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New Approach to Pion Distribution Amplitude

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Abstract

The usual pion distribution amplitude is typically constructed using functions based on one of the trial functions from the 't Hooft singular integral equation, specially of the form $x(1-x)$ type. However, these of function are not exact solutions of the 't Hooft equation, which means a pion distribution amplitude constructed in this manner may not accurately represent a real pion as described by Light Front QCD. In contrast, we have developed a pion distribution amplitude using solutions that correspond to zero-mass state wave functions of a bound system of 1 + 1 dimensional QCD derived from the 't Hooft model. Our distribution amplitude shows $\frac{1-x}{x}$ behavior as x approaches 1. Notably, asymptotic form of our distribution amplitude reveals an intriguing property: peak of our distribution amplitude shifts towards $x=1$ as the coupling constant g^2 increases.

Keywords

Distribution Amplitude, 't Hooft Model

1. Introduction

Recently, several surprising experiment results have emerged in hadron physics.

Among these, Dlamini *et al.* found that the neutral pion's electromagnetic transition form factor is Gaussian-like rather than Regge-like [1], whereas Huber and Horn reported that the charged pion's electromagnetic form factor remains Regge-like, despite employing the same error analysis [2] [3]. In the case of proton, Experiments by H1 and Zeus [4] and Atlas [5] have shown that the fundamental proton structure resembles see-quarks in both electron-proton collision and in proton-proton collision. Given these findings, Arrington *et al.* highlighted that the lightest pseudo-scalar mesons, particularly the pion and kaon, are crucial for deepening our understanding of mass emergence and structural mechanisms in strongly interacting matter. Consequently, unraveling the structure of pions and kaons has become increasingly important. Proper structure functions are essential for interpreting comprehend experimental findings. For instance, Raya *et al.* demonstrated the valence u-quark of the pion distribution function, which directly related to pion distribution amplitude [6] [7]. This underscores the significance of obtaining an accurate pion distribution amplitude. Several notable efforts to determine the pion distribution amplitude include work by Zhang *et al.*, who proposed a function of the form $x^{\alpha_1}(1-x)^{\alpha_2}$ [8], Chang *et al.*, who predicted it using by Dyson-Schwinger Equation [9] and Raya et al., who accounted for pion radius effects [7]. The latter two employed the functions of the form $\left[x(1-x)\right]^{\alpha}P(x)$ type functions. These efforts utilize characteristic functions akin to trial functions used in solutions of ‘t Hooft singular integral equation [10]. To construct distribution amplitude, Light Front QCD (4 dimension) is commonly applied, which makes it, at least analogically, reasonable to use a type of solutions of ‘t Hooft singular integral equation that derived in 2 or 1 + 1 dimensions, given that his equation uses light-corn gauge. However, if we extend this analogy further, aforementioned function types may not suffice. Litvinov *et al.* [11] presented a method to construct exact analytical solutions to the ‘t Hooft equation, which are not of the form $x^{\beta_1}(1-x)^{\beta_2}$ type but rather are sum of functions of the form $\left(\frac{x}{1-x}\right)^{\gamma_i}$ and combinations of sine and cosine functions of the form $\left(\left(\lambda_i+\frac{N}{2}\right)\ln\left(\frac{x}{1-x}\right)\right)$. These wave functions, corresponding to eigenvalues, are identified in the space $\nu\in\left[-\infty,\infty\right]$ and fundamentally represented by the residue of simple poles.

In this paper, we assume that the aforementioned analogy applied to the pion distribution amplitude and explore it within the framework of hadronic operator formalism proposed by Suura [12], rather than using solutions from the ‘t Hooft singular integral equation. There are two main reasons for choosing this framework. First, we successfully derived both neutral and charged pion wave functions and their electromagnetic (transition) form factors within Suura’s hadronic operator framework in 3 + 1 dimensions [13]-[15]. Second, we applied Suura’s hadronic operator to the ‘t Hooft model in 1 + 1 dimensions, obtaining mass spectra similar to those derived from ‘t Hooft singular integral equation [16]. As demonstrated in ref. [16], our set of equations is not exactly equivalent to the ‘t Hooft singular integral equation. Nevertheless, we managed to obtain a zero-mass solution, while the ‘t Hooft singular integral equation does not yield zero-mass solution in massless quark case (chiral limit) as shown in ref. [11]. Since Peccei *et al.* argued that zero-mass solution exists in the large N limit of two dimensional QCD massless quark case [17], we believe that our approach is significant and meaningful.

2. Formulation

As mentioned in Section 1, to construct a pion distribution amplitude, Light Front

QCD (LFQCD) is commonly applied. Therefore, we may consider momentum space wave functions of the large N limit two dimensional QCD massless quarks case as a base argument to construct a pion distribution amplitude. We derived configuration space wave functions of this case in the framework of hadronic operator proposed by Suura as shown in ref. [16]. Here we briefly show the derivation of configuration space wave functions in ref. [16] with small revision.

For 1 + 1 dimensional bound system, the Bethe-Salpeter-like amplitude defined by Suura becomes as

$$X(1,2)=\langle 0|q(1,2)|P\rangle \tag{1}$$

($|0\rangle$ and $|phy\rangle$ are vacuum and physical states, respectively)

where the gauge-invariant operator $q(1,2)$ is defined as

$$q_{\zeta\eta}(1,2)=T_r^c q_\eta^\dagger(x)P\exp\left(ig\int_1^2 \mathrm{d}xA_1^a(x)\frac{\lambda_a}{2}\right)q_\zeta(1) \tag{2}$$

Here ζ and η denote Dirac indices. P denotes the path ordering and $\frac{\lambda_a}{2}$ components are adjoint representation of the SU(N) color gauge group. T_r^c indicates that the trace is taken for color spin a. Above definition is formulated in the time axial gauge ($A_0=0$ gauge). Path is an exactly straight line because we are interested in 1 + 1 dimension case. The equation of motion for $q(1,2)$ becomes

$$i\partial_t q(1,2)=-i\alpha\partial(2)q(1,2)-q(1,2)i\alpha\partial(1)+g\int_1^2 dxq_E(1,2:x) \tag{3}$$

where

$$q_E(1,2:x)=T_r^c q^\dagger(2)U(2,x)E^a(x)U(x,1)q(1)$$

$$U(2,1)=P\exp\left(ig\int_1^2 \mathrm{d}xA_1^a(x)\frac{\lambda_a}{2}\right)$$

Note that $E^a(x)$ and $A^a(x)$ indicate $E_1^a(x)$ and A_1^a because we are working in 1 + 1 dimensions.

