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The Morita Equivalence for the A_∞-Algebras

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Abstract

This paper explores the foundational and advanced aspects of A_∞-algebras. We present a comprehensive study of A_∞-algebras and their homology, emphasizing the construction of long exact sequences in simplicial homology and their implications for short exact sequences of A_∞-algebras. Furthermore, we investigate the trace and inclusion maps in matrix A_∞-algebras, proving their mutual invertibility and highlighting their role in preserving homological properties. These results underscore the utility of A_∞-algebras in simplifying complex algebraic systems while maintaining essential structural invariants.

Keywords

A-Infinity, Homology, Inclusion Map, Trace Map

1. Introduction

The concept of A_∞-algebras (or strongly homotopy associative algebras) and the associated structures such as A_∞-modules have evolved significantly since their inception. The journey began with Jim Stasheff, who introduced the idea of A_∞-spaces within the context of homotopy theory. Stasheff created these structures to describe spaces equipped with homotopy associative multiplications, thereby laying the groundwork for A_∞-algebras [1]. This breakthrough not only expanded our understanding of spaces with relaxed associativity conditions but also opened doors to developing algebraic structures that capture higher homotopy invariants.

In the 1990s, the study of A_∞-algebras gained renewed attention, especially due to their applications in theoretical physics and geometry. At the 1994 International Congress of Mathematicians, Maxim Kontsevich introduced the concept of categorical mirror symmetry—a revolutionary idea that highlighted the importance of A_∞-structures in understanding dualities between geometric and algebraic objects. Around the same time, Bernhard Keller extended the use of A_∞-

algebras to noncommutative algebra and representation theory, showing their utility in derived categories and homological properties of algebras. Keller's work demonstrated that A_∞-structures were not just theoretical constructs but practical tools for analyzing complex algebraic systems.

The development of these ideas continued into the early 2000s, with contributions from S. V. Lapin. Lapin explored multiplicative A_∞-structures within the framework of spectral sequences, connecting fibrations and their homotopy properties to differential algebraic structures. Simultaneously, the Hochschild complex became an essential tool for studying the algebraic aspects of A_∞-algebras. J. P. Serre's foundational work explored the role of Hochschild homology and cohomology in understanding both algebraic and quantum structures, while Lapin's subsequent research offered insights into cyclic homology and simplicial realizations of spaces. These efforts revealed the deep connections between homological invariants, algebraic operations, and topological structures.

In the following decade, significant advancements were made in the study of A_∞-modules over A_∞-algebras. Alaa Hassan Noreldeen Mohamed provided a rigorous treatment of the homology of chain complexes endowed with A_∞-structures. Mohamed demonstrated that the homology $H^*(A)$ of a differential algebra naturally admits a graded D_∞-algebra structure, while graded A_∞-module structures and morphisms emerge in $H^*(M)$, the homology of associated modules. These results showed that the framework of A_∞-algebras offers a natural generalization of classical differential algebraic systems, allowing for a systematic study of higher homotopy invariants [2] and [3].

The study of A_∞-algebras has proven to be an indispensable tool for understanding homotopy properties, spectral sequences, and algebraic operations. Concepts such as Massey products, Hochschild homology, and graded modules have emerged as powerful techniques for analyzing these structures. In this paper, we delve deeper into the study of A_∞-algebras and their role in homology algebra.

2. Homology Theory of A_∞-Algebras

In this section, we delve into the foundational concepts and definitions pertinent to the homology theory of A_∞-algebras. We start by defining the basic structures of A_∞-algebras, followed by an exploration of their simple homology ([4] [5] and [6]).

