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Dual Solution of MHD Stagnation-Point Flow towards a Stretching Surface

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Abstract

The effect of a uniform transverse magnetic field on two-dimensional stagnation-point flow of an incompressible viscous electrically conducting fluid over a stretching surface is investigated when the surface is stretched in its own plane with a velocity proportional to the distance from the stagnation-point. This magnetohydrodynamic (MHD) flow problem is governed by the parameter b representing the ratio of the strain rate of the stagnation-point flow to that of the stretching sheet and the magnetic field parameter M. It is known from a previous paper [9] that if b > 1, the steady solution to the problem is monotonic increasing and the solution is also unique. But when $0 < b < b_c$ (where b_c (< 1) depends on M), there exists a dual solution which is non-monotonic in addition to a monotonic decreasing solution. It is found in this paper that bc decreases as M increases. Numerically it is shown that if M > 0.23919, the non-monotonic solution cannot exist and so in this case, the only solution is monotonic decreasing. A stability analysis reveals that when $0 < b < b_c$, the solutions along the upper branch corresponding to the monotonic solution are linearly stable while those along the lower branch for the non-monotonic solution are linearly unstable. It is also shown that the decay rate of a disturbance increases with increasing M for the stable solution but the growth rate of instability for the non-monotonic solution decreases with increasing M.

1. Introduction

Flow of an incompressible viscous fluid over a stretching surface has important bearing on several industrial processes. For instance, in the extrusion of a polymer in a melt-spinning process, the extrudate from the die is usually drawn and simultaneously stretched into a sheet which is then cooled gradually by direct contact with water. The property of the final product depends on the rate of heat transfer at the surface of the sheet. Further the study of the flow of an electrically conducting viscous fluid caused by the deformation of the walls of the vessel containing this fluid in the presence of a magnetic field is of great interest in modern metallurgical and metal-working processes. Crane [1] obtained a similarity solution in closed analytical form for steady two dimensional flow of an incompressible viscous fluid caused solely by the stretching of an elastic sheet which moves in its own plane with a velocity

varying linearly with distance from a fixed point. Pavlov [2] gave a similarity solution in exact analytical form of the MHD boundary layer equations for steady two-dimensional flow of an electrically conducting incompressible viscous fluid due to the stretching of a plane elastic surface in its own plane in the presence of a uniform transverse magnetic field. This problem was extended by Chakrabarti and Gupta [3] to include suction or injection at the stretching surface. The temperature distribution in the flow was also found by them in the case of constant surface temperature. Andersson [4] investigated the MHD flow of a viscoelastic fluid past a stretching surface in the presence of a uniform transverse magnetic field.

Recently Chiam [5] studied steady two-dimensional stagnation-point flow of an incompressible viscous fluid towards a stretching surface in the case when the parameter b representing the ratio of the strain rate of the stagnation-point flow to that of the stretching surface is

equal to unity. By removing this highly restrictive assumption (b = 1), Mahapatra and Gupta [6] analyzed the steady two dimensional orthogonal stagnation-point flow of an incompressible viscous fluid towards a stretching surface in the general case b ≠ 1. They found that the structure of the boundary layer depends crucially on the value of b. Temperature distribution in the flow was also obtained by them. Earlier Mahapatra and Gupta [7] investigated the steady two- dimensional orthogonal stagnation-point flow of an incompressible viscous electrically conducting fluid towards a stretching surface, the flow being permeated by a uniform transverse magnetic field. The steady and unsteady equations governing stagnation-point flow of an electrically nonconducting incompressible viscous fluid over a stretching surface along with the existence and uniqueness of the solutions of these equations were studied by Paullet and Weidman [8].

