The Distribution of the Value of the Firm and Stochastic Interest Rates

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ABSTRACT

The time evolution of the value of a firm is commonly modeled by a linear, scalar stochastic differential equation (SDE) of the type $dV_t = r_t V_t dt + \sigma_v(t) V_t dW_v$ where the coefficient r_t in the drift term denotes the (exogenous) stochastic short term interest rate and $\sigma_v(t)$ is the given volatility of the value process. In turn, the dynamics of the short term interest rate, dr_t, are modeled by a scalar SDE. It is shown that V_t exhibits a lognormal distribution when r_t is a normal/Gaussian process defined by a common variety of narrow sense linear SDEs. The results can be applied to different financial situations where modeling value of the firm is critical. For example, with the context of the structural models, using this result one can readily compute the probability of default of a firm.

Keywords: Alue of Firm; Lognormal; Interest Rate Process

1. Introduction

Modeling the value of the firm is one of the more important research topics in finance. The value of an unlevered firm is the value of expected future cash flows discounted at a rate appropriate for an all-equity firm whereas the value of a levered firm is commonly expressed as the value of an unlevered firm plus the gain from leverage due to a tax shield provided by the debt. Including business disruption costs, the optimal capital structure can then be characterized as a trade-off between the interest tax shield and disruption costs. Recent analysis by Hackbarth, Hennessy and Leland [1] extends this line of research by examining an optimal mixture of debt; that is, the optimal mixture of bank debt and market debt (bonds).

Improved models for value of the firm are potentially useful in several contexts. For one example, consider models of credit spreads. Leland and Toft [2] develop an ambitious model of firm value that addresses optimal capital structure, optimal debt maturity, and the term structure of credit spreads. Recently, Qi [3] has modified the Leland and Toft [2] model by setting the lower bankruptcy boundary to be a fraction of bond face value. The importance of good structural models for credit spreads has been enhanced with the growth of credit derivatives and the credit crisis of 2007 and 2008. More specifically, notional amounts of credit derivatives grew by over 100%

for every year from 2004 through 2006. At the end of 2006, there was 34.5 trillion outstanding (see Saha-Bubna and Barrett [4]). The weakened credit quality of many financial firms in 2007 and 2008 caused high volatility in equity markets and, also, large changes in the value of credit spreads and credit default swaps.

Our purpose is to derive distributions of V_t whose evolution critically depends on the models for the short term interest rate process, r_t. Models for r_t can be broadly classified as (a) general linear and (b) non linear models. General linear models are also popularly known as affine models. In this context, we refer to Duffie, Filipovic and Schachermayer [5]; Duffie and Singleton [6]; and Lamberton and Lapeyre [7]. In this paper we are particularly interested in a special class of the general linear models called narrow sense linear models described by Arnold [8].

The next section describes the processes for value of the firm and short term interest rates. Next, we discuss a general framework for the solution of the distribution of V_t. Then, we describe solutions in the cases where r_t processes are *narrow sense* linear. Such r_t processes are quite popular for models of credit risk. The shapes of the V_t distributions are shown to be sensitive to such parameters as the correlation between the V_t and r_t processes. For example, a positive correlation displays a V_t distribution with fatter tails than one with negative

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correlation.

2. The Processes for Value of the Firm and Interest Rates

The time evolution of the value, V_t, of a firm is routinely modeled under the risk neutral measure by a linear, scalar, stochastic differential equation (SDE)

$$\frac{\mathrm{d}V_t}{V_t} = r_t \mathrm{d}t + \sigma_v(t)\mathrm{d}W_v, \qquad (1.1)$$

where the instantaneous drift r_t denotes the (exogenous) stochastic variable known as the short-term interest rate process and $\sigma_v(t)$ is a deterministic function representing the instantaneous volatility as in Acharya and Carpenter [9].

This general form of value process has been used in numerous important structural models of credit risk. For example, see Merton [10] and Acharya and Carpenter [9] where any dividends and coupon payments, outflows of γ from the firm to investors, are subtracted from the r_t drift term. Many firms do not pay dividends and our model is one of zero coupon debt so that a γ of zero is reasonable. We note that Longstaff and Schwartz [11] similarly have a zero γ.