2.1. Definitions of Algebra and Graded Spaces

An algebra over a field R is a linear vector space X equipped with a multiplication function $T: X \times X \to X$, denoted by $(v,u) \mapsto vu$. The operation T is distributive and linear in both variables, satisfying the following for all $v,u,w \in X$ and $\alpha \in R$:

$$w(v+u) = wv + wu, (v+u)w = vw + uw \text{, and } \alpha(vu) = (\alpha v)u = v(\alpha u).$$

To extend this concept to graded vector spaces, consider a vector space X indexed by a set I. The I-graded vector space is defined as $X=\oplus_{i\in I}X_i$, where each X_i is a vector space. Elements $x\in X_i$ are called homogeneous elements of degree i, denoted by $\deg x=i$ or $|x|=i$ [7].

The tensor product of vector spaces X and Y over a field F is the vector space $X\otimes Y$, with a basis consisting of symbols $\{x_i\otimes y_l \mid i\in I, l\in L\}$. A bilinear map $R:X\times Y\to M$ induces a unique linear map $R':X\otimes Y\to M$, satisfying $R=R'\circ\phi$, where ϕ is the canonical inclusion map [8].

2.2. Graded Algebras and Tensor Algebras

A graded algebra M over a field R is defined as $M=\oplus_{i\in I}M_i$, with a multiplication map satisfying $\deg(mn)=\deg m+\deg n$. From a vector space M, the tensor algebra $T(M)$ is constructed as:

$$T(M)=R\oplus M\oplus M^{\otimes 2}\oplus M^{\otimes 3}\oplus\cdots$$

This algebra is naturally graded, with multiplication defined via the tensor product [9].

2.3. Differential Graded Algebras (DGAs)

A differential graded algebra (DGA) is a graded algebra M equipped with a degree +1 chain map $d:M\to M$, satisfying $d^2=0$ and the Leibniz rule:

$d(mn)=d(m)n+(-1)^{|m|}md(n)$, for all $m,n\in M$.

If $mn=(-1)^{|m||n|}nm$, the DGA is commutative [10].

2.4. A_∞-Algebras

An A_∞-algebra over a field RR is a graded vector space $A=\oplus_{p\in Z}A^p$ equipped with homogeneous maps $r_n:A^{\otimes n}\to A$ of degree $2-n$, satisfying the Stasheff identities $SL(n)$:

$$\sum(-1)^{m+st}r_q\left(id^{\otimes m}\otimes r_s\otimes id^{\otimes t}\right)=0$$

where the sum is taken over all decompositions $n=m+s+t$, with $m,t\geq 0$ and $s\geq 1$. The first few identities ensure differential properties, derivations, and associativity up to homotopy [11] and [12].

2.5. Morphisms and Quasi-Isomorphisms

A morphism $h:A\to B$ between A_∞-algebras is a family of graded maps $h_n:A^{\otimes n}\to B$ of degree $1-n$, satisfying specific compatibility conditions. If h_1 induces an isomorphism at the homology level, h is called a quasi-isomorphism. Two A_∞-algebras are quasi-isomorphic if there exists a quasi-isomorphism between them [13].

A minimal model of an A_∞-algebra is a representative with $r_1=0$, facilitating the study of its homotopy properties and equivalence classes.

3. Main Result

The next theorem establishes the existence of a long exact sequence in simplicial homology for a short exact sequence of A_∞-algebras. This result is crucial because it allows us to analyze how homological properties are preserved and transferred across components in exact sequences of A_∞-algebras. By connecting the homology of the algebra A, its extension B, and the quotient C, this sequence becomes a powerful tool for decomposing complex homological structures. In the following, we will denote the simplicial homology of any algebra X by $HH_n(X)$.

Theorem 2.1

If there is a short exact sequence $0 \to A \to B \to C \to 0$ for DGAs of A_∞-algebras over a field such that $B = A \oplus C$, then the following long exact sequence of simplicial homology holds:

$$\begin{aligned}&\cdots \to HH_n(A) \to HH_n(B) \to HH_n(C) \to HH_{n-1}(A)\\&\to HH_{n-1}(B) \to HH_{n-1}(C) \to \cdots.\end{aligned}$$

Proof:

Short Exact Sequence Setup:

Consider the exact sequence $0 \to A \to B \to C \to 0$, where A and C are A_∞-algebras, and B is their direct sum $A \oplus C$. This ensures that any element in B can be uniquely written as a pair (a, c) with $a \in A$ and $c \in C$.