In this paper we investigate the dual solution of steady two-dimensional orthogonal stagnation-point flow of an incompressible viscous electrically conducting fluid towards a stretching surface in the presence of a uniform transverse magnetic field. We provided a mathematical proof in [9] that if b > 1, then the solution of this problem is monotonic increasing and further this solution is unique. But for 0 < b < 1, it was shown that there exists a dual solution which is non-monotonic in addition to a monotonic decreasing solution. We have found numerically in this paper that this dual solution exists upto a certain value of the magnetic parameter M. A linear stability analysis of these two solutions is also presented.

2. Flow Analysis

Consider the steady two-dimensional stagnation point flow of an incompressible viscous and electrically conducting fluid permeated by a uniform transverse magnetic field towards a flat stretching surface coinciding with the plane y = 0, the flow being confined to the region y > 0. Two equal and opposite forces are applied on the stretching surface along x-axis so that the surface is stretched with a velocity proportional to the distance from the stagnation-point which is taken as the origin. This is shown in **Figure 1**.

Using boundary layer approximations, the equations for steady MHD two-dimensional flow are in usual notation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}(\mathbf{j}\times\mathbf{B_0})_x, \qquad (1)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} , \qquad (2)$$

where $\mathbf{j}$ is the electric current density, $\mathbf{B_0}$ is the imposed uniform magnetic field (acting along the y-direction). In writing (1), we have neglected the induced magnetic
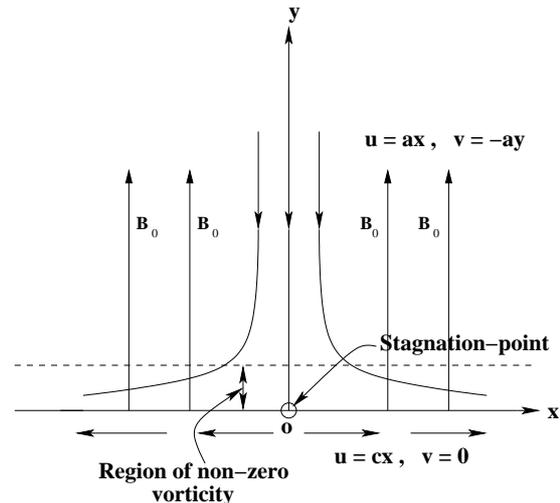

Figure 1. A sketch of the physical problem.

field since the magnetic Reynolds number R_M for the flow is assumed to be very small. Such an assumption is justified for the flow of electrically conducting fluids such as liquid metals e.g., liquid sodium, mercury etc. [10].

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (3)$$

Clearly the electric current flows parallel to the z-axis which is normal to the plane of the flow. Hence by Ohm's law

$$j_x = 0, \ j_y = 0, \ j_z = \sigma[E_z + uB_0], \qquad (4)$$

where σ is the electrical conductivity of the fluid which is assumed constant and E_z is the electric field along z-axis. It is assumed that the current lines in this two-dimensional problem close in a manner consistent with the experiment. Now since the flow is steady, Maxwell's equation gives

$$\nabla\times\mathbf{E} = \mathbf{0}, \qquad (5)$$

where $\mathbf{E}$ is the electric field (which acts parallel to the z-direction). This gives

$$\frac{\partial E_z}{\partial x} = 0 \ \text{ and } \ \frac{\partial E_z}{\partial y} = 0$$

so that E_z is a function of z only.

Since the induced magnetic field is neglected in view of the assumption $R_M \ll 1$, electric current in the flow is determined from Ohm's law and not from $\nabla\times\mathbf{B} = \mu_e\,\mathbf{j}$, μ_e being the magnetic permeability. But the consequence $\nabla\cdot\mathbf{j} = 0$ of this equation must be satisfied [10]. Noting that the physical variables are functions of *x* and *y*, $\nabla\cdot\mathbf{j} = 0$ then gives from (4), $E_z =$ constant. Thus using (4), we find from (1),

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \qquad (6)$$
$$-\frac{\sigma B_0^2 u}{\rho}-\frac{\sigma B_0 E_z}{\rho}.$$