The V_t drift of r_t indicates our model is risk neutral. We could assume different firms have different drifts due to such things as different expected returns in their industry as well as different riskiness of assets and future projects. However, such an assumption is arbitrary and yields a model that is not arbitrage free. We believe it is much more theoretically credible to posit a risk neutral, arbitrage free model.

The dynamics of the short term interest rate are modeled, under the same risk neutral measure, by a (scalar) SDE of the type

$$\mathrm{d}r_t = \alpha(r_t, t)\mathrm{d}t + \sigma(r_t, t)\mathrm{d}W_r, \qquad (1.2)$$

where the instantaneous drift, $\alpha(r_t, t)$ and the volatility, $\sigma(r_t, t)$ are smooth functions. It is further assumed that the Wiener increment processes $\mathrm{d}W_v$ and $\mathrm{d}W_r$ are correlated; that is,

$$E\left[(\mathrm{d}W_v)(\mathrm{d}W_r)\right] = \rho\mathrm{d}t \qquad (1.3)$$

with $|\rho| \le 1$. It is worth noting that in this set up the flow of information is only one way — r_t affects V_t and not vice versa. By combining several well known results from the literature, in this paper we characterize the distribution of the value process V_t for different choices of the r_t processes.

All the known stochastic interest rate models can be broadly classified into two classes—single factor models (SFMs) and multi-factor models (MFMs). We refer to Cairns [12] and Privault [13] for details. In this paper, we are primarily interested in the SFMs. These SFMs can be divided into linear and nonlinear models. Following Arnold [8], linear models can be further subdivided into two subclasses. The SFM in (1.2) is called a narrow sense linear model if

$$\alpha(r, t) = a_1(t)r_t + a_2(t), \qquad (1.4)$$

and

$$\sigma(r, t) = \sigma_r(t). \qquad (1.5)$$

A general linear model, on the other hand, has $\alpha(r, t)$ in the form (1.4) and

$$\sigma(r, t) = b_1(t)r_t + b_2(t), \qquad (1.6)$$

where $a_i(t)$, $b_i(t)$, $i = 1, 2$ and $\sigma_r(t)$ are smooth functions of time t. The general linear models are also known as affine models as in Duffie, Filipovic, and Schachermayer [5]; Duffie and Singleton [6]; and Lamberton and Lapeyre [7]. The SFM in (1.2) is called a nonlinear model if either $\alpha(r_t, t)$ and/or $\sigma(r_t, t)$ are nonlinear functions of the short rate r_t.

Refer to **Tables 1(a)-(c)** for examples of these models. The narrow sense linear models of Merton [14], Vasicek [15], Ho and Lee [16], and Hull and White [17] are special cases of the Heath, Jarrow and Morton [18] model and define normal/Gaussian processes.

We first solve the scalar SDE (1.2) for r_t, and using it in (1.1), we then recover V_t. It is well known that V_t is a lognormal process when $r_t \equiv r$ a constant. See Kloeden and Platen [19]. We extend this result by first showing that V_t also inherits this lognormal distribution where r_t is a normal process defined by the narrow sense linear models in **Table 1**.

This problem of quantifying the probability distribution of V_t is critical to credit risk analysis. For a review of various approaches to credit risk refer to the books by Duffie and Singleton [6], Bielecki and Rutkowski [20], and Jarrow *et al.* [21]. Clearly computation of the default probability in structural models requires knowledge of the probability distribution of V_t contingent on the chosen model for the interest rate.

3. A framework for the Solution

In this section we develop a framework for solving (1.1)-(1.2). Setting $g_t = \ln V_t$ and applying Ito's lemma, Equation (1.1) becomes.

$$\mathrm{d}g_t = \left(r_t - \frac{1}{2}\sigma_v^2(t)\right)\mathrm{d}t + \sigma_v(t)\mathrm{d}W_{v,t}. \qquad (2.1)$$

See Kloeden and Platen [19].