We employ the metric system and γ-matrices in 2 dimensions as follows, which are also employed by Casher *et al.* [18]

$$g^{00}=1,\ \ g^{11}=-1$$

$$\gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \gamma^1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ \alpha=\gamma^0\gamma^1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\sigma_3$$

After evaluating $q_E(1,2:x)$ and regrouping the charge vertex term, we obtained the final form of the equation of motion for $q(1,2:x)$ as

$$\begin{aligned}i\partial_t q(1,2)=&-i\partial(2)q(1,2)-q(1,2)i\partial(1)\\&-\frac{g^2}{4}\int_1^2 \mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}x\varepsilon(x-z)q(1,x)q(x,2)+O\left(\frac{1}{N}\right)\end{aligned} \tag{4}$$

where $\varepsilon(x)$ is step function defined as $\varepsilon(x)=+1(x\geq 0)$ and $-1(x<0)$.

Recalling $X(1,2)$ as $X(1,2)=\langle 0|q(1,2)|phy\rangle$ ($|0\rangle$ and $|phy\rangle$ are vacuum and physical states, respectively) and taking the new variables as

$$G=\frac{1}{2}\left(x(1)+x(2)\right) \text{ and } r=x(2)-x(1),$$

we consider the following form of $X(1,2)$ as

$$X(1,2)=\mathrm{e}^{-iP_0t}\mathrm{e}^{iP_1G}X(r)$$

After factoring out the phase part, Equation (4) becomes

$$\begin{aligned}P_0X(r)=&\left\{\frac{1}{2}\alpha P_1,X(r)\right\}-\left[i\alpha\frac{\partial}{\partial r},X(r)\right]\\&+\frac{g^2}{8}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{iP_1\frac{x-x(1)}{2}}\left(\left|x(2)-x\right|-\left|x-x(1)\right|\right)\left[S(1,x),X\left(x(2)-x\right)\right]\\&+O\left(\frac{1}{N}\right)\end{aligned} \quad (5)$$

where $S(1,x)=\langle 0|q(1,x)|0\rangle$.

Recalling the fact that $\alpha=i\sigma_3$ and then taking the decomposition

$$X(1,2)=1X_0(1,2)+(i\sigma_3)X_1(1,2)+(\sigma_2)X_2(1,2)+(\sigma_1)X_3(1,2) \quad (6)$$

where 1 denotes unit matrix, we obtain the set of equations:

$$P_0X_0(r)=iP_1X_1(r) \quad (7)$$

$$P_0X_1(r)=-iP_1X_0(r)+O\left(\frac{1}{N}\right) \quad (8)$$

$$W_0X_2(r)=2\frac{\partial X_3}{\partial r}-\frac{g^2}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}x'\frac{|r-x'|-|x'|}{x'}X_3(r-x') \quad (9)$$

$$W_0X_3(r)=-2\frac{\partial X_2}{\partial r}+\frac{g^2}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}x'\frac{|r-x'|-|x'|}{x'}X_2(r-x') \quad (10)$$

Here we employ the form $S(r)=(i\sigma_3)P_r\frac{1}{2\pi}\frac{1}{r}$ (P_r denotes Principal Value) because Schwinger suggested that the propagator function behaves like a free propagator with small values of r (Schwinger parametrization) [19]. We justify this form on **Appendix A**. Important point is that Equation (9) and Equation (10) are only necessary to consider the mass spectra of this system under the condition of $P_1=0$ (in rest frame) because equations are closed for X_2 and X_3 under this condition.

Because we are interested in the ‘t Hooft model, we investigate solutions of Equation (9) and Equation (10).

Changing the variables in the integral as $r-x'=s$ and using the notation $r=x$, and taking $X_\pm=X_3\pm iX_2$, Equation (9) and Equation (10) change to following two closed forms.

$$W_0X_+(x)=2i\partial_xX_+(x)-i\frac{g^2}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}s\frac{|s|-|x-s|}{x-s}X_+(s) \quad (11)$$

$$W_0X_-(x)=-2i\partial_xX_-(x)+i\frac{g^2}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}s\frac{|s|-|x-s|}{x-s}X_-(s) \quad (12)$$

First, we seek a solution of $X_-(x)$ (this corresponds to $X_+(x)$ in ref. [16]).

Note that we could obtain the same form as Equation (11) if we take $-W_0$ for W_0 of Equation (12).

Thus, once we found a solution of $X_-(x)$, we could obtain a solution of $X_+(x)$ by replacing W_0 to $-W_0$ in a solution of $X_-(x)$.

Considering absolute value and using step function $\varepsilon(x)$ defined above, integral term becomes as

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}s\frac{|s|-|x-s|}{x-s}X_-(s)\\&=-\int_{-\infty}^{\infty}\mathrm{d}s\varepsilon(s)X_-(s)-x\int_{-\infty}^{\infty}\mathrm{d}s\frac{\varepsilon(s)X_-(s)}{s-x}+\int_{-\infty}^{\infty}\mathrm{d}s\varepsilon(x-s)X_-(s)\end{aligned}$$

Multiplying $\varepsilon(x)$ on both sides of Equation (12) and considering the new function $F_-(x)$ defined as $F_-(x)=\varepsilon(x)X_-(x)$ and notifying the fact that $X_-(x)=\varepsilon(x)F_-(x)$ ($(\varepsilon(x))^2=1$), Equation (12) becomes

$$\begin{aligned}W_0F_-(x)&=-2i\partial_xF_-(x)+4i\delta(x)F_-(x)\varepsilon(x)-i\frac{g^2}{8\pi}\varepsilon(x)x\int_{-\infty}^{\infty}\mathrm{d}s\frac{F_-(s)}{s-x}\\&\quad+i\frac{g^2}{8\pi}\varepsilon(x)\int_{-\infty}^{\infty}\mathrm{d}sF_-(s)\varepsilon(s)\varepsilon(x-s)-i\frac{g^2}{8\pi}\varepsilon(x)\int_{-\infty}^{\infty}\mathrm{d}sF_-(s)\end{aligned}\tag{13}$$

The δ-function appears because of the fact $\partial_x\varepsilon(x)=2\delta(x)$.

Here we consider the positive region ($x>0$ region for which $\varepsilon(x)=+1$). Taking the derivative with respect to x on both side of Equation (13), Equation (13) becomes

$$\begin{aligned}W_0\partial_xF_-(x)&=-2i\partial_x^2F_-(x)+4i\partial_x\left(\delta(x)F_-(x)\right)-i\frac{g^2}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}s\frac{F_-(s)}{s-x}\\&\quad-i\frac{g^2}{8\pi}x\int_{-\infty}^{\infty}\mathrm{d}s\frac{\frac{\mathrm{d}F_+}{\mathrm{d}s}}{s-x}+i\frac{g^2}{4\pi}F_-(x)\end{aligned}\tag{14}$$

To obtain the last term, we use the fact that $\partial_x\varepsilon(x-s)=2\delta(x-s)$ and that $X_-(x)=\varepsilon(x)F_-(x)=F_-(x)$ (for $x>0$).