Construction of Associated Tensor Algebra:

Let TB denote the tensor algebra associated with B, and let AB be the ideal in TB. The short exact sequence of DGAs $0 \to AB \to TB \to B \to 0$ provides a framework to analyze homology.

Deriving Long Exact Sequences:

Using the homological properties of DGAs, we derive the first long exact sequence (Cartan & Eilenberg 1956):

$$\begin{aligned}&\cdots \to HH_n(AB) \to HH_n(TB) \to HH_n(B) \to HH_{n-1}(AB)\\&\to HH_{n-1}(TB) \to HH_{n-1}(B) \to \cdots.\end{aligned} \tag{1}$$

Similarly, the short exact sequence $0 \to K \to TB \to C \to 0$ yields:

$$\begin{aligned}&\cdots \to HH_n(K) \to HH_n(TB) \to HH_n(C) \to HH_{n-1}(K)\\&\to HH_{n-1}(TB) \to HH_{n-1}(C) \to \cdots.\end{aligned} \tag{2}$$

Identifying Relationships between Components:

From (1) and (2), we observe that:

In (1) we have $HH_n(TB) \to HH_n(B) \to HH_{n-1}(AB)$, then we get $HH_n(B) \cong HH_{n-1}(AB)$. Similarly, in (2) we have $HH_n(TB) \to HH_n(C) \to HH_{n-1}(K)$. So $HH_n(C)$ is equivalence for $HH_n(B)$, then $HH_n(B) \cong HH_{n-1}(K)$:

$$HH_n(B) \cong HH_{n-1}(AB), \quad HH_n(B) \cong HH_{n-1}(K). \tag{3}$$

Substituting into the short exact sequence $0 \to AB \to K \to A \to 0$, we derive:

$$\begin{aligned}&\cdots \to HH_n(AB) \to HH_n(K) \to HH_n(A) \to HH_{n-1}(AB)\\&\to HH_{n-1}(K) \to HH_{n-1}(A) \to \cdots.\end{aligned} \tag{4}$$

Combining Results:

Using (3) and (4), we link $HH_n(A), HH_n(B)$, and $HH_n(C)$ to construct the desired long exact sequence:

$$\cdots \to HH_n(A) \to HH_n(B) \to HH_n(C) \to HH_{n-1}(A)$$
$$\to HH_{n-1}(B) \to HH_{n-1}(C) \to \cdots.$$

This completes the proof.

The following theorem introduces the trace map, a homomorphism that simplifies the study of homology in matrix DGAs. The trace map relates the Hochschild homology of a matrix algebra $M_m(L)$ to that of the underlying algebra L, showing they are isomorphic. This result is foundational in reducing the complexity of calculations, as it implies that the homology of a matrix algebra retains the same structure as the base algebra.

Theorem 2.2

If L is an A_∞-DGA over a module k, and $M_m(L)$ is the DGA of matrices over L, then for all $n \geq 0$ and $m \geq 1$, the trace map:

$$Tr_* : HH_n(M_m(L)) \to HH_n(L)$$

is an isomorphism.

Proof:

Matrix Representation in A_∞-Algebras:

Let $M_m(L)$ denote the algebra of $m \times m$ matrices over L. The trace map Tr_* acts as a homomorphism that collapses matrix structures into their diagonal elements, preserving the homological relationships.