Setting $\mathrm{d}W_{r,t} = \mathrm{d}W_{1,t}$ and
$\mathrm{d}W_{v,t} = \rho\mathrm{d}W_{1,t} + \sqrt{1-\rho^2}\mathrm{d}W_{2,t}$ (Shreve [22]) we can re-

Table 1. Alternative Models of r_t. The single factor model in (1.2) is called a narrow sense linear model if $\alpha(r_t,t) = a_1(t)r_t + a_2(t)$ and $\sigma(r_t,t) = \sigma_r(t)$. In contrast, the model is called an affine model or a general linear model if $\alpha(r,t)$ is of the above form and $\sigma(r_t,t) = b_1(t)r_t + b_2(t)$. The model is called nonlinear if either $\alpha(r_t,t)$ and/or $\sigma(r_t,t)$ are nonlinear functions of the short rate r_t. (a) Narrow sense linear models; (b) General linear or affine models; (c) Nonlinear models.

(a)		
Merton [14]	$dr_t = \theta dt + \sigma_r dW_r$	r_t - Gaussian
Vasicek [15]	$dr_t = (\theta - cr_t)dt + \sigma_r dW_r$	r_t - Gaussian
Ho-Lee [16]	$dr_t = \theta(t)dt + \sigma_r dW_r$	r_t - Gaussian
Hull and White [17]	$dr_t = (\theta(t) - cr_t)dt + \sigma_r dW_r$	r_t - Gaussian
Hull [23]	$dr_t = (\theta(t) - c(t)r_t)dt + \sigma_r(t)dW_r$	r_t - Gaussian

(b)	
Dothan [24]	$dr_t = \theta r_t dt + \sigma_r r_t dW_r$
Brennan-Schwartz [25]	$dr_t = (\theta - cr_t)dt + \sigma_r r_t dW_r$

(c)	
Cox-Ingersoll-Ross [26]	$dr_t = \theta(\mu - r_t)dt + \sigma_r\sqrt{r_t}dW_r$
Pearson-Sun [27]	$dr_t = \theta(\mu - r_t)dt + \sigma_r\sqrt{r_t - \beta}dW_r$ where β limits the short rate.
Black, Derman and Toy [28]	$d\eta_t = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\eta_t\right)dt + \sigma_r(t)dW_r$ where $\eta_t = \ln r_t$
Black-Karasinski [29]	$d\eta_t = (\theta(t) - c(t)\eta_t)dt + \sigma(t)dW_r$ where $\eta_t = \ln r_{tt}$

write the pair of Equations (1.2) and (2.1) as

$$dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)dW_{1,t} \qquad (2.2)$$

and

$$dg_t = \left[r_t - \frac{1}{2}\sigma_v^2(t)\right]dt + v_1(t)dW_{1,t} + v_3(t)dW_{2,t} \quad (2.3)$$

where $dW_{1,t}$ and $dW_{2,t}$ are two independent Wiener increment processes and

$$v_1(t) = \rho\sigma_v(t) \text{ and } v_3(t) = \sqrt{1 - \rho^2}\sigma_v(t). \qquad (2.4)$$

Integrating (2.3), we obtain

$$g_t = g_0 + \int_0^t r_s ds - \frac{1}{2}\int_0^t \sigma_v^2(s)ds + x_t + z_t \qquad (2.5)$$

where

$$x_t = \int_0^t v_1(s)dW_{1,s} \qquad (2.6)$$

and

$$z_t = \int_0^t v_3(s)dW_{2,s}. \qquad (2.7)$$

From the properties of the Ito integral, Mikosch [30]

and Shreve [22], it follows that

$$x_t \sim N(0, T_1(t)) \text{ and } z_t \sim N(0, T_3(t)) \qquad (2.8)$$

where

$$T_1(t) = \int_0^t v_1^2(s)ds \text{ and } T_3(t) = \int_0^t v_3^2(s)ds. \qquad (2.9)$$

Since $W_{1,t}$ and $W_{2,t}$ are independent Wiener processes, it readily follows that $(x_t + z_t) \sim N(0, T_1(t) + T_3(t))$. Thus, the distribution of g_t and hence of $V_t = e^{g_t}$ critically depend on the properties of the r_t process in (2.2).