For the singular integral term, we apply the slightly modified Sokhotsky formula [20]:

$$\frac{1}{i\pi}\int_{-\infty}^{\infty}\mathrm{d}s\frac{F_-(s)}{s-x}=a_1\phi_-^{(+)}(x)+a_2\phi_-^{(-)}(x)\tag{15}$$

$$F_-(x)=a_1\phi_-^{(+)}(x)-a_2\phi_-^{(-)}(x)\tag{16}$$

where $\phi_-^{(+)}(x)$ is the value result when x asymptotically approaches real axis in the upper-half hemisphere, while $\phi_-^{(-)}(x)$ is the value result when x asymptotically approaches real axis in the lower-half hemisphere, and a_1 and a_2 are non-zero real constants.

Here, we have modified the original form of the Sokhotsky formula by changing $\phi_-^{(+)}$ to $a_1\phi_-^{(+)}$ and $\phi_-^{(-)}$ to $a_2\phi_-^{(-)}$, which will be used in a later argument.

Note that s and x are real values.

Subsequently, Equation (14) is written as

$$a_1\left[\frac{\partial^2\phi_-^{(+)}}{\partial x^2}+\left[\left(\frac{W_0}{2i}\right)-\frac{g^2}{8}\frac{1}{2i}x\right]\frac{\partial\phi_-^{(+)}}{\partial x}+\left[-\left(\frac{g^2}{8\pi}\right)-\left(\frac{g^2}{8}\right)\frac{1}{2i}\right]\phi_-^{(+)}-2\partial_x\left(\delta(x)\phi_-^{(+)}\right)\right]$$
$$=a_2\left[\frac{\partial^2\phi_-^{(-)}}{\partial x^2}+\left[\left(\frac{W_0}{2i}\right)+\frac{g^2}{8}\frac{1}{2i}x\right]\frac{\partial\phi_-^{(-)}}{\partial x}+\left[-\left(\frac{g^2}{8\pi}\right)+\left(\frac{g^2}{8}\right)\frac{1}{2i}\right]\phi_-^{(-)}-2\partial_x\left(\delta(x)\phi_-^{(-)}\right)\right] \quad (17)$$

In order to find the solutions for both $\phi_-^{(+)}$ and $\phi_-^{(-)}$, we set both sides of Equation (17) to be equal to zero. Because a_1 and a_2 are non-zero constants, we can independently obtain the differential equation for $\phi_-^{(+)}$ and $\phi_-^{(-)}$.

To address the derivative of the δ-function term, we integrate successively twice from ∞ to *x*. Note that here we consider the range $x>0$ while $\phi_-^{(\pm)}(\infty)=0$ and $\partial\phi_-^{(\pm)}/\partial x\big|_{x=\infty}=0$; specifically, we are seeking solutions that satisfy the previous conditions. Subsequently, we can see that

$$\int_\infty^x \mathrm{d}x'\delta(x')\phi_-^{(\pm)}(x')=0.$$

(see **Appendix B**)

Taking the derivative successively twice, we obtain the following differential equations.

$$\frac{\partial^2\phi_-^{(+)}}{\partial x^2}+\left[\left(\frac{W_0}{2i}\right)-\frac{g^2}{8}\frac{1}{2i}x\right]\frac{\partial\phi_-^{(+)}}{\partial x}+\left[-\left(\frac{g^2}{8\pi}\right)-\left(\frac{g^2}{8}\right)\frac{1}{2i}\right]\phi_-^{(+)}=0 \quad (18)$$

$$\frac{\partial^2\phi_-^{(-)}}{\partial x^2}+\left[\left(\frac{W_0}{2i}\right)+\frac{g^2}{8}\frac{1}{2i}x\right]\frac{\partial\phi_-^{(-)}}{\partial x}+\left[-\left(\frac{g^2}{8\pi}\right)+\left(\frac{g^2}{8}\right)\frac{1}{2i}\right]\phi_-^{(-)}=0 \quad (19)$$

The solutions satisfying the condition that both function and its first derivative at infinity are zero are as follows (the derivation is given in **Appendix C**).

$$\phi_-^{(+)}(x)=\mathrm{e}^{\frac{1}{4}\alpha x^2}\mathrm{e}^{-\frac{1}{2}\beta x}2^{-\frac{1}{4}-\frac{i}{\pi}}\left(\sqrt{\alpha}x-\frac{\beta}{\sqrt{\alpha}}\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\left(\sqrt{\alpha}x-\frac{\beta}{\sqrt{\alpha}}\right)^2\right) \quad (20)$$

$$\phi_-^{(-)}(x)=\mathrm{e}^{-\frac{1}{4}\alpha x^2}\mathrm{e}^{-\frac{1}{2}\beta x}2^{-\frac{1}{4}+\frac{i}{\pi}}\left(i\left(\sqrt{\alpha}x+\frac{\beta}{\sqrt{\alpha}}\right)\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{1}{2}\left(\sqrt{\alpha}x+\frac{\beta}{\sqrt{\alpha}}\right)^2\right) \quad (21)$$

where

$$\alpha=\frac{g^2}{8}\frac{1}{2i},\quad \beta=\frac{W_0}{2i}$$

$W_{\kappa,\mu}(x)$ is the Whittaker function.

As mentioned before, a solution of $X_+(x)$ can obtain by replacing W_0 to $-W_o$ in Equation (20) and Equation (21) as

$$X_+(x)=a_1\phi_+^{(+)}(x)-a_2\phi_+^{(-)}(x)$$

$$\phi_+^{(+)}(x)=\mathrm{e}^{\frac{1}{4}\alpha x^2}\mathrm{e}^{\frac{1}{2}\beta x}2^{-\frac{1}{4}-\frac{i}{\pi}}\left(\sqrt{\alpha}x+\frac{\beta}{\sqrt{\alpha}}\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\left(\sqrt{\alpha}x+\frac{\beta}{\sqrt{\alpha}}\right)^2\right) \quad (22)$$

$$\phi_{+}^{(-)}(x)=\mathrm{e}^{-\frac{1}{4}\alpha x^2}\mathrm{e}^{\frac{1}{2}\beta x}2^{-\frac{1}{4}+\frac{i}{\pi}}\left(i\left(\sqrt{\alpha}x-\frac{\beta}{\sqrt{\alpha}}\right)\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{1}{2}\left(\sqrt{\alpha}x-\frac{\beta}{\sqrt{\alpha}}\right)^2\right) \quad (23)$$

Boundary condition for our case is that $X_-=0$ at $x=0$ ($X_+=0$ at $x=0$) and $X_-=0$ at $x=\infty$ ($X_+=0$ at $x=\infty$). Because both $X_-(x)$ and $X_+(x)$ are represented by linear combination of Whittaker function $W_{\kappa,\mu}(x)$, the conditions at $x=\infty$ are automatically satisfied.

