(r)$. A finite series expan-

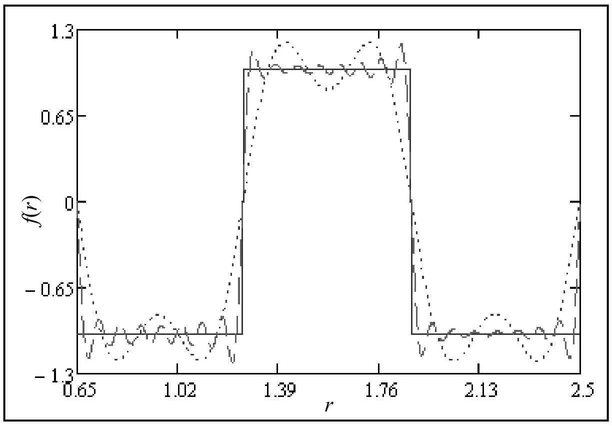

Figure 7. **—— Equation (13), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.**

sion was obtained for arbitrary radial boundaries in [a, b]. Coefficients were found by calculating the zeros of the transcendental equation and by employing the relationship of orthogonality. A number of examples were numerically and graphically discussed.

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