In closing this section consider the special case when $\sigma_v(t) \equiv \sigma_v$, a constant. Then,

$$T_1(t) = \rho^2\sigma_v^2 t \text{ and } T_3(t) = (1 - \rho^2)\sigma_v^2 t. \qquad (2.10)$$

Further, when $\rho = 0$, we obtain

$$x_t \equiv 0 \text{ and } z_t \equiv \sigma_v W_{2,t}. \qquad (2.11)$$

4. Narrow Sense Linear Models for r_t

Setting

$$\alpha(r_t,t) = \theta(t) - c(t)r_t \text{ and } \sigma_r(r_t,t) = \sigma_r(t), \qquad (3.1)$$

in (2.2), we get a narrow sense (time varying) linear model known as the generalized Hull and White [17] model given by

$$dr_t = -c(t)r_t dt + \theta(t)dt + \sigma_r(t)dW_{1,t}. \quad (3.2)$$

Since all the other narrow sense linear models in **Table 1** are special cases of (3.2), we first concentrate on solving (3.2). Defining

$$\bar{c}(t) = \int_0^t c(s)ds, \quad (3.3)$$

we get

$$\Phi(t) = e^{-\bar{c}(t)}. \quad (3.4)$$

This is known as the fundamental solution of (3.2). Hence the solution of (3.2) is given by (Arnold [8], Gard [31], Kuo [32], Lamberton and Lapeyre [7])

$$r_t = r_t(\det) + r_t(ran), \quad (3.5)$$

where

$$r_t(\det) = e^{-\bar{c}(t)}r_0 + \int_0^t e^{-[\bar{c}(t)-\bar{c}(s)]}\theta(s)ds, \quad (3.6)$$

$$r_t(ran) = \int_0^t v_2(u)dW_{1,u}, \quad (3.7)$$

and

$$v_2(u) = e^{-[\bar{c}(t)-\bar{c}(u)]}\sigma_r(u). \quad (3.8)$$

Hence,

$$r_t \sim N\left(r_t(\det), T_2(t)\right) \quad (3.9)$$

where

$$T_2(t) = \int_0^t v_2^2(u)du. \quad (3.10)$$

Now combining (3.5)-(3.10), it follows that

$$\int_0^t r_s ds = \int_0^t r_s(\det)ds + \int_0^t r_s(ran)ds. \quad (3.11)$$

Applying integration by parts to the second integral on the right hand side of (3.11) and using (3.7), it follows that

$$\bar{r}_t(ran) = \int_0^t r_s(ran)ds = tr_t(ran) - \int_0^t sd\left(r_s(ran)\right)$$
$$= \int_0^t (t-s)v_2(s)dW_{1,s} \quad (3.12)$$

Is also a Gaussian process with mean zero and variance given by

$$\bar{T}_2(t) = \text{var}\left(\bar{r}_t(ran)\right) = \int_0^t (t-s)^2 v_2^2(s)ds. \quad (3.13)$$

Substituting (3.11)-(3.12) in (2.5), we get

$$g_t - g_0 = g_t(\det) + g_t(ran) \quad (3.14)$$

where

$$g_t(\det) = \int_0^t r_s(\det)ds - \frac{1}{2}\int_0^t \sigma_v^2(s)ds \quad (3.15)$$

and

$$g_t(ran) = \left[x_t + \bar{r}_t(ran)\right] + z_t. \quad (3.16)$$

Substituting (2.6), (3.12) and (2.7) in (3.16), the latter becomes

$$g_t(ran) = \int_0^t \left[v_1(s) + (t-s)v_2(s)\right]dW_{1,s} + \int_0^t v_3(s)dW_{2,s} \quad (3.17)$$

Since $dW_{1,t}$ and $dW_{2,t}$ are independent, it follows that $g_t(ran) \sim N\left(0, \text{var}(g_t)\right)$ where

$$\text{var}(g_t) = \int_0^t \left[v_1(s) + (t-s)v_2(s)\right]^2 ds + \int_0^t v_3^2(s)ds \quad (3.18)$$

$$= \sigma_1^2(t) + \sigma_2^2(t), \quad (3.19)$$

where

$$\sigma_1^2(t) = T_1(t) + 2\int_0^t (t-s)v_1(s)v_2(s)ds + \bar{T}_2(t)$$

and

$$\sigma_2^2(t) = T_3(t).$$

Combining (3.15)-(3.18) with (3.14), we finally obtain

$$g_t - g_0 \sim N\left(\mu(t), \sigma^2(t)\right) \quad (3.20)$$

where $\mu(t) = g_t(\det)$ given by (3.16) and $\sigma^2(t) = \text{var}(g_t)$ given by (3.19).