The condition at $x=0$ gives mass spectra other than zero-mass of our description of the ‘t Hooft model as shown in ref. [16]. The condition at $x=0$ gives the following equation.

$$\left(-i\left(\frac{2W_0}{g}\right)^2\right)^{-\frac{1}{4}}\left[i^{-\frac{1}{4}}a_1W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(-i\frac{1}{2}\left(\frac{2W_0}{g}\right)^2\right)-i^{\frac{1}{4}}a_2W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(i\frac{1}{2}\left(\frac{2W_0}{g}\right)^2\right)\right]=0 \quad (24)$$

Because Equation (18) and Equation (19) include zero-mass case, we can obtain mass spectra including zero-mass by Equation (24).

To determine coefficients a_1 and a_2, we only need to consider zero-mass case that is $W_0=0$ ($\beta=0$).

Setting a value of $W_0=0$ ($\beta=0$), Equation (20) and Equation (21) become as

$$\phi_{-}^{(+)}(x)=2^{-\frac{1}{4}-\frac{i}{\pi}}\left(\mathrm{e}^{-i\frac{\pi}{4}}\sqrt{|\alpha|}x\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\mathrm{e}^{-i\frac{\pi}{2}}|\alpha|x^2\right) \quad (25)$$

$$\phi_{-}^{(-)}(x)=2^{-\frac{1}{4}+\frac{i}{\pi}}\left(\mathrm{e}^{i\frac{\pi}{4}}\sqrt{|\alpha|}x\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\mathrm{e}^{i\frac{\pi}{2}}|\alpha|x^2\right) \quad (26)$$

Recalling that Whittaker functions $W_{\kappa,\mu}$ of both equations have μ value of $-\frac{1}{4}$, and that $W_{\kappa,\mu}$ is represented by linear combination of $M_{\kappa,\mu}$ and $M_{\kappa,-\mu}$ as [21]

$$W_{\kappa,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}M_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}M_{\kappa,-\mu}(z) \quad (27)$$

where $M_{\kappa,\mu}$ is defined as

$$M_{k,\mu}(z)=z^{\mu+\frac{1}{2}}\mathrm{e}^{-\frac{z}{2}}{}_1F_1(\mu-\kappa+1,2\mu+1;z) \quad (28)$$

Each κ and μ values of $\phi_{-}^{(+)}$ and $\phi_{-}^{(-)}$ are $\kappa=-\frac{1}{4}-\frac{i}{\pi}$, $\mu=-\frac{1}{4}$ and $\kappa=-\frac{1}{4}+\frac{i}{\pi}$, $\mu=-\frac{1}{4}$, respectively. Because only the first term of $M_{\kappa,\mu}$ dominates as x asymptotically approaches 0, behavior of $\phi_{-}^{(\pm)}(x)$ as $x\to 0$ are following.

$$\phi_{-}^{(+)}(x)\to 2^{-\frac{1}{4}-\frac{i}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{i}{\pi}\right)}\left(\frac{1}{2}\right)^{\frac{1}{4}}$$

$$\phi_{-}^{(-)}(x) \to 2^{-\frac{1}{4}+\frac{i}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-\frac{i}{\pi}\right)} \left(\frac{1}{2}\right)^{\frac{1}{4}}$$

Therefore, in order to satisfy the condition that $X_{-}=0$ at $x=0$, it is sufficient for coefficients to set that $a_1 = 2^{\frac{i}{\pi}}\Gamma\left(1+\frac{i}{\pi}\right)$ and $a_2 = 2^{-\frac{i}{\pi}}\Gamma\left(1-\frac{i}{\pi}\right)$.

These coefficient values are also true for X_{+} case because $\phi_{+}^{(+)}(x)=\phi_{-}^{(+)}(x)$ and $\phi_{+}^{(-)}(x)=\phi_{-}^{(-)}(x)$ in the case of zero-mass ($\beta=0$).

Therefore, the wave functions for zero-mass case ($\beta=0$) become as

$$\begin{aligned} X_{\mp}(x) = 2^{-\frac{1}{4}} \Bigg[& \Gamma\left(1+\frac{i}{\pi}\right) e^{\frac{1}{4}\alpha x^2} \left(\sqrt{\alpha} x\right)^{-\frac{1}{2}} W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\alpha x^2\right) \\ & -\Gamma\left(1-\frac{i}{\pi}\right) e^{-\frac{1}{4}\alpha x^2} \left(i\sqrt{\alpha} x\right)^{-\frac{1}{2}} W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{1}{2}\alpha x^2\right) \Bigg] \end{aligned} \tag{29}$$

Thus, we obtain the wave function of $X_3(x)$ as

$$\begin{aligned} X_3(x) &= \frac{X_{-}(x)+X_{+}(x)}{2} \\ &= 2^{-1} \Bigg[\Gamma\left(1+\frac{i}{\pi}\right) e^{\frac{1}{4}\alpha x^2} \left(\frac{1}{2}\sqrt{\alpha} x\right)^{-\frac{1}{2}} W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\alpha x^2\right) \\ &\quad -\Gamma\left(1-\frac{i}{\pi}\right) e^{-\frac{1}{4}\alpha x^2} \left(i\sqrt{\alpha} x\right)^{-\frac{1}{2}} W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{1}{2}\alpha x^2\right) \Bigg] \end{aligned} \tag{30}$$

$$X_2(x) = \frac{X_{+}(x)-X_{-}(x)}{2i} = 0 \tag{31}$$

where $\alpha = \frac{g^2}{8}\frac{1}{2i}$.

We derived a charged pion wave function and an electromagnetic form factor in 3 + 1 dimensions in ref. [13] and showed that a charged pion wave function is represented by χ_3 ($\beta\left(i\vec{\alpha}\cdot\hat{r}\right)$ component) only because χ_2 (β component) is zero. Equation (30) and Equation (31) correspond to these results.

3. Distribution Amplitude

In this section, we construct a charged pion distribution amplitude. To achieve this aim, we apply one dimensional Fourier transform to $X_3(x)$.

