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A Common Fixed Point Result for Generalized Cyclic Contraction Pairs Involving Altering Distance and Control Functions in Partial Metric Spaces

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Abstract

In this paper, our purpose is to establish a common fixed point result for a pair of self-mappings satisfying some generalized cyclic contraction type conditions involving altering distance and control function with two variables in partial metric spaces. Moreover, we provide an example in support of our main result.

Keywords

Partial Metric, Cyclic Contraction, 0-Completeness, Common Fixed Point

1. Introduction

The Banach contraction principle, a cornerstone in fixed point theory, has been extensively generalized and applied across various branches of mathematics. Among these generalizations, the concept of cyclic contractions, introduced by Kirk *et al.* [1], has received significant attention. This framework, which allows for mappings defined over cyclically ordered subsets, differs from classical contractions by not requiring continuity. Numerous fixed point results in this setting have followed, see [2]-[9].

Parallel to these developments, partial metric spaces, introduced by Matthews [10] in the context of denotational semantics of computation, have emerged as a powerful generalization of metric spaces. In partial metric spaces, the self-distance of a point need not be zero, a feature well-suited for modeling in computer science and domain theory. Matthews also established a version of the Banach contraction principle in this framework and introduced a class of open p-balls generating a

T_0 topology. Since then, numerous fixed point theorems have been obtained in this setting, see for examples [11]-[25].

The integration of cyclic contractions with partial metric spaces has further enriched the theory, yielding fixed point results under weaker or generalized contractive conditions [26]-[31].

A notable direction in this context is the use of altering distance functions, introduced by Khan *et al.* [32], which provide a flexible framework for defining contractions using control functions. These functions have been extensively utilized to establish fixed point theorems in both metric and generalized spaces [33]-[37].

Building upon these foundational ideas, recent works have explored fixed point results for cyclic mappings involving generalized control functions with two variables, particularly in 0-complete partial metric spaces [5] [38]-[48]. These results not only unify but also extend several classical and contemporary fixed point theorems.

In this paper, we contribute to this line of research by establishing a fixed point theorem for mappings satisfying cyclic weaker-type contraction conditions involving a two-variable control function in 0-complete partial metric spaces. We present to demonstrate the applicability and novelty of our results.

2. Preliminaries

In this section, we begin with some basic facts and properties of partial metric spaces.

Definition 2.1 ([10]). A **partial metric** on a nonempty set X is a function $p: X \times X \to \mathbb{R}^+$ satisfying for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$;

(p2) $p(x,x) \leq p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a **partial metric space**.

Remark 2.2. If $p(x,y) = 0$, then $x = y$. However, $x = y$ does not necessarily imply $p(x,y) = 0$.

Example 2.3. ([10]) Let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space.

Example 2.4. ([10]) Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ and $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.

Remark 2.5. [29] Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If $U \in \tau_p$ and $x \in U$, there exists $r > 0$ such that $B_p(x,r) \subseteq U$.

Remark 2.6. [29] A sequence (x_n) converges to x in τ_p if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Definition 2.7 ([10]) Let (X,p) be a partial metric space.

- A sequence (x_n) **converges** to x if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

- (x_n) is **Cauchy** if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
- (X, p) is **complete** if every Cauchy sequence converges to some $x \in X$ with $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Definition 2.8 ([24]).

(a) A sequence (x_n) is **0-Cauchy** if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

(b) (X, p) is **0-complete** if every 0-Cauchy sequence converges to x with $p(x,x) = 0$.

Lemma 2.9. *Let (X, p) be a partial metric space.*

(a) [20] [49] *If $p(x_n, z) \to p(z,z) = 0$, then $p(x_n, y) \to p(z, y)$ for all $y \in X$.*

(b) [24] *If (X, p) is complete, then it is 0-complete.*

Example 2.10 ([24]). The space $X = [0,\infty) \cap \mathbb{Q}$, equipped with the partial metric $p(x,y) = \max\{x, y\}$, is 0-complete but not complete. The constant sequence $x_n = 1$ is Cauchy but not 0-Cauchy.

Remark 2.11. [29] Every closed subset of a 0-complete partial metric space is 0-complete.

Definition 2.12 ([1]) Let X be a nonempty set, $q \in \mathbb{N}$, and $f : X \to X$. A **cyclic representation** of X w.r.t. f is $X = \bigcup_{i=1}^{q} A_i$, where:

- A_i are nonempty subsets of X,
- $f(A_1) \subseteq A_2, \cdots, f(A_q) \subseteq A_1$.

Definition 2.13 ([50] [51]).

- A **coincidence point** of T and S is $x \in X$ such that $Tx = Sx$.
- $y = Tx = Sx$ is called a **point of coincidence**.

Definition 2.14 ([20] [51]). Mappings $T, S : X \to X$ are **weakly compatible** if $T(Sx) = S(Tx)$ whenever $Sx = Tx$.

Proposition 2.15 ([50] [51]). *If T and S are weakly compatible and have a unique point of coincidence y, then y is their unique common fixed point.*

Definition 2.16 ([32]). A function $\gamma : [0,\infty) \to [0,\infty)$ is an **altering distance function** if:

- γ is continuous and nondecreasing,
- $\gamma(t) = 0 \Leftrightarrow t = 0$.

We denote the set of altering distance functions by Γ.