We summarize the above developments in the following:

Theorem 3.1: Let the interest rate r_t evolve according to a narrow sense linear, scalar, SDE of the type (3.2). Then, r_t is a Gaussian process and consequently $g_t - g_0$ in (2.5) is also a Gaussian process given by (3.20).

Since $V_t = e^{g_t}$, from (3.14)-(3.20), we get

$$\left(\frac{V_t}{V_0}\right) = V_t(\det)V_t(ran) \quad (3.21)$$

where

$$V_t(\det) = \exp\left[\int_0^t r_s(\det)ds - \frac{1}{2}\int_0^t \sigma_v^2(s)ds\right] \quad (3.22)$$

and

$$V_t(ran) = \exp\left[\left(x_t + \bar{r}_t(ran)\right) + z_t\right]. \quad (3.23)$$

The following corollary is immediate.

Corollary 3.2: Since $g_t - g_0 \sim N\left(\mu(t), \sigma^2(t)\right)$, $\left(\frac{V_t}{V_0}\right)$ is a lognormal process whose probability density function, as a function of time, is given by

$$\text{Prob}\left[\frac{V_t}{V_0}\right] = \frac{1}{\sqrt{2\pi}\sigma(t)\left(\frac{V_t}{V_0}\right)} \exp\left[-\frac{\left[\ln\left(\frac{V_t}{V_0}\right) - \mu(t)\right]^2}{2\sigma^2(t)}\right].$$

(3.24)

It can be verified (Johnson *et al.* [33]) that the time evolution of the mean and variance of the value process $\frac{V_t}{V_0}$ are given by

$$E\left[\frac{V_t}{V_0}\right] = \exp\left[\mu(t) + \frac{\sigma^2(t)}{2}\right]$$

(3.25)

and

$$Var\left[\frac{V_t}{V_0}\right] = \exp\left[\sigma^2(t) - 1\right]\exp\left[2\mu(t) + \sigma^2(t)\right].$$ (3.26)

We now enlist a number of nested corollaries by considering special cases of interest rate models.

Case 1: Let $\sigma_v(t) \equiv \sigma_v$, a constant. Then $v_1(t) \equiv \rho\sigma_v$, $v_3(t) \equiv \sqrt{1-\rho^2}\sigma_v$ and $v_2(s)$ is given by (3.8). From (3.15) and (3.20), the mean is

$$\mu(t) = \int_0^t r_s(\det)\,ds - \frac{1}{2}\sigma_v^2 t$$

(3.27)

where $r_t(\det)$ is given in (3.6). From (3.19)-(3.20), the variance is

$$\sigma^2(t) = \sigma_v^2 t + \int_0^t (t-s)^2 v_2^2(s)\,ds \\ + 2\rho\sigma_v \int_0^t (t-s)v_2(s)\,ds.$$

(3.28)

Case 2: Hull and White [17] model: In this model, $c(t) \equiv c$ and $\sigma_r(t) \equiv \sigma_r$ where $c > 0$ and $\sigma_r > 0$ are constants. Thus, $\bar{c}(t) = ct$, $v_2(s) = \sigma_r e^{-c(t-s)}$ and

$$r_t(\det) = e^{-ct}r_0 + e^{-ct}\int_0^t e^{cs}\theta(s)\,ds.$$

(3.29)

Hence the mean is

$$\mu(t) = \int_0^t r_s(\det)\,ds - \frac{1}{2}\sigma_v^2 t$$

(3.30)

and the variance is

$$\sigma^2(t) = \sigma_v^2 t + \sigma_r^2 \int_0^t (t-s)^2 e^{-2c(t-s)}\,ds \\ + 2\rho\sigma_v\sigma_r \int_0^t (t-s)e^{-c(t-s)}\,ds$$

(3.30a)

$$= \sigma_v^2 t + \frac{\sigma_r^2}{4c^3}\left[1 - e^{-2ct}\left(2c^2t^2 + 2ct + 1\right)\right] \\ + \frac{2\rho\sigma_v\sigma_r}{c^2}\left[1 - e^{-ct}\left(ct + 1\right)\right].$$ (3.30b)