In the case of our κ and μ values, $\phi_{-}^{(+)}(x)$ and $\phi_{-}^{(-)}(x)$ can be described as

$$\phi_{-}^{(+)}(x) = c_1 \frac{\left(\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right)^{\frac{1}{4}}}{\left(e^{-i\frac{\pi}{4}}\sqrt{|\alpha|}x\right)^{\frac{1}{2}}} {}_1F_1\left(1+\frac{i}{\pi},\frac{1}{2};\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right)$$

$$+c_2\frac{\left(\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right)^{\frac{3}{4}}}{\left(e^{-i\frac{\pi}{4}}\sqrt{|\alpha|}x\right)^{\frac{1}{2}}}{}_1F_1\left(\frac{3}{2}+\frac{i}{\pi},\frac{3}{2};\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right)$$
$$=c_1'{}_1F_1\left(1+\frac{i}{\pi},\frac{1}{2};\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right)+c_2'\sqrt{|\alpha|}\sqrt{x^2}{}_1F_1\left(\frac{3}{2}+\frac{i}{\pi},\frac{3}{2};\frac{1}{2}e^{-i\frac{\pi}{2}}|\alpha|x^2\right) \tag{32}$$

$$\phi_-^{(-)}(x)=d_1'{}_1F_1\left(1-\frac{i}{\pi},\frac{1}{2};\frac{1}{2}e^{i\frac{\pi}{2}}|\alpha|x^2\right)+d_2'\sqrt{|\alpha|}\sqrt{x^2}{}_1F_1\left(\frac{3}{2}-\frac{i}{\pi},\frac{3}{2};\frac{1}{2}e^{i\frac{\pi}{2}}|\alpha|x^2\right) \tag{33}$$

Equation (32) and Equation (33) show $\phi_-^{(+)}(x)$ and $\phi_-^{(-)}(x)$ are clearly even functions. Thus, $X_3(x)$ is an even function.

$$F_3(|q|)=\int_{-\infty}^{\infty}dxe^{-iqx}X_3(x)=\int_{-\infty}^{0}dxe^{-iqx}X_3(x)+\int_0^{\infty}dxe^{-iqx}X_3(x) \tag{34}$$

For the first term of the second line of Equation (34), by changing the variable x to $-x'$, the first term becomes

$$\text{first term}=\int_{\infty}^{0}-dx'e^{-iqx'}X_3(-x')=\int_0^{\infty}dx'e^{iqx'}X_3(x')=\int_0^{\infty}dx'e^{-i|q|x'}X_3(x')$$

For the second line, we recall the fact that X_3 is an even function. Recalling the fact that the measured momentum is always Q^2 in experiment and $\sqrt{Q^2}=\pm|Q|$, we take $q=-|q|$ in the last line. By taking $q=+|q|$ in the second term of the second line of Equation (34), Equation (34) becomes

$$F_3(|q|)=2\int_0^{\infty}dxe^{-i|q|x}X_3(x) \tag{35}$$

Recalling that the distribution amplitude is considered in 4 (3 + 1) dimensions and that using solutions of the ‘t Hooft model obtained in 2 (1 + 1) dimensions is only analogy, we may consider that qx is actually $|q||x|$ in this case. Then we obtain the same result for $F_3(|q|)$ without using above argument.

To proceed further, we evaluate the integral as follows.

$$F_3^{(1)}(|q|)=\Gamma\left(1+\frac{i}{\pi}\right)\int_0^{\infty}dxe^{-i|q|x}e^{\frac{1}{4}\alpha x^2}\left(\frac{1}{2}\sqrt{\alpha}\sqrt{x^2}\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\alpha x^2\right) \tag{36}$$

$$F_3^{(2)}(|q|)=\Gamma\left(1-\frac{i}{\pi}\right)\int_0^{\infty}dxe^{-i|q|x}e^{-\frac{1}{4}\alpha x^2}\left(i\frac{1}{2}\sqrt{\alpha}\sqrt{x^2}\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{1}{2}\alpha x^2\right) \tag{37}$$

Recalling that $\alpha=\frac{g^2}{8}\frac{1}{2i}=|\alpha|e^{-i\frac{\pi}{2}}$, we change the variables as $x=e^{-i\frac{3\pi}{4}}\frac{1}{\sqrt{|\alpha|}}z$ for Equation (36) and $x=e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\overline{z}$ for Equation (37), respectively. Then Equation (36) and Equation (37) become

$$F_3^{(1)}(|q|)=\Gamma\left(1+\frac{i}{\pi}\right)e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^{e^{i\frac{3\pi}{4}}|\alpha|\infty}dz\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z(1-i)\right)e^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) \tag{38}$$

$$F_3^{(2)}\left(|q|\right)=\Gamma\left(1-\frac{i}{\pi}\right)\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^{\mathrm{e}^{i\frac{\pi}{4}}|\alpha|\infty}\mathrm{d}\overline{z}\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}\overline{z}\left(1+i\right)\right)\mathrm{e}^{\frac{1}{4}\overline{z}^2}\left(\frac{1}{2}\overline{z}\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}\overline{z}^2\right) \quad (39)$$

Note that, for Equation (36), $\mathrm{e}^{-i\frac{3\pi}{4}}\left(\frac{1}{2}\sqrt{|\alpha|}\mathrm{e}^{-i\frac{\pi}{4}}\mathrm{e}^{-i\frac{3\pi}{4}}\frac{1}{\sqrt{|\alpha|}}z\right)^{-\frac{1}{2}}=\mathrm{e}^{-i\frac{\pi}{4}}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}$.

We consider the integral contour of **Figure 1** for Equation (38) and that of **Figure 2** for Equation (39), respectively.

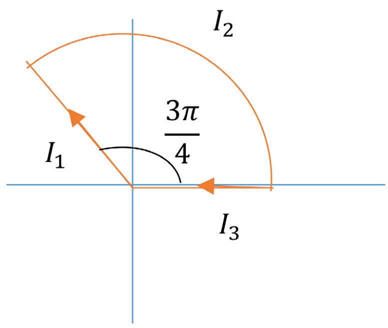

Figure 1. Integral contour for Eq. (38).

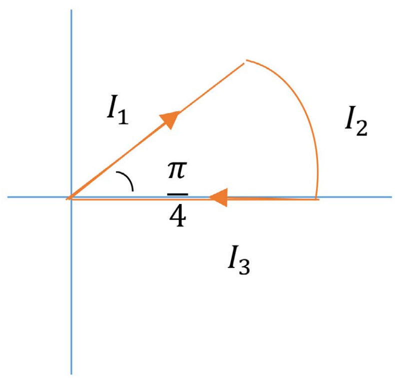

Figure 2. Integral contour for Eq. (39).

Because there are no poles inside of contour area for both Equation (38) and Equation (39) cases, we obtain $I_1+I_2+I_3=0$. Thus $I_1=-I_3$ because of $I_2=0$ (see **Appendix D**). This means that we can change the integral range of both cases from 0 to ∞.