Definition 2.17 ([8]). *Let (X, d) be a metric space. An operator $f : X \to X$ is a **cyclic weaker ψ-contraction** if:*

1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation w.r.t. f,

2) There exists a continuous, nondecreasing $\psi : [0,1] \to [0,1)$ with $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$, such that

$$d(fx, fy) \le d(x,y) - \psi(d(x,y)) \quad \forall x \in A_i, y \in A_{i+1}.$$

Theorem 2.18 ([8]). *Every cyclic weaker ψ-contraction on a complete metric space has a fixed point in $\bigcap_{i=1}^{m} A_i$.*

Definition 2.19 ([35]). A mapping $T : X \to X$ is a **weak C-contraction** if

$$d(Tx,Ty) \le \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]-\psi\left(d(x,Ty),d(y,Tx)\right),$$

where $\psi:[0,\infty)^2 \to [0,\infty)$ is continuous and $\psi(x,y)=0 \Leftrightarrow x=y=0$.

Theorem 2.20 ([35]). *Every weak C-contraction on a complete metric space has a unique fixed point.*

Example 2.21 ([24]). *$X=[0,\infty)\cap\mathbb{Q}$ with $p(x,y)=\max\{x,y\}$ is 0-complete but not complete.*

Definition 2.22 ([21]) Let (X,p) be a PMS, $C\subset X$ and $\varphi:C\to\mathbb{R}^+$ a function on C. Then, the function φ is called *lower semi-continuous* (*l.s.c.*) on C whenever

$$\lim_{n\to\infty} p(x_n,x)=p(x,x) \Rightarrow \varphi(x) \le \liminf_{n\to\infty}\varphi(x_n)=\sup_{n\ge 1}\inf_{m\ge n}\varphi(x_m).$$

In 2013, Nashine *et al.* [47] introduced a class of generalized control functions as follows:

Let Ψ denote the class of all functions $\psi:[0,\infty)^2\to[0,\infty)$ satisfying the following conditions:

(a) ψ is lower semicontinuous;

(b) $\psi(s,t)=0$ if and only if $s=t=0$.

In 2021, Mohanta and Patra [29] established the following coincidence point and common fixed point result for a pair of self-mappings satisfying some generalized cyclic contraction type conditions involving a control function with two variables in partial metric spaces

Theorem 2.22. *Let (X,p) be a 0-complete partial metric space, $q\in\mathbb{N}$ and $A_1,A_2,\cdots,A_q$ be nonempty subsets of X. Suppose the mappings $T,f:X\to X$ are such that $f(A_1),f(A_2),\cdots,f(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

(C1) $T(A_i)\subseteq f(A_{i+1})$ *for* $i=1,2,\cdots,q$, *where* $A_{q+1}=A_1$;

(C2) *there exists* $\psi\in\Psi$ *such that*

$$p(Tx,Ty)\le M(fx,fy)-\psi\left(p(fx,fy),p(fx,Tx)\right)$$

for any $(fx,fy)\in f(A_i)\times f(A_{i+1})$, $i=1,2,\cdots,q$ *with* $A_{q+1}=A_1$, *where*

$$M(fx,fy)=\max\left\{p(fx,fy),p(fx,Tx),p(fy,Ty),\frac{p(fx,Ty)+p(Tx,fy)}{2}\right\}. \quad (2.1)$$

Then T and f have a unique point of coincidence u in $\bigcap_{i=1}^q f(A_i)$ with $p(u,u)=0$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in $\bigcap_{i=1}^q f(A_i)$.

In the next section, we prove a coincidence point and common fixed point theorem for a pair of self-mappings on a 0-complete partial metric space, under a generalized contractive condition involving an altering distance function and a two-variable control function. This result generalizes Theorem 3.1 of [29].

3. Main Results

Theorem 3.1. *Let (X,p) be a 0-complete partial metric space, and let $q\in\mathbb{N}$.*

Suppose $A_1, A_2, \cdots, A_q$ *are nonempty subsets of* X, *and let* $T, f : X \to X$ *be two mappings such that the images* $f(A_1), f(A_2), \cdots, f(A_q)$ *are closed subsets of* (X, p). *Assume the following conditions are satisfied:*

(C1) $T(A_i) \subseteq f(A_{i+1})$ *for all* $i = 1, 2, \cdots, q$, *where* $A_{q+1} := A_1$.

(C2) *There exist functions* $\psi \in \Psi$ *and* $\gamma \in \Gamma$ *such that for all* $(fx, fy) \in f(A_i) \times f(A_{i+1})$, $i = 1, 2, \cdots, q$ *with* $A_{q+1} = A_1$, *the following inequality holds:*

$$\gamma(p(Tx, Ty)) \le \gamma(M(fx, fy)) - \psi(\gamma(p(fx, fy)), \gamma(p(fx, Tx))),$$

where

$$M(fx, fy) = \max\left\{p(fx, fy), p(fx, Tx), p(fy, Ty), \frac{p(fx, Ty) + p(Tx, fy)}{2}\right\}. \quad (3.1)$$

Then T *and* f *have a unique point of coincidence* $u \in \bigcap_{i=1}^{q} f(A_i)$ *with* $p(u, u) = 0$. *Furthermore, if* T *and* f *are weakly compatible, then* u *is their unique common fixed point in* $\bigcap_{i=1}^{q} f(A_i)$.