Case 3: Ho-Lee [16] model: In this model, $c(t) \equiv 0$ and $v_2(s) = \sigma_r$. Then

$$r_t(\det) = r_0 + \int_0^t \theta(s)\,ds.$$

(3.31)

The mean is

$$\mu(t) = \int_0^t r_s(\det)\,ds - \frac{1}{2}\sigma_v^2 t$$

(3.32)

and the variance is

$$\sigma^2(t) = \sigma_v^2 t + \sigma_r^2 \int_0^t (t-s)^2\,ds \\ + 2\rho\sigma_v\sigma_r \int_0^t (t-s)\,ds$$

(3.33a)

$$= \frac{\sigma_r^2}{3}t^3 + \rho\sigma_v\sigma_r t^2 + \sigma_v^2 t.$$

(3.33b)

Case 4: Vasicek [15] model: In this model, $c(t) \equiv c$, $\theta(t) \equiv \theta$ and $\sigma_r(t) \equiv \sigma_r$. Then, $v_2(s) = \sigma_r e^{-c(t-s)}$ and

$$r_t(\det) = e^{-ct}r_0 + \frac{\theta}{c}\left[1 - e^{-ct}\right].$$

(3.34)

Hence, the mean is

$$\mu(t) = \int_0^t r_s(\det)\,ds - \frac{1}{2}\sigma_v^2 t$$

(3.35a)

$$= \left(\frac{\theta}{c} - \frac{1}{2}\sigma_v^2\right)t + \left(\frac{r_0}{c} - \frac{\theta}{c^2}\right)\left(1 - e^{-ct}\right)$$

(3.35b)

and the variance is

$$\sigma^2(t) = \sigma_v^2 t + \sigma_r^2 \int_0^t (t-s)^2 e^{-2c(t-s)}\,ds \\ + 2\rho\sigma_v\sigma_r \int_0^t (t-s)e^{-c(t-s)}\,ds$$

(3.36a)

$$= \sigma_v^2 t + \frac{\sigma_r^2}{4c^3}\left[1 - e^{-2ct}\left(2c^2t^2 + 2ct + 1\right)\right] \\ + \frac{2\rho\sigma_v\sigma_r}{c^2}\left[1 - e^{-ct}\left(ct + 1\right)\right].$$ (3.36b)

Case 5: Merton [14]: In this case, $\theta(t) \equiv \theta$, $c(t) \equiv 0$ and $\sigma_r(t) \equiv \sigma_r$. Then $v_2(s) = \sigma_r$ and

$$r_t(\det) = r_0 + \theta t.$$

(3.37)

Hence, the mean is

$$\mu(t) = \int_0^t r_s(\det)\,ds - \frac{1}{2}\sigma_v^2 t \\ = \left(r_0 - \frac{1}{2}\sigma_v^2\right)t + \frac{\theta}{2}t^2$$

(3.38)

and the variance is

$$\sigma^2(t) = \sigma_v^2 t + \sigma_r^2 \int_0^t (t-s)^2\,ds \\ + 2\rho\sigma_v\sigma_v \int_0^t (t-s)\,ds$$

(3.39a)

$$= \frac{\sigma_r^2}{3}t^3 + \rho\sigma_v\sigma_r t^2 + \sigma_v^2 t . \qquad (3.39b)$$

Case 6: Let $r_t \equiv r$, a constant and $\sigma_t \equiv 0$. Then $v_2(s) \equiv 0$, $r_t(ran) \equiv 0$ and $r_t(det) = r$. The mean

$$\mu(t) = \left(r - \frac{1}{2}\sigma_v^2\right)t \qquad (3.40)$$

and the variance

$$\sigma^2(t) = \sigma_v^2 t . \qquad (3.41)$$

We now provide sample plots of the V_t distribution, when r_t follows the Vasicek [15] model, for three different values of the correlation ($\rho = 0$, 0.9 and −0.9) in **Figures 1**-**3** respectively. In each case the distribution of V_T for $T = 5, 10, 15$ and 20 are given. From these figures it follows that as T increases both the mean and variance of V_T increases. Further, comparing **Figures 1** and **2**, it follows that the effect of the positive correlation ($\rho > 0$) is to reduce the peak while making the tails fatter compared to the case when $\rho = 0$. Similarly from **Figures 1** and **3**, we readily see the negative correlation has the opposite effect of increased peak and thinner tails compared to $\rho = 0$.