Then, we obtain the Fourier transform of $X_3\left(x\right)$, namely $F_3\left(|q|\right)$, except constants as follows:

$$\begin{aligned}F_3\left(|q|\right)=&\Gamma\left(1+\frac{i}{\pi}\right)\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^{\infty}\mathrm{d}z\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z\left(1-i\right)\right)\mathrm{e}^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)\\&-\Gamma\left(1-\frac{i}{\pi}\right)\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^{\infty}\mathrm{d}z\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z\left(1+i\right)\right)\mathrm{e}^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)\end{aligned} \quad (40)$$

The absolute value of $F_3\left(|q|\right)$, namely $\left|F_3\left(|q|\right)\right|$, is our charged pion distribution amplitude in the range of $[0,\infty]$. Note that over all coefficient is unimportant because actual distribution amplitude should be normalized.

In order to construct a charged pion distribution amplitude in commonly used range, we have to replace $|q|$ to $\frac{q_1}{1-q_1}$. Recalling the fact that q_1 is

$\text{Momentum/Maximal Momentum}$ and that

$\text{Momentum} = \bar{x}\cdot\text{Maximal Momentum}$ $\left(\bar{x}\in[0,1]\right)$, we obtain our charged pion distribution amplitude as

$$F_3(\bar{x}) = \Gamma\left(1+\frac{i}{\pi}\right)e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^\infty \mathrm{d}z\exp\left(-\frac{1-i}{\sqrt{2|\alpha|}}\frac{\bar{x}}{1-\bar{x}}z\right)e^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)$$
$$-\Gamma\left(1-\frac{i}{\pi}\right)e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\int_0^\infty \mathrm{d}z\exp\left(-\frac{1+i}{\sqrt{2|\alpha|}}\frac{\bar{x}}{1-\bar{x}}z\right)e^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) \quad (41)$$

Again we have to take absolute value of $F_3(\bar{x})$ and normalizaon is necessary.

In order to observe the behavior of our distribution amplitude, we construct an approximate form of $F_3(|q|)$.

In order to evaluate integral, we divided integral region two parts, namely range 1 is $[0,u]$ and range 2 is $[u,\infty]$.

When $|q|$ is very large, only the first term of Whittaker function is sufficient to evaluate integral in range 1 and an asymptotic form of $W_{\kappa,\mu}$ is sufficient to range 2 integral. Because Whittaker function $W_{\kappa,\mu}$ is defined as Equation (27), integral becomes

$$I = e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\left(\int_0^u \mathrm{d}z\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z(1-i)\right)+\int_u^\infty \mathrm{d}z\frac{1}{z^{1+\frac{i}{\pi}}}\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z(1-i)\right)\right)$$
$$-e^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\left(\int_0^u \mathrm{d}z\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z(1+i)\right)+\int_u^\infty \mathrm{d}z\frac{1}{z^{1-\frac{i}{\pi}}}\exp\left(-\frac{|q|}{\sqrt{2|\alpha|}}z(1+i)\right)\right) \quad (42)$$

Here we use an asymptotic form of $W_{\kappa,\mu}(z)$ described in Equation (43) [21] for the second and the fourth terms.

$$W_{\kappa,\mu}(z)\sim e^{-\frac{z}{2}}z^{\kappa}\left[1+\sum_{n=1}^{\infty}\frac{\left[\mu^2-\left(\kappa-\frac{1}{2}\right)^2\right]\cdots\left[\mu^2-\left(\kappa-n+\frac{1}{2}\right)^2\right]}{n!z^n}\right] \quad (43)$$

Note that Γ-function are cancelled out by coefficient of $M_{\kappa,\mu}$ of Whittaker function $W_{\kappa,\mu}$ (refer Equation (27).

Changing variable $\frac{z}{\sqrt{2|\alpha|}}$ to z', integral becomes

$$I = e^{-i\frac{\pi}{4}}\left[\left(\sqrt{2}\int_0^{\frac{u}{\sqrt{2|\alpha|}}}\mathrm{d}z'\exp\left(-|q|z'(1-i)\right)+\left(\frac{1}{2|\alpha|}\right)^{\frac{i}{2\pi}}\int_{\frac{u}{\sqrt{2|\alpha|}}}^\infty \mathrm{d}z'\frac{1}{z'^{1+\frac{i}{\pi}}}\exp\left(-|q|z'(1-i)\right)\right)\right.$$

$$-\left(\sqrt{2}\int_0^{\frac{u}{\sqrt{2|\alpha|}}}\mathrm{d}z'\exp\left(-|q|z'(1+i)\right)+\left(\frac{1}{2|\alpha|}\right)^{-\frac{i}{2\pi}}\int_{\frac{u}{\sqrt{2|\alpha|}}}^{\infty}\mathrm{d}z'\frac{1}{z'^{1-\frac{i}{\pi}}}\exp\left(-|q|z'(1+i)\right)\right)\Bigg] \quad (44)$$

We can obtain the results of integral in Equation (44) by employing the integral formula [22]

$$\int_0^{\frac{u}{\sqrt{2|\alpha|}}}\mathrm{d}z\exp\left(-|q|z(1\mp i)\right)=\frac{1}{|q|(1\mp i)}\gamma\left(1,\frac{|q|(1\mp i)u}{\sqrt{2|\alpha|}}\right) \quad (45)$$

where $\gamma(\lambda',p)$ is incomplete gamma function defined as

$$\gamma(\lambda',p)=\int_0^p\mathrm{d}t\,\mathrm{e}^{-t}t^{\lambda'-1}$$

and its series expansion is shown as [21]

$$\gamma(\lambda',p)=\mathrm{e}^{-p}\sum_{n=0}^{\infty}\frac{p^{\lambda'+n}}{\lambda'(\lambda'+1)\cdots(\lambda'+n)}$$

For $\lambda'=1$ case, we can evaluate summation as follows:

$$\gamma(1,p)=\mathrm{e}^{-p}\sum_{n=0}^{\infty}\frac{p^{n+1}}{12\cdots n(n+1)}$$

Taking $n+1=\bar{n}$, summation begins $\bar{n}=1$, then summation becomes as

$$\text{sum}=\sum_{\bar{n}=1}^{\infty}\frac{p^{\bar{n}}}{\bar{n}!}=\sum_{\bar{n}=0}^{\infty}\frac{p^{\bar{n}}}{\bar{n}!}-1=\mathrm{e}^{p}-1$$

Then, $\gamma(1,p)=1-\mathrm{e}^{-p}$.