Since from (2), the pressure p is a function of x only, the pressure gradient $\frac{\partial p}{\partial x}$ can now be obtained from (6) in the inviscid region as

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=-U(x)\frac{dU}{dx}-\frac{\sigma B_0^2 U}{\rho}-\frac{\sigma B_0 E_z}{\rho}, \qquad (7)$$

where $U(x)$ is the free stream velocity and use is made of the fact that E_z is constant. Hence eliminating $\frac{\partial p}{\partial x}$ from (6) and (7), we get

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+v\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B_0^2}{\rho}(U-u). \qquad (8)$$

We note that the electric field does not directly affect the boundary layer equation although it has an influence on the relation between the free stream velocity and the pressure distribution given by (7).

Thus the governing equations for the velocity components u and v are (3) and (8). The boundary conditions are

$$u=cx, \quad v=0 \quad at \quad y=0,$$
$$u\rightarrow U(x)=ax, \quad v\rightarrow -ay \qquad (9)$$
$$as \quad y\rightarrow\infty, \qquad (10)$$

where a and c are positive constants. Under the similarity transformation

$$u=cxF'(\eta), \quad v=-(cv)^{1/2}F(\eta),$$
$$\eta=y(c/v)^{1/2}, \qquad (11)$$

where a prime denotes derivative with respect to the similarity variable η, we find that (3) is identically satisfied.

Substitution of (11) in (8) then gives

$$F'''(\eta)+F(\eta)F''(\eta)-F'^2(\eta)$$
$$-M^2 F'(\eta)+M^2 b+b^2=0, \qquad (12)$$

where $b=a/c$ and M is the dimensionless magnetic parameter $B_0(\sigma/\rho c)^{1/2}$.

The boundary conditions for (12) then follow from (9) and (10) as

$$F(0)=0, \quad F'(0)=1, \quad F'(\infty)=b. \qquad (13)$$

A detailed proof of the uniqueness of the solution for the same problem for an electrically non-conducting ($M=0$) fluid was given by Paullet and Weidman [8]. Subsequently the present authors [9] proved the uniqueness of the solution for an electrically conducting fluid in the presence of a magnetic field ($M\neq 0$) using a similar approach as that in [8]. In [9] we have shown that if $b>1$, the solution of the boundary value problem (BVP) given by (12) and (13) is monotonic increasing for all magnetic parameter M and further this solution is unique. But if $0<b<1$, there is a dual solution to the above BVP, which is non-monotonic in addition to a solution which is monotonic decreasing. Note that when $b=1$, the above BVP admits of the exact solution $F(\eta)=\eta$, which is the inviscid solution near the stretching surface as found by Mahapatra and Gupta [7]. However this solution is not frictionless in a strict sense because in this case the friction is uniformly distributed and does not therefore affect the motion.

3. Numerical Solution

Since there is no general analytical solution of the non-linear differential equation (12) with the boundary conditions (13) for the case $b\neq 1$, so this system is solved numerically by an efficient shooting method for different values of M and b. To do this, we first transform the non-linear differential equation (12) to a system of three first order differential equations as:

$$y_1'=y_2, \quad y_2'=y_3,$$
$$y_3'=y_2^2-y_1 y_3+M^2(y_2-b)-b^2, \qquad (14)$$

where $y_1=F(\eta)$, $y_2=F'(\eta)$, $y_3=F''(\eta)$ and a prime denotes differentiation with respect to the independent variable η.

