Open Journal of Discrete Mathematics, **2012, 2, 1-4**
http://dx.doi.org/10.4236/ojdm.2012.21001 Published Online January 2012 (http://www.SciRP.org/journal/ojdm)

The Code of the Symmetric Net with *m* = 4 and *μ* = 2

Ahmad N. Al-Kenani
Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
Email: aalkenani10@hotmail.com

Received September 29, 2011; revised November 1, 2011; accepted November 12, 2011

ABSTRACT

In this paper, we investigated the code over $GF(2)$ which is generated by the incidence matrix of the symmetric (2,4) -net $\mathcal{D}$. By computer search, we found that this binary code of $\mathcal{D}$ has rank 13 and the minimum distance is 8.

Keywords: Symmetric Nets; Codes

1. Introduction

A $t-(v,k,\lambda)$ design $\mathcal{D}$ is an incidence structure with v points, k points on each block and any subset of t points is contained in exactly λ blocks, where $v>k,\lambda>0$. the number of blocks is b and the number of blocks on a point is r.

The design $\mathcal{D}$ is *resolvable* if its blocks can be partitioned into r *parallel classes*, such that each parallel class partitions the point set of $\mathcal{D}$. Blocks in the same parallel class are parallel. Clearly each parallel class has $m=v/k$ blocks. $\mathcal{D}$ is *affine resolvable*, or simply *affine*, if it can be resolved so that any two non-parallel blocks meet in μ points, where $\mu=k/m=k^2/v$ is constant. Affine 1-designs are also called nets. The dual design of a design $\mathcal{D}$ is denoted by $\mathcal{D}^\star$. If $\mathcal{D}$ and $\mathcal{D}^\star$ are both affine, we call $\mathcal{D}$ a *symmetric net*. We use the terminology of Jungnickel [1] (see also [2-5]). In this case $v=b=\mu m^2$ and $k=r=\mu m$. That is, $\mathcal{D}$ is an affine $1-(\mu m^2,\mu m,\mu m)$ design whose dual $\mathcal{D}^\star$ is also affine with the same parameters. For short we call such a symmetric net a (μ,m)-net.

If $\mathcal{D}$ is a symmetric net we shall refer to the parallel classes of $\mathcal{D}$ as block classes of $\mathcal{D}$ and to the parallel classes of $\mathcal{D}^\star$ as point classes of $\mathcal{D}$.

For any finite structure $\mathcal{D}$ with point set $\mathcal{P}$ and block set $\mathcal{B}$, the code $C_p(\mathcal{D})$ of $\mathcal{D}$ over prime field F_p is the subspace of the space $F_p^{\mathcal{P}}$ of all functions from $\mathcal{P}$ to F_p that is spanned by the incidence vectors of the blocks of $\mathcal{D}$. This code is equivalent to the code given by the column space of any incidence matrix of the incidence structure, where we use the blocks to index the columns (and the points the rows) of the incidence matrix.

2. The Symmetric Net with $m=4$ and $\mu=2$

The symmetric net that we shall be concerned with in this paper is the one with $m=4$ and $\mu=2$. As a design it has parameters

$$1-(32,8,8).$$

Its incidence matrix is (1).

A computer search has shown that to within isomorphism there is only one symmetric net with these parameters. We denote this symmetric net by $\mathcal{D}$.

Butson [6] showed that there exist symmetric nets with m any prime and $\mu=2$. This was extended to m any prime power by Jungnickel [7]. Therefore $\mathcal{D}$ is one of the family of symmetric nets constructed by Jungnickel.

3. The Codes

The columns of the incidence matrix of $\mathcal{D}$ can be considered as vectors of the 32-dimensional vector space over any finite prime field F. The subspace they generate is the code of the net $\mathcal{D}$ over F. By computer we found that the binary code (that is, the code over the field of order 2) of $\mathcal{D}$ has rank 13. The weight distribution of its codewords is given below. The all one vector is in the code since it is obtained as the sum of the 4 columns corresponding to the blocks of any parallel class in the incidence matrix. Therefore the code is self-complementary in that the complement of a codeword is also a codeword, see [8] or [9]. Hence we only list the number of codewords of weight up to 16.

$$M=\begin{pmatrix}
1&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&1&0&1\\
0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1&0&0&1&1&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&1\\
0&0&0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0\\
0&0&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&1&0&0&0\\
1&0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0&0&0&0&1&1&0&0&0&0\\
0&1&0&0&0&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&1\\
1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&1&0&1&0\\
0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&1&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&1&1&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&1&1\\
0&0&1&1&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&0&1&0&0\\
0&0&1&0&1&0&0&0&1&1&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&1&0&1&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&0&0&1&0&0
\end{pmatrix} \tag{1}$$

Weight	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Number	1	0	0	0	0	0	0	0	60	0	0	0	1792	0	0	0	4486

Since the minimum distance is 8, the binary code is 3-error correcting.