The primary motivation for characterizing the distribution of V_t is to compute the probability of default. Within the framework of structural models, there has been an evolution of the definition of default. In the now classic paper, Merton [10] defines default as the event $\{V_T \le K\}$ where K is the face value of the discount bond with maturity T. Using the results described above, we could readily compute the probability default according to this classical definition[1].
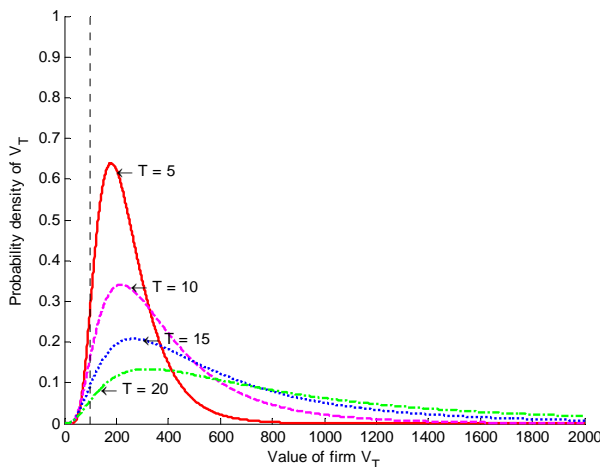

Figure 1. Probability density of V_T for Vasicek with $\sigma_r = 0.1$, $\rho = 0$, $r_0 = 0.1$, $\theta = 0.05$, $c = 0.5$, $\sigma_v = 0.2$, $V_0 = 150$.

[1]We note that Shimko, Tejima, and Van Deventer [42] build upon the Merton [11] model and solve for bond and equity prices as opposed to value of the firm.
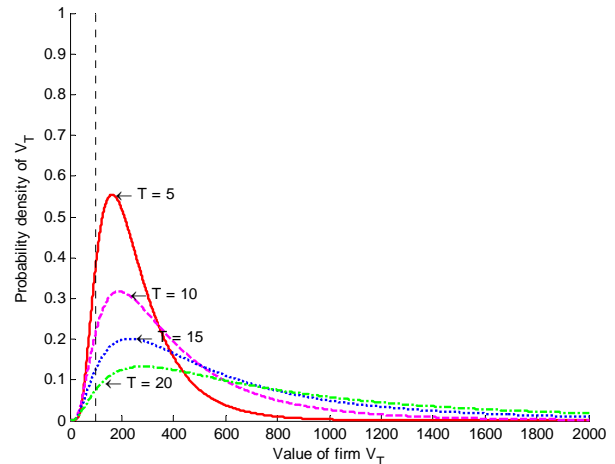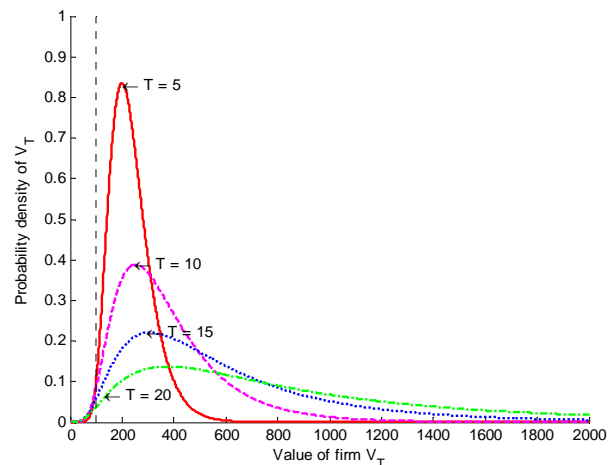

Figure 2. Probability density of V_T for Vasicek with $\sigma_r = 0.1$, $\rho = 0.9$, $r_0 = 0.1$, $\theta = 0.05$, $c = 0.5$, $\sigma_v = 0.2$, $V_0 = 150$.

Figure 3. Probability density of V_T for Vasicek with $\sigma_r = 0.1$, $\rho = -0.9$, $r_0 = 0.1$, $\theta = 0.05$, $c = 0.5$, $\sigma_v = 0.2$, $V_0 = 150$.