Proof. Let $Y = \bigcup_{i=1}^{q} A_i$, and let $x_0 \in Y$ be arbitrary. Then there exists $i_0 \in \{1, 2, \cdots, q\}$ such that $x_0 \in A_{i_0}$. Since $T(A_{i_0}) \subseteq f(A_{i_0+1})$, there exists $x_1 \in A_{i_0+1}$ such that $u_1 = Tx_0$. Continuing this process, we construct a sequence (x_n) with $u_n = Tx_{n-1}$, for $n = 1, 2, 3, \cdots$, where $x_n \in A_{i_0+n}$ and $A_{q+k} := A_k$. Define $u_n := f(x_n)$. Then $u_n \in f(A_{j_n})$, and $T(x_n) = u_{n+1}$. If $p(u_n, u_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $u_n = u_{n+1} = Tx_n$, so u_{n+1} is a point of coincidence of T and f. Assume $p(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since $\psi(s, t) > 0$ for $s + t > 0$ and $\gamma(t) > 0$ for $t > 0$, we have:

$$\psi(\gamma(p(u_n, u_{n+1})), \gamma(p(u_n, u_{n+1}))) > 0, \ \forall n \in \mathbb{N}. \quad (3.2)$$

For each $n \in \mathbb{N}$, there exists $i \in \{1, 2, \cdots, q\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$, so $(u_n, u_{n+1}) \in f(A_i) \times f(A_{i+1})$. Applying (C2):

$$\begin{aligned} \gamma(p(u_{n+1}, u_{n+2})) &= \gamma(p(Tx_n, Tx_{n+1})) \\ &\le \gamma(M(u_n, u_{n+1})) - \psi(\gamma(p(u_n, u_{n+1})), \gamma(p(u_n, Tx_n))) \quad (3.3) \\ &= \gamma(M(u_n, u_{n+1})) - \psi(\gamma(p(u_n, u_{n+1})), \gamma(p(u_n, u_{n+1}))), \end{aligned}$$

where

$$\begin{aligned} &M(u_n, u_{n+1}) \\ &= \max\left\{p(u_n, u_{n+1}), p(u_n, Tx_n), p(u_{n+1}, Tx_{n+1}), \frac{p(u_n, Tx_{n+1}) + p(Tx_n, u_{n+1})}{2}\right\} \\ &\le \max\left\{p(u_n, u_{n+1}), p(u_{n+1}, u_{n+2}), \frac{p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2})}{2}\right\} \\ &\le \max\{p(u_n, u_{n+1}), p(u_{n+1}, u_{n+2})\}. \end{aligned}$$

So,

$$M(u_n, u_{n+1}) \le \max\{p(u_n, u_{n+1}), p(u_{n+1}, u_{n+2})\}.$$

Since γ is non-decreasing, it preserves inequalities and commutes with the maximum operator due to its monotonicity. Thus, we obtain:

$$\begin{aligned}\gamma\left(M\left(u_n,u_{n+1}\right)\right)&\le\gamma\left(\max\left\{p\left(u_n,u_{n+1}\right),p\left(u_{n+1},u_{n+2}\right)\right\}\right)\\&=\max\left\{\gamma\left(p\left(u_n,u_{n+1}\right)\right),\gamma\left(p\left(u_{n+1},u_{n+2}\right)\right)\right\}.\end{aligned}\tag{3.4}$$

Let $a_n:=\gamma\left(p\left(u_n,u_{n+1}\right)\right)$. Then from (3.3) using (3.4), we obtain:

$$a_{n+1}\le\max\left\{a_n,a_{n+1}\right\}-\psi\left(a_n,a_n\right).\tag{3.5}$$

If $\max\left\{a_n,a_{n+1}\right\}=a_{n+1}$, then:

$$a_{n+1}\le a_{n+1}-\psi\left(a_n,a_n\right)<a_{n+1},$$

a contradiction. Thus $\max\left\{a_n,a_{n+1}\right\}=a_n$, and (3.5) becomes:

$$a_{n+1}\le a_n-\psi\left(a_n,a_n\right)<a_n.\tag{3.6}$$

Hence, the sequence $\{a_n\}$ is decreasing and bounded below by zero, and thus converges to some limit $L\ge 0$. Taking the limit in (3.6) and using the continuity of ψ, we obtain:

$$L\le L-\lim_{n\to\infty}\psi\left(a_n,a_n\right)\le L-\psi\left(L,L\right),$$

which implies $\psi\left(L,L\right)=0$, and hence $L=0$. Therefore,

$$\lim_{n\to\infty}\gamma\left(p\left(u_n,u_{n+1}\right)\right)=0.\tag{3.7}$$

Since γ is continuous and $\gamma(t)=0$ if and only if $t=0$, it follows from (3.7) that

$$\lim_{n\to\infty}p\left(u_n,u_{n+1}\right)=0.\tag{3.8}$$

To show that the sequence (u_n) is 0-Cauchy, suppose the contrary. Then there exists $\epsilon>0$ and subsequences $\left(u_{m_i}\right)$ and $\left(u_{n_i}\right)$ with $n_i>m_i>i$ such that:

$$p\left(u_{m_i},u_{n_i}\right)\ge\epsilon\ \text{and}\ p\left(u_{m_i},u_{n_i-1}\right)<\epsilon.\tag{3.9}$$

Using conditions (3.9), and property (p4), we obtain:

$$\begin{aligned}\epsilon&\le p\left(u_{m_i},u_{n_i}\right)\\&\le p\left(u_{m_i},u_{n_i-1}\right)+p\left(u_{n_i-1},u_{n_i}\right)-p\left(u_{n_i-1},u_{n_i-1}\right)\\&<\epsilon+p\left(u_{n_i-1},u_{n_i}\right),\end{aligned}$$

which implies:

$$\epsilon\le p\left(u_{m_i},u_{n_i}\right)<\epsilon+p\left(u_{n_i-1},u_{n_i}\right).$$

Taking the limit as $i\to\infty$ and using condition (3.8), we get:

$$\lim_{i\to\infty}p\left(u_{m_i},u_{n_i}\right)=\epsilon.\tag{3.10}$$

Note that for each i, there exists $r_i\in\{1,2,\cdots,q\}$ such that $n_i-m_i+r_i\equiv 1(\operatorname{mod} q)$. Hence, for large i, $x_{m_i-r_i}$ and x_{n_i} lie in different, consecutively indexed sets A_j and A_{j+1} (modulo q). Thus,

$$\left(u_{m_i-r_i},u_{n_i}\right)\in f\left(A_j\right)\times f\left(A_{j+1}\right).$$