Thus, we obtain integral results as

$$I^{(1)}=\frac{1}{|q|(1\mp i)}\left(1-\exp\left(-\frac{|q|(1\mp i)u}{\sqrt{2|\alpha|}}\right)\right)$$

$$\begin{aligned}I^{\gamma}&=\frac{1}{|q|}\left[\left(\frac{1}{1-i}-\frac{1}{1+i}\right)-\exp\left(-\frac{|q|u}{\sqrt{2|\alpha|}}\right)\left(\frac{1}{1-i}\exp\left(\frac{iqu}{\sqrt{2|\alpha|}}\right)-\frac{1}{1+i}\exp\left(-\frac{i|q|u}{\sqrt{2|\alpha|}}\right)\right)\right]\\&=\frac{i}{|q|}\left[1-\exp\left(-\frac{|q|u}{\sqrt{2|\alpha|}}\right)\left(\sin\left(\frac{|q|u}{\sqrt{2|\alpha|}}\right)+\cos\left(\frac{|q|u}{\sqrt{2|\alpha|}}\right)\right)\right]\end{aligned} \quad (46)$$

$$\begin{aligned}I^{\Gamma}&=\int_{\frac{u}{\sqrt{2|\alpha|}}}^{\infty}\mathrm{d}z\frac{1}{z^{1\pm\frac{i}{\pi}}}\exp\left(-|q|z(1\mp i)\right)\\&=\int_{\frac{u}{\sqrt{2|\alpha|}}}^{\infty}\mathrm{d}z\frac{1}{z}\exp\left(-|q|z(1\mp i)\right)\left(\cos\left(\frac{1}{\pi}\ln(z)\right)\mp i\sin\left(\frac{1}{\pi}\ln(z)\right)\right)\end{aligned} \quad (47)$$

Because sin-function and cosine-function exist inside the integral and integration is taken up to infinity. We can conclude that I^{Γ} is much smaller than I^{γ}. Thus, we neglect contribution of I^{Γ} to pion distribution amplitude.

For small $|q|$ case, we consider Equation (40) that is exact form of $F_3(|q|)$.

Equation (40) is rewritten as

$$F_3(|q|)=\Gamma\left(1+\frac{i}{\pi}\right)\mathrm{e}^{-i\frac{\pi}{4}}\frac{1}{\sqrt{|\alpha|}}\left[\int_0^\infty \mathrm{d}z\exp\left(-\frac{|q|z}{\sqrt{2|\alpha|}}(1-i)\right)\mathrm{e}^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}-\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)\right.$$
$$\left.-\Gamma\left(1-\frac{i}{\pi}\right)\mathrm{e}^{-i\frac{\pi}{4}}\int_0^\infty \mathrm{d}z\exp\left(-\frac{|q|z}{\sqrt{2|\alpha|}}(1+i)\right)\mathrm{e}^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)\right] \tag{48}$$

For integral, we recall an integral representation of Whittaker function $W_{\kappa,\mu}(z)$.

In our case, because the condition $\mathrm{Re}\left(\mu-\kappa+\frac{1}{2}\right)>0$ is satisfied, we can employ the following integral representation for $W_{\kappa,\mu}(z)$ [21]:

$$W_{\kappa,\mu}(z)=\frac{\mathrm{e}^{-\frac{z}{2}}z^\kappa}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{t}{z}\right)^{\mu+\kappa-\frac{1}{2}} \tag{49}$$

Recalling that $\mu=-\frac{1}{4},\kappa=-\frac{1}{4}\mp\frac{i}{\pi}$

$$\mathrm{e}^{\frac{1}{4}z^2}\left(\frac{1}{2}z\right)^{-\frac{1}{2}}W_{-\frac{1}{4}\mp\frac{i}{\pi},-\frac{1}{4}}\left(\frac{1}{2}z^2\right)=\frac{1}{\Gamma\left(\frac{1}{2}\pm\frac{i}{\pi}\right)}2^{\frac{3}{4}\pm\frac{i}{\pi}}z\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{-\frac{1}{2}\pm\frac{i}{\pi}}\left(\frac{1}{z^2+2t}\right)^{1\pm\frac{i}{\pi}} \tag{50}$$

Integral part of $F_3(|q|)$ apart from constant becomes

$$I_{F3}=\int_0^\infty \mathrm{d}z\exp\left(-\frac{|q|z}{\sqrt{2|\alpha|}}(1\mp i)\right)z\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{-\frac{1}{2}\pm\frac{i}{\pi}}\left(\frac{1}{z^2+2t}\right)^{1\pm\frac{i}{\pi}}$$
$$=\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{-\frac{1}{2}\pm\frac{i}{\pi}}\int_0^\infty \mathrm{d}z\exp(-\lambda z)z\left(z^2+2t\right)^{-1\mp\frac{i}{\pi}} \tag{51}$$

We can refer the following formula for z integral [22].

$$\int_0^\infty \mathrm{d}x x^{2\nu-1}\left(w^2+x^2\right)^{\xi-1}\mathrm{e}^{-\lambda x}=\frac{w^{2\nu+2\xi-2}}{2\sqrt{\pi}\Gamma(1-\xi)}G_{13}^{31}\left(\frac{\lambda^2w^2}{4}\Big|\begin{matrix}1-\nu & & \\ 1-\xi-\nu & 0 & \frac{1}{2}\end{matrix}\right) \tag{52}$$

In our case, $\nu=1$, $\xi=\mp\frac{i}{\pi}$, $\lambda=\frac{|q|(1\mp i)}{\sqrt{2|\alpha|}}$, $w=\sqrt{2t}$, z integral becomes

$$I_z=\frac{(2t)^{\mp2\frac{i}{\pi}}}{2\sqrt{\pi}\Gamma\left(1\pm\frac{i}{\pi}\right)}G_{13}^{31}\left(\frac{\lambda^2 2t}{4}\Big|\begin{matrix}0 & & \\ \pm\frac{i}{\pi} & 0 & \frac{1}{2}\end{matrix}\right) \tag{53}$$

Then, I_{F3} becomes apart from constant

$$I_3=\frac{2^{-1\mp2\frac{i}{\pi}}}{\sqrt{\pi}\Gamma\left(1\pm\frac{i}{\pi}\right)}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{-\frac{1}{2}\mp\frac{i}{\pi}}G_{13}^{31}\left(\frac{\lambda^2 2t}{4}\Big|\begin{matrix}0 & & \\ \pm\frac{i}{\pi} & 0 & \frac{1}{2}\end{matrix}\right) \tag{54}$$

where G_{pq}^{mn} is Meijer's function defined as [22]

$$G_{pq}^{mn}\left(x\Big|_{br}^{ar}\right)=\sum_{h=1}^m\frac{\prod_{j=1}^m\Gamma\left(b_j-b_h\right)\prod_{j=1}^n\Gamma\left(1+b_h-a_j\right)}{\prod_{j=m+1}^q\Gamma\left(1+b_h-b_j\right)\prod_{j=n+1}^p\Gamma\left(a_j-b_h\right)}x^{b_h}$$

$$\times {}_pF_{q-1}\left(1+b_h-a_1,\cdots,1+b_h-a_p;1+b_h-b_1,\cdots,*,\cdots,1+b_h-b_q:(-1)^{p-m-n}x\right)$$

This definition is given under the condition such that either $p<q$ or $p=q$ and $|x|<1$.