However, Longstaff and Schwartz [11] define default by the event $\left\{ \min_{0 \le t \le T} V_t \le K \right\}$. Recently, Giesecke [34] has expanded on this theme and has defined the default by the compound event $\left\{ \min_{0 \le t \le T} V_t < D \text{ or } V_T < K \right\}$ for $D < K$. Recall that the probability of these later events can be readily calculated using the "reflection principle" if V_t is a standard Wiener process or by using the Girsanov theorem if V_t is a Wiener process with a drift. (See Elliott and Kopp [35] and Giesecke [34]). To enable computation of default probability according to Giesecke [34], in the following, we seek to express $(g_t - g_0)$ in (3.14) as the sum of a drift term and a (time changed) Wiener process.

To this end recall that every Ito integral is equivalent to a time changed Wiener process. (See Shiryaev [36], Oksendal [37], Karatzas and Shreve [25]). Accordingly, from (3.17) we obtain

$$g_t\left(ran\right) = B_1\left(\sigma_1\left(t\right)\right) + B_2\left(\sigma_2\left(t\right)\right) \qquad (3.42)$$

where $B_1\left(t\right)$ and $B_2\left(t\right)$ are two independent Wiener process with

$$B_1\left(\sigma_1\left(t\right)\right) = \int_0^t \left[v_1\left(s\right) + \left(t-s\right)v_2\left(s\right)\right]\mathrm{d}W_{1,s} \ ,$$

$$B_2\left(\sigma_2\left(t\right)\right) = \int_0^t v_3\left(s\right)\mathrm{d}W_{2,s}$$

and $\sigma_1^2\left(t\right)$ and $\sigma_2^2\left(t\right)$ are given in (3.19). Since $B_1\left(t\right)$ and $B_2\left(t\right)$ are independent, there exists a Wiener process $B\left(t\right)$ such that

$$g_t\left(ran\right) = B\left(\sigma\left(t\right)\right) \qquad (3.43)$$

where

$$\sigma^2\left(t\right) = \sigma_1^2\left(t\right) + \sigma_2^2\left(t\right) = Var\left(g_t\right)$$

as given by (3.18)-(3.19).

Combining (3.43) with (3.14), it follows that

$$g_t - g_0 = \mu\left(t\right) + B\left(\sigma\left(t\right)\right) \qquad (3.44)$$

where $\mu\left(t\right) = g_t\left(\det\right)$ as in (3.20).

5. Conclusions

We have analyzed the impact of r_t on V_t when r_t evolves according to a narrow sense linear model in **Table 1(a)**. Consider the case when r_t evolves according to a general linear model, such as for example, the Brennan-Schwartz [38] model in **Table 1(b)**. In this case the explicit form of the solution for r_t is well known (Arnold [8], Gard [15], Lamberton and Lapeyre [7]) and is given by

$$r_t = r_0\Phi\left(t\right) + \theta\int_0^t \Phi\left(t-u\right)\mathrm{d}u$$

where the process $\Phi\left(t\right)$ is given by

$$\Phi\left(t\right) = \exp\left(\sigma_r W_t + \alpha t\right) \text{ with } \alpha = -\left(c + \frac{1}{2}\sigma_r^2\right). \text{ Hence,}$$

$\int_0^t r_s\mathrm{d}s$ involves a process

$$A_t = \int_0^t \Phi\left(s\right)\mathrm{d}s = \int_0^t \exp\left(\sigma_r W_s + \alpha s\right)\mathrm{d}s$$

which is an integral of the exponential functionals of the Wiener process. Processes of the type A_t routinely arise in the evaluation of Asian type options (Vorst [39]). By relating $\Phi\left(t\right)$ to a Bessel process, Yor [40] and Geman and Yor [41] have provided a complete characterization of the distribution of the A_t process. Combining these results with (2.5) to derive the distribution of g_t is an interesting open problem. Similarly, computing the dis-tribution g_t when r_t evolves according to the nonlinear models is **Table 1(c)** is also wide open. Solutions to these problem will shed further light on the impact of the choice of interest rate models on default probability and hence on credit risk analysis.

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