The boundary conditions (13) become

$$y_1=0, \quad y_2=1 \quad at \quad \eta=0,$$
$$y_2\rightarrow b \quad as \quad \eta\rightarrow\infty. \qquad (15)$$

The values of y_1 and y_2 are given at the starting point $\eta=0$. But the value of y_2 as $\eta\rightarrow\infty$ is replaced by $y_2=b$ at a finite value $\eta=\eta_\infty$ to be determined later. The value of y_3 at $\eta=0$ is guessed in order to initiate the integration scheme. Starting from the given values of y_1 and y_2 at $\eta=0$ and the guessed value of y_3 at $\eta=0$, we integrate the first order equations (14) by using a fourth-order Runge-Kutta method up to the end-point $\eta=\eta_\infty$. The computed value of y_2 at $\eta=\eta_\infty$ is then compared with $y_2=b$. The absolute difference between this computed value and b should be as small as possible. To this end we use a Newton-Raphson iteration procedure to assure quadratic convergence of the iterations. The value of η_∞ is then increased till y_2 attains the value b asymptotically.

4. Dual Solution of MHD Stagnation-Point Flow

Our numerical results reveal that for $0<b<1$, in addition to a monotonic decreasing solution which was al-

ready found by Mahapatra and Gupta [7], there exists a dual solution which is non-monotonic. We have found that in the absence of magnetic field ($M = 0$), the non-monotonic solution exists in the range $0 < b \leq 0.16906$. This result in the nonmagnetic case ($M = 0$) was also obtained in [8]. For $M = 0.1$, the dual solution exists in the range $0 < b \leq 0.14845$, and for $M = 0.2$, such solution is found in the range $0 < b \leq 0.08234$. Numerically we have found that when $M \leq 0.23919$, the non-monotonic solution exists in the range $0 < b < 1$. Thus we see that as M increases, the range of b (such that $0 < b < 1$) for which the dual (*i.e.*, non-monotonic) solution exists progressively decreases. The novel result which emerges from the analysis is that when $M > 0.23919$, there is no dual solution in the range $0 < b < 1$ and the solution in this range is monotonic decreasing and unique.

Figure 2 gives the variation of the dimensionless skin-friction coefficient F''(0) with b for several values of *M*. Here the upper branch corresponds to the monotonic solution while the lower branch corresponds to the non-monotonic one. It can be seen that for the monotonic solution, |F''(0)| increases with increase in *M*. This result can be physically explained as follows. It is clear from the momentum equation (8) that when $b < 1$ (*i.e.*, $a<c$), the Lorentz force given by the last term is negative because the stagnation-point velocity U(= ax) is less than the stretching velocity cx of the sheet so that u > U. Hence this magnetic force decelerates the flow inside the boundary layer. Since the velocity of a fluid particle parallel to the surface is diminished relative to that of the stretching surface, the velocity gradient at the surface increases with increase in the magnetic field B_0 (characterized by the magnetic parameter *M*). This results in increase in the dimensionless skin-friction coefficients |F''(0)| with increase in *M*. Note that the above argument is valid for the monotonic solution for the velocity distribution because in this case the velocity inside the boundary layer decreases monotonically from the surface velocity cx to ax (since $a < c$) so that u > U.

Figures 3-**5** display self-similar velocity profiles of F'(η) for different values of *M* at selected values of *b*. The non-monotonic behavior of the velocity distribution is evident from these figures. It is observed from these figures that for a fixed value of *b*, the velocity parallel to the stretching surface at a point in the boundary layer decreases with increase in M. From a physical point of view, this follows from the fact that when $b < 1$, increase in *M* leads to increase in the magnitude of the retarding Lorentz force acting on the fluid resulting in enhanced deceleration of the flow.