Using condition (C2), we get:

$$\begin{aligned}\gamma\left(p\left(u_{m_i-r_i+1},u_{n_i+1}\right)\right)&=\gamma\left(p\left(Tx_{m_i-r_i},Tx_{n_i}\right)\right)\\&\le\gamma\left(M\left(u_{m_i-r_i},u_{n_i}\right)\right)-\psi\left(\gamma\left(p\left(u_{m_i-r_i},u_{n_i}\right)\right),\gamma\left(p\left(u_{m_i-r_i},Tx_{m_i-r_i}\right)\right)\right),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}M\left(u_{m_i-r_i},u_{n_i}\right)=\max\Bigg\{&p\left(u_{m_i-r_i},u_{n_i}\right),p\left(u_{m_i-r_i},Tx_{m_i-r_i}\right),p\left(u_{n_i},Tx_{n_i}\right),\\&\frac{p\left(u_{m_i-r_i},Tx_{n_i}\right)+p\left(Tx_{m_i-r_i},u_{n_i}\right)}{2}\Bigg\}.\end{aligned}\tag{3.12}$$

We now show:

$$\lim_{i\to\infty}p\left(u_{m_i-r_i},u_{m_i}\right)=0.\tag{3.13}$$

By repeated use of (p4), we have:

$$p\left(u_{m_i-r_i},u_{m_i}\right)\le\sum_{l=0}^{r_i-1}p\left(u_{m_i-r_i+l},u_{m_i-r_i+l+1}\right)\le\sum_{l=0}^{q-1}p\left(u_{m_i-r_i+l},u_{m_i-r_i+l+1}\right).$$

Taking the limit and using (3.8), we obtain (3.13). From (p4), we also have:

$$\begin{aligned}p\left(u_{m_i-r_i},u_{n_i}\right)&\le p\left(u_{m_i-r_i},u_{m_i}\right)+p\left(u_{m_i},u_{n_i}\right)-p\left(u_{m_i},u_{m_i}\right)\\&\le p\left(u_{m_i-r_i},u_{m_i}\right)+p\left(u_{m_i},u_{n_i}\right).\end{aligned}$$

Hence,

$$\limsup_{i\to\infty}p\left(u_{m_i-r_i},u_{n_i}\right)\le\epsilon.\tag{3.14}$$

Also,

$$\epsilon\le p\left(u_{m_i},u_{n_i}\right)\le p\left(u_{n_i},u_{m_i-r_i}\right)+p\left(u_{m_i-r_i},u_{m_i}\right)-p\left(u_{m_i-r_i},u_{m_i-r_i}\right),$$

which implies

$$\epsilon\le\limsup_{i\to\infty}p\left(u_{m_i-r_i},u_{n_i}\right)\le\epsilon,$$

and hence

$$\lim_{i\to\infty}p\left(u_{m_i-r_i},u_{n_i}\right)=\epsilon.\tag{3.15}$$

Using similar arguments and (3.8), we also derive:

$$\lim_{i\to\infty}p\left(u_{n_i+1},u_{m_i-r_i}\right)=\epsilon,\tag{3.16}$$

$$\lim_{i\to\infty}p\left(u_{n_i},u_{m_i-r_i+1}\right)=\epsilon,\tag{3.17}$$

$$\lim_{i\to\infty}p\left(u_{n_i+1},u_{m_i-r_i+1}\right)=\epsilon.\tag{3.18}$$

From (3.12) and the limits above, we have:

$$\lim_{i\to\infty} M\left(u_{m_i - r_i}, u_{n_i}\right) = \epsilon. \tag{3.19}$$

Now take the limit in (3.1), and apply the continuity of γ and the lower semicontinuity of ψ, we get:

$$\gamma(\epsilon) \le \gamma(\epsilon) - \psi(\gamma(\epsilon), 0),$$

which implies:

$$\psi(\gamma(\epsilon), 0) \le 0.$$

But by assumption, $\psi(s,t) > 0$ for $s + t > 0$, so:

$$\gamma(\epsilon) = 0 \Rightarrow \epsilon = 0,$$

a contradiction. Hence, (u_n) is a 0-Cauchy sequence in $f(Y)$. Since $f(Y) = \bigcup_{i=1}^{q} f(A_i)$ and $f(Y)$ is closed in the 0-complete space (X, p), it follows that $f(Y)$ is 0-complete. So, (u_n) converges to a point $u \in f(Y)$ such that:

$$\lim_{n\to\infty} f(gx_n, u) = p(u,u) = 0. \tag{3.20}$$

We now prove:

$$u \in \bigcap_{i=1}^{q} f(A_i). \tag{3.21}$$

As $x_0 \in A_{i_0}$, by (C1), the sequence $(u_{nq})_{n\ge 0} \subseteq f\left(A_{i_0}\right)$. Since $f\left(A_{i_0}\right)$ is closed, by (3.20), $u \in f\left(A_{i_0}\right)$. By (C1), $\left(u_{nq+1}\right)_{n\ge 0} \subseteq f\left(A_{i_0+1}\right)$. Repeating this for q steps (modulo q), we obtain:

$$u \in f\left(A_{i_0}\right) \cap f\left(A_{i_0+1}\right) \cap \cdots \cap f\left(A_{i_0+q}\right) = \bigcap_{i=1}^{q} f(A_i).$$