Short Exact Sequence in Simplicial Context:

Define a short exact sequence of $Z/2$-complexes:

$$0 \to C_*(M_m(L)) \to \Lambda(M_m(L)) \to \Lambda(M_m(L))[-2] \to 0,$$

where $\Lambda(M_m(L))[-2]$ shifts the degree by -2, *i.e.* $\Lambda(M_m(L))_n \to \Lambda(M_m(L))_{n-2}$. The associated long exact sequence in simplicial homology is:

$$\cdots \to H_n(Z/2, C_*(M_m(L))) \to H_n(Z/2, \Lambda(M_m(L)))$$
$$\to H_n(Z/2, \Lambda(M_m(L))[-2]) \to \cdots.$$

Connecting Homology Groups:

Let Tr denote the homomorphism between $H_n(\Lambda(M_m(L)))$ and $H_n(\Lambda(L))$. By definition:

$$H_n(\Lambda(M_m(L))) = HH_n(Mm(L)),\ H_n(\Lambda(L)) = HH_n(L).$$

The trace map $Tr_* : HH_n(M_m(L)) \to HH_n(L)$ ensures a natural isomorphism between these groups.

Conclusion:

The trace map acts as a projection of matrix components onto their corresponding elements in L, making Tr_* bijective and hence an isomorphism.

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In the following, the inclusion map plays a complementary role to the trace map. This theorem demonstrates that embedding an A_∞-algebra L into a matrix algebra $M_m(L)$ preserves homological properties. The isomorphism established by the inclusion map confirms that the algebraic structure of L is fully retained within the matrix algebra, ensuring consistency between the two representations.

Theorem 2.3: Inclusion Map for A_∞-Algebras

If L is a DGA over K, and $M_m(L)$ is the DGA of matrices over L, then for all $n \geq 0$ and $m \geq 1$, the inclusion map:

$$inc : HH_n(L) \to HH_n(M_m(L))$$

is an isomorphism.

Proof:

Embedding of Elements:

The inclusion map inc embeds elements of L into $M_m(L)$ by representing them as diagonal matrices. This process preserves the additive and multiplicative structure of the algebra.

Exact Sequence Setup:

Using the same short exact sequence as in Theorem 2.2, we derive the associated long exact sequence. By naturality, the inclusion map inc respects the homological structure.

Conclusion:

The map inc is the inverse of the trace map Tr, completing the isomorphism between $HH_n(L)$ and $HH_n(M_m(L))$.

In following theorem, we delve deeper into the relationship between trace and inclusion maps by proving that they are inverses of each other. This result highlights the duality between these two operations, ensuring that one can seamlessly move between the homology of $L \otimes I$ and $M_m(L) \otimes M_m(I)$. This is a key result for applications where matrix algebras are used to model or simplify complex algebraic systems.

Theorem 2.4: Trace and Inclusion Maps as Inverses

Statement:

If L is an A_∞-algebra, and I is H-unital over K, then the trace and inclusion maps:

$$tr : HH_n(L \otimes I) \to HH_n(Mm(L) \otimes Mm(I)),$$

$$inc : HH_n(Mm(L) \otimes Mm(I)) \to HH_n(L \otimes I),$$

are inverses of each other.

Proof:

Commutative Diagram Construction:

Using the following commutative diagram:

$$0 \to I \to M_m(I) \to \cdots,$$

$$0 \to I_+ \to M_m(I_+) \to \cdots,$$

where I_+ is the unital extension of I, we observe that the rows are exact.

Morita Invariance:

Morita invariance ensures that the trace and inclusion maps are isomorphisms for $M_m(L)$ and $M_m(I)$. Consequently, $tr \circ inc$ and $inc \circ tr$ are identity maps.

Conclusion:

The trace map tr and inclusion map inc are mutually inverse, establishing a one-to-one correspondence.

4. Conclusion

The study of A_∞-algebras continues to provide profound insights into homological and algebraic structures. Our exploration establishes key results, including the construction of long exact sequences for simplicial homology and the isomorphism between the homology of matrix A_∞-algebras and their underlying algebras via trace and inclusion maps. These findings affirm the robustness of A_∞-algebra frameworks in decomposing and analyzing complex algebraic and homological systems. By demonstrating the interplay between trace and inclusion maps as inverses, this research bridges the gap between theoretical constructs and practical algebraic applications. The results offer a versatile toolkit for advancing the study of homotopy invariants, spectral sequences, and higher algebraic structures, paving the way for future developments in both mathematics and theoretical physics.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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