An interesting result which emerges from this analysis is that for a given value of *M*, the skin-friction coefficient |F''(0)| for the dual solution in the case $0 < b < 1$ is not a monotonic function of *b* but has a maximum at a particular value of *b* (see **Figure 2**). This result is to be contrasted with the corresponding result for the monotonic
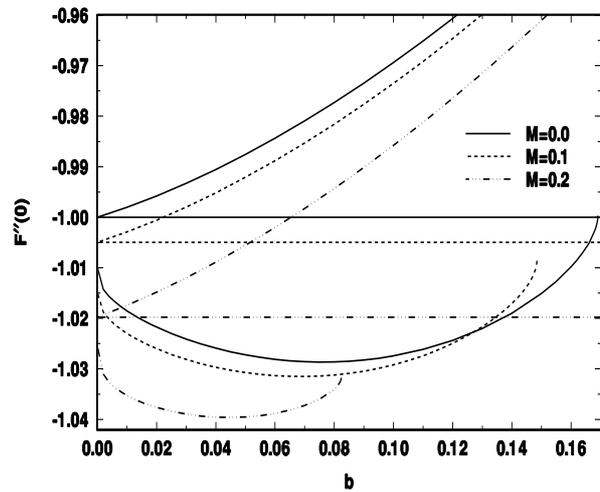

Figure 2. Reduced skin friction coefficient F''(0) with *b* for several values of *M* showing the upper and lower curves.
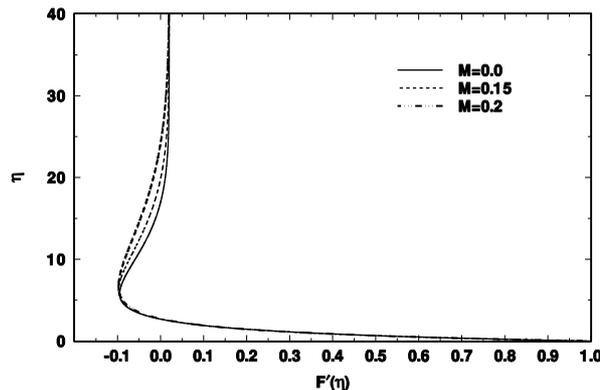

Figure 3. Self-similar velocity profiles (for non-monotonic solution) for *b* = 0.02 with different values of *M*.
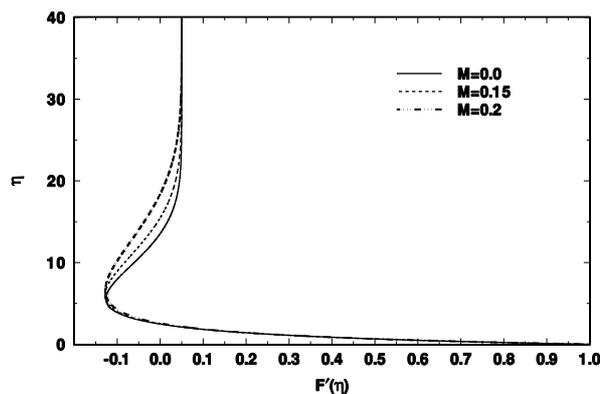

Figure 4. Self -similar velocity profiles (for non-monotonic solution) for *b* = 0.05 with different values of *M*.

solution for $0 < b < 1$, where |F''(0)| decreases monotonically with increase in b as shown in **Figure 2**. Physically this stems from the fact that for the dual solution, the
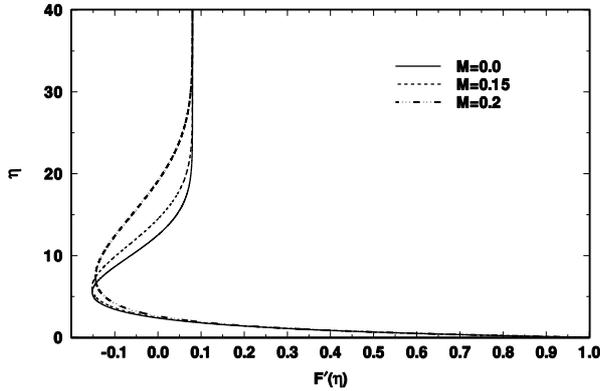
Figure 5. Self -similar velocity profiles (for non-monotonic solution) for $b = 0.08$ with different values of M.

self-similar velocity profiles $F'(\eta)$ are non-monotonic (see **Figures 3-5**).