Now we show that u is a point of coincidence of T and f. Since $u \in f(Y)$, there exists $t \in Y$ such that $u = f(t)$. Let $x_n \in A_i$ for some $i \in \{1, 2, \cdots, q\}$. Then, since $u \in \bigcap_{i=1}^{q} f(A_i)$, we have

$$\left(f(t), u_n\right) = (u, u_n) \in f(A_{i-1}) \times f(A_i),$$

where $A_0 := A_q$. Applying condition (C2), we get an inequality involving the mappings. Now choose $z \in X$ such that $f(z) = u$. We claim that $T(z) = u$. For each $n \in \mathbb{N}$, apply condition (C2) with $x = z$ and $y = x_n$:

$$\begin{aligned} \gamma\left(p(Tz, u_{n+1})\right) &= \gamma\left(p(Tz, Tx_n)\right) \\ &\le \gamma\left(M(fz, fx_n)\right) - \psi\left(\gamma\left(p(fz, fx_n)\right), \gamma\left(p(fz, Tz)\right)\right) \\ &= \gamma\left(M(u, u_n)\right) - \psi\left(\gamma\left(p(u, u_n)\right), \gamma\left(p(u, Tz)\right)\right). \end{aligned} \tag{3.22}$$

As $n \to \infty$, we have $p(u, u_n) \to 0$, so by continuity of γ,

$$\gamma\left(p(u, u_n)\right) \to 0 \text{ and } \gamma\left(M(u, u_n)\right) \to \gamma\left(p(u, fz)\right) = \gamma\left(p(u, u)\right) = 0.$$

Taking the upper limit as $n \to \infty$ in inequality (3.22), and using Lemma 2.9

and the lower semicontinuity of γ, we obtain:

$$\gamma\left(p\left(Tz,u\right)\right)\leq\gamma\left(p\left(u,Tz\right)\right)-\psi\left(0,\gamma\left(p\left(u,Tz\right)\right)\right).$$

Suppose $p\left(Tz,u\right)>0$. Then $\gamma\left(p\left(u,Tz\right)\right)>0$, and by the assumption on ψ, we have $\psi\left(0,\gamma\left(p\left(u,Tz\right)\right)\right)>0$. This leads to:

$$\gamma\left(p\left(Tz,u\right)\right)<\gamma\left(p\left(u,Tz\right)\right)=\gamma\left(p\left(Tz,u\right)\right),$$

a contradiction. Hence, $p\left(Tz,u\right)=0$, so $T\left(z\right)=u=f\left(z\right)$. Therefore, u is a point of coincidence of T and f, with $u\in\bigcap_{i=1}^{q}f\left(A_i\right)$ and $p\left(u,u\right)=0$. To prove uniqueness, assume that there exists another point of coincidence $v\in\bigcap_{i=1}^{q}f\left(A_i\right)$ with $p\left(v,v\right)=0$. Then there exists $w\in X$ such that $v=f\left(w\right)=T\left(w\right)$. Since both $u\in f\left(A_i\right)$ and $v\in f\left(A_{i+1}\right)$ for some i, applying (C2) yields:

$$\begin{aligned}\gamma\left(p\left(u,v\right)\right)&=\gamma\left(p\left(Tz,Tw\right)\right)\\&\leq\gamma\left(M\left(fz,fw\right)\right)-\psi\left(\gamma\left(p\left(fz,fw\right)\right),\gamma\left(p\left(fz,Tz\right)\right)\right)\\&=\gamma\left(M\left(u,v\right)\right)-\psi\left(\gamma\left(p\left(u,v\right)\right),\gamma\left(p\left(u,u\right)\right)\right)\\&=\gamma\left(p\left(u,v\right)\right)-\psi\left(\gamma\left(p\left(u,v\right)\right),0\right).\end{aligned}$$

Hence, $\psi\left(\gamma\left(p\left(u,v\right)\right),0\right)\leq 0$, which implies $\psi\left(\gamma\left(p\left(u,v\right)\right),0\right)=0$. By the properties of ψ, this gives $\gamma\left(p\left(u,v\right)\right)=0$, and thus $p\left(u,v\right)=0$, implying $u=v$. Therefore, T and f have a unique point of coincidence $u\in\bigcap_{i=1}^{q}f\left(A_i\right)$ with $p\left(u,u\right)=0$. Finally, if T and f are weakly compatible, then by Proposition (2.15), they have a unique common fixed point in $\bigcap_{i=1}^{q}f\left(A_i\right)$. □

Remark 3.2. If we take $\gamma\left(t\right)=t$ in Theorem 3.1, we recover Theorem 3.1 of [29]. Moreover, Corollaries 3.2 through 3.7 follow directly as special cases of Theorem (0.23).

Remark 3.3. If we set $g=I$ and $\gamma\left(t\right)=t$ in Theorem 3.1, we obtain Theorem 13 of [47]. Furthermore, as a special case of Corollary 3.6, several classical fixed point results in partial metric spaces can be deduced, including the Matthews version of Banach's contraction principle [10].

The following example illustrates the importance of using an altering distance function γ in fixed point theory within partial metric spaces. Specifically, we show that a fixed point result may fail under a standard contraction but holds when modified with a suitable altering function.

Example 3.4. Let $X=\left[0,1\right]$ be equipped with the partial metric $p\left(x,y\right)=\max\left\{x,y\right\}$. Define the subsets:

$$A_1=\left[0,0.6\right],\ A_2=\left[0.4,1\right],$$

and the mappings $T,f:X\to X$ as follows:

$$Tx=\begin{cases}0.4+\dfrac{x}{10}, & x\in A_1,\\ \dfrac{x}{15}, & x\in A_2,\end{cases}\qquad fx=x.$$

Let the altering distance functions be $\gamma(t)=t^3$ and $\psi(s,t)=\frac{s+t}{8}$.