There doesn't seem to be an easy proof that the dimension of the code is 13 over the binary field. The dimension of the code of $\mathcal{D}$ for odd characteristic is 25. This we prove in this paper.

The incidence matrix of $\mathcal{D}$ may be put in the form:

$$M=\begin{pmatrix}
I&I&I&I&I&I&I&I\\
I&I&A&A&B&B&C&C\\
I&A&B&C&I&A&B&C\\
I&A&C&B&B&C&A&I\\
I&B&I&B&C&A&C&A\\
I&B&A&C&A&C&I&B\\
I&C&B&A&C&I&A&B\\
I&C&C&I&A&B&B&A
\end{pmatrix}$$

where

$$I=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ A=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$B=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\ \text{and } C=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

$G=\{I,A,B,C\}$ is an elementary abelian group of order 4.

First suppose that the characteristic of the field is not 2.

The matrices in G can be simultaneously diagonalised by

$$P=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Conjugating by diagonal (P,P,P,P,P,P,P,P) and then by Q, the permutation matrix which moves rows (and columns) $1,5,9,13,17,21,25,29$ to the first eight positions, rows (and columns) $2,6,10,14,18,22,26,30$ to the next eight positions, rows (and columns) $3,7,11,15,19,23,27,31$ to the next eight positions, rows (and columns) $4,8,12,16,20,24,28,32$ to the last eight positions, we get M conjugate to

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-&-&-&-&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-&-&1&1&-&-&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-&1&-&-&1&-&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-&1&-&1&-&1&-&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-&-&1&-&1&1&-&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-&-&1&1&-&-&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&-&-&-&-&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&-&-&-&-&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&-&-&1&1&-&-&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&-&-&-&-&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&1&-&-&1&-&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&1&-&1&-&1&-&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&-&1&-&1&1&-&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&-&1&1&-&-&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&-&-&1&1&-&-\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&1&-&1&-&1&-\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&-&1&1&-&-&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&-&-&-&-\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&-&-&-&-&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-&1&-&-&1&-&1
\end{pmatrix}$$

where "–" denotes to -1.

The diagonal block matrices

$$\begin{pmatrix}
1&1&1&1&1&1&1&1\\
1&1&-1&-1&-1&-1&1&1\\
1&-1&-1&1&1&-1&-1&1\\
1&-1&1&-1&-1&1&-1&1\\
1&-1&1&-1&-1&1&1&-1\\
1&-1&-1&1&-1&1&1&-1\\
1&1&-1&-1&1&1&-1&-1\\
1&1&1&1&-1&-1&-1&-1
\end{pmatrix}\begin{pmatrix}
1&1&1&1&1&1&1&1\\
1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&1&1&-1&-1\\
1&1&-1&-1&-1&-1&1&1\\
1&-1&1&-1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1\\
1&-1&-1&1&-1&1&1&-1\\
1&-1&-1&1&1&-1&-1&1
\end{pmatrix} \text{ and } \begin{pmatrix}
1&1&1&1&1&1&1&1\\
1&1&-1&-1&1&1&-1&-1\\
1&-1&1&-1&1&-1&1&-1\\
1&-1&-1&1&1&-1&-1&1\\
1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&-1&-1&1&1\\
1&-1&1&-1&-1&1&-1&1\\
1&-1&-1&1&-1&1&1&-1
\end{pmatrix}$$

have determinant 4096. In fact they are Hadamard matrices. Hence the rank of M is $1+3\times 8=25$, if the characteristic is not 2.

4. Acknowledgements

The author would like to thank the Deanship of Scientific Research at King Abdulaziz University for supporting a project no. 169/428, where this paper is a part of that project.

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