5. Stability Analysis

In order to ascertain which of the two solutions in the region $0 < b < b_c$ (where as shown above, b_c depends on M), is expected to appear asymptotically, we test the stability of the above two solutions.

To this end, we consider the unsteady form of the Equation (8) as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + v\frac{\partial^2 u}{\partial y^2} \qquad (16)$$
$$+ \frac{\sigma B_0^2}{\rho}(U - u),$$

together with the equation of continuity (3). We seek the unsteady similarity solution in the form

$$u = cx f_\eta(\eta, \tau), \ v = -(cv)^{1/2} \ f(\eta, \tau),$$
$$\eta = y(c/v)^{1/2}, \qquad (17)$$

where the subscript η denotes partial derivative with respect to η and τ is the dimensionless time ct. Note that with u and v given above, the equation of continuity (3) is identically satisfied. Substitution of (17) in (16) then gives

$$f_{\eta\eta\eta} + ff_{\eta\eta} - f_\eta^2 - M^2 f_\eta + M^2 b$$
$$+ b^2 - f_{\eta\tau} = 0, \qquad (18)$$

where the subscripts η and τ denote partial derivatives with respect to η and τ respectively. The boundary conditions are

$$u = cx, \quad v = 0 \quad at \quad y = 0, \qquad (19)$$
$$u \to U(x) = ax, \quad v \to -ay$$
$$as \ y \to \infty. \qquad (20)$$

In terms of $f(\eta, \tau)$, these conditions are

$$f(0, \tau) = 0, \ f_\eta(0, \tau) = 1, \ f_\eta(\infty, \tau) = b. \qquad (21)$$

Following Merkin [11], we put

$$f(\eta, \tau) = F(\eta) + e^{-\gamma\tau}g(\eta), \qquad (22)$$

where $F(\eta)$ satisfies (12) subject to (13) and corresponds to the steady state solution. Further $g(\eta)$ and all its derivatives are assumed small compared with the steady solution $F(\eta)$ and its derivatives. Such an assumption is made because we are studying the linear stability of the basic flow $F(\eta)$ so that the disturbance $g(\eta)$ is small, γ being the growth (or decay) rate of the disturbance. Substituting (22) in (18) and linearizing, we get

$$g'' + Fg'' + (\gamma - 2F' - M^2)g' + F''g = 0. \qquad (23)$$

Here prime denotes derivative with respect to η. The boundary conditions for $g(\eta)$ follow from (13), (21) and (22) as

$$g(0) = 0, \ g'(0) = 0, \ g'(\infty) = 0. \qquad (24)$$

Clearly the homogeneous linear Equation (23) subject to the homogeneous boundary conditions (24) constitutes an eigenvalue problem with γ as the eigenvalue. Without loss of generality, we can take $g''(0) = 1$. Solutions of (23) and (24) give an infinite set of eigenvalues $\gamma_1 < \gamma_2 < \gamma_3 < \ldots$ If the smallest eigenvalue γ_1 is negative, then there is an initial growth of disturbances and the flow is unstable. On the other hand, when γ_1 is positive, there is an initial decay and the flow is stable.

Figure 6 shows the variation of the smallest eigenvalue γ_1 for the monotonic solutions (corresponding to upper curves in **Figure 2**) with b for several values of the magnetic parameter M. Since γ_1 is real and positive for these curves, it follows that the monotonic solutions are linearly stable. Further, for a given value of b, γ_1 increases with increase in M. Thus we may say that for these stable solutions, disturbances decay more quickly with increase in M.