Verification of Conditions. We observe that Condition (C1) (Cyclic Representation) is satisfied. The images of the sets under T satisfy:

$$T(A_1)=[0.4,0.46]\subseteq A_2=[0.4,1],$$

$$T(A_2)=\left[\frac{0.4}{15},\frac{1}{15}\right]\approx[0.0267,0.0667]\subseteq A_1=[0,0.6].$$

Next we show that for all $x\in A_1$, $y\in A_2$, the contractive condition

$$\gamma\big(p(Tx,Ty)\big)\le\gamma\big(M(fx,fy)\big)-\psi\Big(\gamma\big(p(fx,fy)\big),\gamma\big(p(fx,Tx)\big)\Big)\tag{3.23}$$

is satisfied, where

$$M(fx,fy)=\max\left\{p(fx,fy),p(fx,Tx),p(fy,Ty),\frac{p(fx,Ty)+p(Tx,fy)}{2}\right\}.$$

Key Observations.

1) For $x\in A_1$, $Tx\in[0.4,0.46]$; for $y\in A_2$, $Ty\in[0.0267,0.0667]$.

2) Since $Tx>Ty$, we have:

$$p(Tx,Ty)=Tx\ \Rightarrow\ \gamma\big(p(Tx,Ty)\big)=(Tx)^3.$$

3) Lower bound estimate for M:

$$M\ge\max\{\max\{x,Tx\},y\},\ \text{ as } y>Ty.$$

Analytical Verification.

Case 1: If $x\le 0.4+\frac{x}{10}$ (*i.e.*, $x\le\frac{4}{9}\approx 0.4444$), then $\max\{x,Tx\}=Tx$, so:

$$M\ge\max\{Tx,y\}\ \text{ and }\ \text{RHS}\ge\big(\max\{Tx,y\}\big)^3-\frac{[p(x,y)]^3+(Tx)^3}{8}.$$

Case 2: If $x>\frac{4}{9}$, then $\max\{x,Tx\}=x$, and:

$$M\ge\max\{x,y\},\ \text{RHS}\ge\big(\max\{x,y\}\big)^3-\frac{[p(x,y)]^3+x^3}{8}.$$

Case 3: Take critical values $x=y=0.5$:

$$Tx=0.4+0.05=0.45,\ Ty=\frac{0.5}{15}\approx 0.0333,$$

$$p(Tx,Ty)=0.45,\ \gamma\big(p(Tx,Ty)\big)=0.45^3=0.091125,$$

$$M=\max\{0.5,0.5,0.5,0.5\}=0.5,\ \gamma(M)=0.5^3=0.125,$$

$$\psi=\frac{0.125+0.125}{8}=0.03125,\ \text{RHS}=0.125-0.03125=0.09375,$$

$$\Rightarrow\gamma\big(p(Tx,Ty)\big)=0.091125<0.09375\ \text{(inequality satisfied)}.$$

Next we show that the contractive condition (3.23) fails to hold at $x=y=0.6$ with $\gamma(t)=t$. Let $x=y=0.6$. Then:

$$Tx = 0.4 + \frac{0.6}{10} = 0.46,\ Ty = \frac{0.6}{15} = 0.04,$$

$$p(Tx,Ty) = 0.46,\ \gamma(p(Tx,Ty)) = 0.46,$$

$$M = \max\{0.6, 0.6, 0.6, 0.6\} = 0.6,\ \gamma(M) = 0.6,$$

$$\psi = \frac{0.6+0.6}{8} = 0.15,\ \text{RHS} = 0.6 - 0.15 = 0.45,$$

$$\Rightarrow \gamma(p(Tx,Ty)) = 0.46 > 0.45\ (\text{inequality fails}).$$

Finally, the unique common fixed point of T and f is $z = \frac{4}{9}$.

4. Conclusions and Open Questions

In this paper, we established a common fixed point theorem for generalized cyclic contraction pairs in 0-complete partial metric spaces, incorporating altering distance functions (γ) and control functions (ψ). Our results extend and unify several existing theorems, including those of [29] and [47]. The introduction of nonlinear altering distance functions (e.g., $\gamma(t) = t^3$) allows for fixed point results in cases where traditional linear contractions fail, as demonstrated in Example 3.4.

Future research could explore:

1) **Weaker Contraction Conditions**: Can the assumptions on ψ or γ be relaxed?

2) **Multivalued Mappings**: Do analogous results hold for set-valued cyclic contractions?

This study contributes to the broader landscape of fixed point theory, offering a more flexible framework for analyzing cyclic mappings in generalized metric spaces.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kirk, W.A., Srinivasan, P.S. and Veeramani, P. (2003) Fixed Points for Mappings Satisfying Cyclical Contractive Conditions. *Fixed Point Theory*, **4**, 79-89.

[2] Alemayehu, G.N. (2014) Common Fixed Point Theorems for Cocyclic Weak Contractions in Compact Metric Spaces. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering*, **8**, 411-413.

[3] Agarwal, R.P., Alghamdi, M.A. and Shahzad, N. (2012) Fixed Point Theory for Cyclic Generalized Contractions in Partial Metric Spaces. *Fixed Point Theory and Applications*, **2012**, Article No. 40. https://doi.org/10.1186/1687-1812-2012-40

[4] Karapınar, E. (2011) Fixed Point Theory for Cyclic Weak φ-Contraction. *Applied Mathematics Letters*, **24**, 822-825. https://doi.org/10.1016/j.aml.2010.12.016

[5] Karapınar, E. and Yuce, I.S. (2012) Fixed Point Theory for Cyclic Generalized Weak ϕ-Contraction on Partial Metric Spaces. *Abstract and Applied Analysis*, **2012**, Article ID: 491542. https://doi.org/10.1155/2012/491542