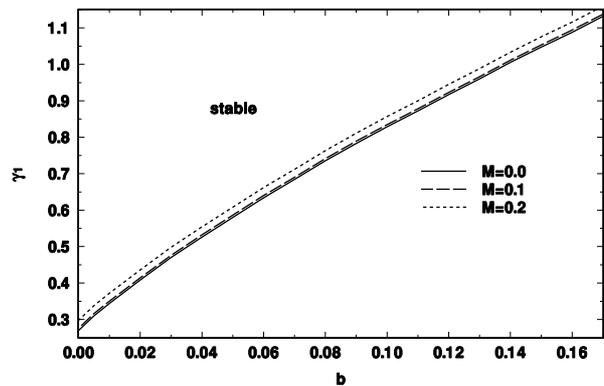On the other hand, **Figure 7** gives a plot of the variation of the lowest eigenvalue γ_1 for the non-monotonic

Figure 6. Plot of lowest eigenvalues γ_1 as a function of b for the upper branch solution with different values of M.
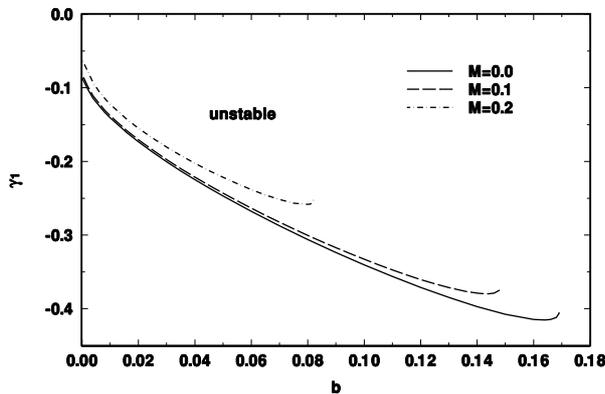
Figure 7. Plot of lowest eigenvalues γ_1 as a function of *b* for the lower branch solution with different values of *M*.

solutions (corresponding to lower curves in **Figure 2**) with b for several values of *M*. Since $\gamma_1 < 0$ for these curves, it is clear that the non-monotonic solutions are linearly unstable. Further for a given *b*, $|\gamma_1|$ decreases with increase in *M* for the non-monotonic solutions. Hence it follows that the growth rate of unstable disturbances decreases with increase in *M*. A physical interpretation of these results is that the magnetic field has a stabilizing influence on the flow. In fact a magnetic field exerts something like a viscous drag on an electrically conducting fluid and also imparts to it some degree of rigidity. Thus magnetic field can be expected to inhibit any tendency to instability. Further from an energetic point of view, a magnetic field is stabilizing since in addition to the dissipation of energy by viscosity, there is dissipation of magnetic energy in the form of Joule heating [12]. Specially, when an electrically conducting fluid is perturbed in the presence of a magnetic field, the disturbances induced by currents, whose energy is dissipated by Joule heating.

It may be noted that the result that the monotonic solution is linearly stable and the non-monotonic solution is linearly unstable in the case of an electrically nonconducting fluid with $M = 0$ was arrived at earlier in [8]. Our present stability analysis is based on the approach in [8].

6. Conclusions

An investigation is made of steady two dimensional MHD stagnation-point flow of an electrically conducting incompressible viscous fluid over a stretching surface in the presence of a uniform transverse magnetic field. The surface is stretched in its own plane with a velocity proportional to the distance from the stagnation-point. The flow problem is governed by the dimensionless parameter b representing the ratio of the strain-rate of the stagnation flow to that of the stretching surface and the

magnetic parameter M. It is shown that in addition to the monotonic increasing solution (which is unique) for $b > 1$ earlier found by Mahapatra and Gupta [7], there exist two solutions in the range $0 < b < b_c$, where $b_c (< 1)$ depends on *M*. In this range one solution is monotonic increasing whilst the dual solution is non-monotonic. It is also found numerically that b_c decreases with increase in *M* and when $M > 0.23919$, the dual solution does not exist.

A linear stability analysis of the two solutions in the range $0 < b < b_c$ reveals that the monotonic decreasing solution is linearly stable but the non-monotonic solution is linearly unstable. Further the magnetic field exerts a stabilizing influence on the flow.

7. Acknowledgements

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