[6] Karapınar, E., Shobkolaei, N., Sedghi, S. and Vaezpour, M. (2012) A Common Fixed Point Theorem for Cyclic Operators on Partial Metric Spaces. *Filomat*, **26**, 407-414. https://doi.org/10.2298/fil1202407k

[7] Nashine, H.K. (2012) Cyclic Generalized-Weakly Contractive Mappings and Fixed Point Results with Applications to Integral Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **75**, 6160-6169. https://doi.org/10.1016/j.na.2012.06.021

[8] Păcurar, M. and Rus, I.A. (2010) Fixed Point Theory for Cyclic-Contractions. *Nonlinear Analysis: Theory, Methods & Applications*, **72**, 1181-1187. https://doi.org/10.1016/j.na.2009.08.002

[9] Sharbu, B., Geremew, A. and Berhane, A. (2017) A Common Fixed Point Theorem for Reich Type Co-Cyclic Contraction in Dislocated Quasi-Metric Paces. *Ethiopian Journal of Science and Technology*, **10**, 81-94. https://doi.org/10.4314/ejst.v10i2.1

[10] Matthews, S.G. (1994) Partial Metric Topology. *Annals of the New York Academy of Sciences*, **728**, 183-197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

[11] Abdeljawad, T. (2011) Fixed Points for Generalized Weakly Contractive Mappings in Partial Metric Spaces. *Mathematical and Computer Modelling*, **54**, 2923-2927. https://doi.org/10.1016/j.mcm.2011.07.013

[12] Altun, I. and Acar, Ö. (2012) Fixed Point Theorems for Weak Contractions in the Sense of Berinde on Partial Metric Spaces. *Topology and Its Applications*, **159**, 2642-2648. https://doi.org/10.1016/j.topol.2012.04.004

[13] Babu, G.V.R. and Ratna Babu, D. (2019) Common Fixed Points of Rational Type and Geraghty-Suzuki Type Contraction Maps in Partial Metric Spaces. *Journal of the International Mathematical Virtual Institute*, **9**, 341-359.

[14] Bari, C.D. and Vetro, P. (2011) Fixed Points for Weak ψ-Contractions on Partial Metric Spaces. *International Journal of Contemporary Mathematical Sciences*, **6**, 5-12.

[15] Bukatin, M., Kopperman, R., Matthews, S. and Pajoohesh, H. (2009) Partial Metric Spaces. *American Mathematical Monthly*, **116**, 708-718. https://doi.org/10.4169/193009709x460831

[16] Ćirić, L., Samet, B., Aydi, H. and Vetro, C. (2011) Common Fixed Points of Generalized Contractions on Partial Metric Spaces and an Application. *Applied Mathematics and Computation*, **218**, 2398-2406. https://doi.org/10.1016/j.amc.2011.07.005

[17] Dwivedi, P.K. (2022) Common Fixed Point Theorem on Partial Metric Space. *International Journal of Mathematics Trends and Technology*, **68**, 30-37. https://doi.org/10.14445/22315373/ijmtt-v68i4p506

[18] Gangopadhyay, M., Saha, M. and Baisnab, A.P. (2013) Some Fixed Point Theorem in Partial Metric Spaces. *TWMS Journal of Applied and Engineering Mathematics*, **3**, 206-213.

[19] Heckmann, R. (1999) Approximation of Metric Spaces by Partial Metric Spaces. *Applied Categorical Structures*, **7**, 71-83. https://doi.org/10.1023/a:1008684018933

[20] Jungck, G. (1996) Common Fixed Points for Noncontinuous Non-Self Maps on Non-Metric Spaces. *Far East Journal of Mathematical Sciences*, **4**, 199-215.

[21] Karapinar, E. (2011) Generalizations of Caristi Kirk's Theorem on Partial Metric Spaces. *Fixed Point Theory and Applications*, **2011**, Article No. 4. https://doi.org/10.1186/1687-1812-2011-4

[22] Karapınar, E. and Erhan, İ.M. (2011) Fixed Point Theorems for Operators on Partial Metric Spaces. *Applied Mathematics Letters*, **24**, 1894-1899. https://doi.org/10.1016/j.aml.2011.05.013

[23] Oltra, S. and Valero, O. (2004) Banach's Fixed Point Theorem for Partial Metric

Spaces. *Rendiconti dell' Istituto di Matematica dell' Università di Trieste*, **36**, 17-26.

[24] Romaguera, S. (2009) A Kirk Type Characterization of Completeness for Partial Metric Spaces. *Fixed Point Theory and Applications*, **2010**, Article ID 493298. https://doi.org/10.1155/2010/493298

[25] Valero, O. (2005) On Banach Fixed Point Theorems for Partial Metric Spaces. *Applied General Topology*, **6**, 229-240. https://doi.org/10.4995/agt.2005.1957

[26] Karapinar, E. (2012) Weak ψ-Contraction on Partial Metric Spaces. *Journal of Computational Analysis and Applications*, **14**, 206-210.

[27] Karapınar, E. and Yüksel, U. (2011) Some Common Fixed Point Theorems in Partial Metric Spaces. *Journal of Applied Mathematics*, **2011**, Article ID 263621. https://doi.org/10.1155/2011/263621

[28] Karapinar, E. (2011) A Note on Common Fixed Point Theorems in Partial Metric Spaces. *Miskolc Mathematical Notes*, **12**, 185-191. https://doi.org/10.18514/mmn.2011.335

[29] Mohanta, S.K. and Biswas, P. (2021) Generalized Cyclic Contractions and Coincidence Points Involving a Control Function on Partial Metric Spaces. *Journal of Nonlinear Analysis and Optimization*, **12**, 61-81.

[30] Nashine, H.K. and Karapinar, E. (2013) Fixed Point Results in Orbitally Complete Partial Metric Spaces. *Bulletin of the Malaysian Mathematical Sciences Society*, **36**, 1185-1193.

[31] Romaguera, S. (2012) Fixed Point Theorems for Generalized Contractions on Partial Metric Spaces. *Topology and its Applications*, **159**, 194-199. https://doi.org/10.1016/j.topol.2011.08.026

[32] Khan, M.S., Swaleh, M. and Sessa, S. (1984) Fixed Point Theorems by Altering Distances between the Points. *Bulletin of the Australian Mathematical Society*, **30**, 1-9. https://doi.org/10.1017/s0004972700001659

[33] Ahmad, A.G.B., Fadail, Z.M., Nashine, H.K., Kadelburg, Z. and Radenović, S. (2012) Some New Common Fixed Point Results through Generalized Altering Distances on Partial Metric Spaces. *Fixed Point Theory and Applications*, **2012**, Article No. 120. https://doi.org/10.1186/1687-1812-2012-120

[34] Aydi, H. (2013) A Common Fixed Point Result by Altering Distances Involving a Contractive Condition of Integral Type in Partial Metric Spaces. *Demonstratio Mathematica*, **46**, 383-394. https://doi.org/10.1515/dema-2013-0450

[35] Choudhury, B.S. (1970) Unique Fixed Point Theorem for Weakly C-Contractive Mappings. *Kathmandu University Journal of Science*, *Engineering and Technology*, **5**, 6-13. https://doi.org/10.3126/kuset.v5i1.2842

[36] Sastry, K.P.R., Naidu, S.V.R., Babu, G.V.R. and Naidu, G.A. (2000) Generalization of Common Fixed Point Theorems for Weakly Commuting Maps by Altering Distances. *Tamkang Journal of Mathematics*, **31**, 243-250. https://doi.org/10.5556/j.tkjm.31.2000.399

[37] Shatanawi, W. and Postolache, M. (2010) Common Fixed Point Results for Mappings under Nonlinear Contraction of Cyclic form in Ordered Metric Spaces. *Fixed Point Theory and Applications*, **2010**, Article ID: 493298.

[38] Abbas, M., Nazir, T. and Romaguera, S. (2011) Fixed Point Results for Generalized Cyclic Contraction Mappings in Partial Metric Spaces. *Revista de la Real Academia de Ciencias Exactas*, *Fisicas y Naturales. Serie A. Matematicas*, **106**, 287-297. https://doi.org/10.1007/s13398-011-0051-5

[39] Altun, I., Sola, F. and Simsek, H. (2010) Generalized Contractions on Partial Metric

Spaces. *Topology and its Applications*, **157**, 2778-2785. https://doi.org/10.1016/j.topol.2010.08.017

[40] Haghi, R.H., Rezapour, S. and Shahzad, N. (2011) Some Fixed Point Generalizations Are Not Real Generalizations. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **74**, 1799-1803. https://doi.org/10.1016/j.na.2010.10.052

[41] Haghi, R.H., Rezapour, S. and Shahzad, N. (2013) Be Careful on Partial Metric Fixed Point Results. *Topology and its Applications*, **160**, 450-454. https://doi.org/10.1016/j.topol.2012.11.004

[42] He, F. and Chen, A. (2016) Fixed Points for Cyclic φ-Contractions in Generalized Metric Spaces. *Fixed Point Theory and Applications*, **2016**, Article No. 67. https://doi.org/10.1186/s13663-016-0558-8

[43] Karapinar, E. and Nashine, H.K. (2013) Fixed Point Theorems for Kanaan Type Cyclic Weakly Contractions. *Nonlinear Analysis and Optimization*, **4**, 29-35.

[44] Mohanta, S.K. (2011) A Fixed Point Theorem via Generalized W-Distance. *Bulletin of Mathematical Analysis and Applications*, **3**, 134-139.

[45] Mohanta, S.K. and Mohanta, S. (2012) A Common Fixed Point Theorem in G-Metric Spaces. *Cubo* (*Temuco*), **14**, 85-101. https://doi.org/10.4067/s0719-06462012000300006

[46] Mohanta, S.K. and Patra, S. (2017) Coincidence Points and Common Fixed Points for Hybrid Pair of Mappings in b-Metric Spaces Endowed with a Graph. *Journal of Linear and Topological Algebra*, **6**, 301-321.

[47] Nashine, H.K. and Kadelburg, Z. (2013) Cyclic Contractions and Fixed Point Results via Control Functions on Partial Metric Spaces. *International Journal of Analysis*, **2013**, Article ID: 726387. https://doi.org/10.1155/2013/726387

[48] Yamaod, O., Sintunavarat, W. and Cho, Y.J. (2015) Common Fixed Point Theorems for Generalized Cyclic Contraction Pairs in B-Metric Spaces with Applications. *Fixed Point Theory and Applications*, **2015**, Article No. 164. https://doi.org/10.1186/s13663-015-0409-z

[49] Abdeljawad, T., Karapınar, E. and Taş, K. (2011) Existence and Uniqueness of a Common Fixed Point on Partial Metric Spaces. *Applied Mathematics Letters*, **24**, 1900-1904. https://doi.org/10.1016/j.aml.2011.05.014

[50] Abbas, M. and Jungck, G. (2008) Common Fixed Point Results for Noncommuting Mappings without Continuity in Cone Metric Spaces. *Journal of Mathematical Analysis and Applications*, **341**, 416-420. https://doi.org/10.1016/j.jmaa.2007.09.070

[51] Babu, G.V.R. and Alemayehu, G.N. (2010) Point of Coincidence and Common Fixed Points of a Pair of Generalized Weakly Contractive Maps. *Journal of Advanced Research in Pure Mathematics*, **2**, 89-106. https://doi.org/10.5373/jarpm.338.010810