In our case, recalling Equation (48), Equation (49) and Equation (54), the first terms of $|F_3(|q|)|$ become apart from real value constants as

$$\begin{aligned}
\text{First term of } |F_3(|q|)| &= \frac{\left|\Gamma\left(\frac{i}{\pi}\right)\right|\Gamma\left(\frac{1}{2}\right)}{\left|\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\right)\right|}\left(\frac{\lambda^2 2t}{4}\right)^0 - \frac{\left|\Gamma\left(-\frac{i}{\pi}\right)\right|\Gamma\left(\frac{1}{2}\right)}{\left|\Gamma\left(\frac{1}{2}-\frac{i}{\pi}\right)\right|}\left(\frac{\lambda^2 2t}{4}\right)^0 \\
&+\frac{\left|\Gamma\left(-\frac{i}{\pi}\right)\right|\left|\Gamma\left(\frac{1}{2}-\frac{i}{\pi}\right)\right|\left|\Gamma\left(1+\frac{i}{\pi}\right)\right|}{\left|\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\right)\right|}\left|\left(\frac{\lambda^2 2t}{4}\right)^{\frac{i}{\pi}}\right| \\
&-\frac{\left|\Gamma\left(\frac{i}{\pi}\right)\right|\left|\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\right)\right|\left|\Gamma\left(1-\frac{i}{\pi}\right)\right|}{\left|\Gamma\left(\frac{1}{2}-\frac{i}{\pi}\right)\right|}\left|\left(\frac{\lambda^2 2t}{4}\right)^{-\frac{i}{\pi}}\right| \\
&+\frac{\Gamma\left(-\frac{1}{2}\right)\left|\Gamma\left(-\frac{1}{2}+\frac{i}{\pi}\right)\right|\Gamma\left(\frac{3}{2}\right)}{\left|\Gamma\left(\frac{1}{2}+\frac{i}{\pi}\right)\right|}\left(\frac{\left(\frac{|q|(1-i)}{\sqrt{2|\alpha|}}\right)^2 2t}{4}\right)^{\frac{1}{2}} \\
&-\frac{\Gamma\left(-\frac{1}{2}\right)\left|\Gamma\left(-\frac{1}{2}-\frac{i}{\pi}\right)\right|\Gamma\left(\frac{3}{2}\right)}{\left|\Gamma\left(\frac{1}{2}-\frac{i}{\pi}\right)\right|}\left(\frac{\left(\frac{|q|(1+i)}{\sqrt{2|\alpha|}}\right)^2 2t}{4}\right)^{\frac{1}{2}} \qquad (55)\\
&= \text{absolute value of }\left(-2\pi i\frac{|q|}{\sqrt{2|\alpha|}}\sqrt{2t}\sqrt{\frac{1}{4}+\left(\frac{1}{\pi}\right)^2}\right)
\end{aligned}$$

The reason why we take absolute value of $\left(\frac{\lambda^2 2t}{4}\right)^{\pm\frac{i}{\pi}}$ is following. This can be rewritten as

$\left(\frac{\lambda^2 2t}{4}\right)^{\pm\frac{i}{\pi}} = t^{\pm\frac{i}{\pi}}\exp\left(\pm\frac{i}{\pi}\ln\left|\frac{\lambda^2}{2}\right|\right) = t^{\pm\frac{i}{\pi}}\left(\cos\left(\frac{1}{\pi}\ln\left|\frac{\lambda^2}{2}\right|\right)\pm i\sin\left(\frac{1}{\pi}\ln\left|\frac{\lambda^2}{2}\right|\right)\right)$. Note that $t^{\pm\frac{i}{\pi}}$ term cancels integral part of $t^{\mp\frac{i}{\pi}}$ term when considering only the first term of Meijer's G-function. Recalling the fact that we must include $|q|=0$ case $(\lambda=0)$, $|F_3(|q|)|$ become meaningless because $\sin(\infty),\cos(\infty)$ do not exist. In addition, the other reason is that, recalling Equation (54), this integral becomes meaningful when t can be taken from 0 to quite large value for a certain fixed λ^2 value ($|q|$ value). That means λ^2 should be quite small ($|q|$ quite small) because of the condition $\left|\frac{\lambda^2 2t}{4}\right|<1$. Then, integral becomes $\int_0^{\overline{\infty}} \mathrm{d}t e^{-t} t^{-\frac{1}{2}} \sin\left(\frac{1}{\pi}\ln\left|\frac{\lambda^2}{2}\right|\right)$ or $\cos\left(\frac{1}{\pi}\ln\left|\frac{\lambda^1}{2}\right|\right) \approx 0$ because sin or cos-function rapidly oscillates. Thus, we cannot evaluate this contribution to $|F_3(|q|)|$ properly when we take sin, cos-functional representation. Note that $\overline{\infty}$ denotes vary large value except ∞ for any finite λ^2 (quite small), and that t-integral becomes $\int_0^{\overline{\infty}} \mathrm{d}t e^{-t} t^{n+\frac{1}{2}\mp\frac{i}{\pi}-1} \approx \Gamma\left(n+\frac{1}{2}\mp\frac{i}{\pi}\right)$ because n-th term of ${}_1F_2$ is $\left(\frac{\lambda^2}{2}\right)^n t^n$. Recalling the fact that we have to take absolute value of Γ-function when considering $|F_3(|q|)|$ as shown later, we notice that this series become well defined when we take the absolute value of $\left(\frac{\lambda^2 2t}{4}\right)^{\pm\frac{i}{\pi}}$ because this absolute value is the first term and also it multiplies to all other terms.

Here we employ formula as

$$\Gamma(iy)=\left[\pi/y\sinh(\pi y)\right]^{\frac{1}{2}}, \Gamma\left(\frac{1}{2}+iy\right)=\left[\pi/\cosh(\pi y)\right]^{\frac{1}{2}},$$

$$\Gamma\left(\frac{1}{2}\pm n+iy\right)=\left(\frac{\pi}{\cosh(\pi y)}\prod_{k=1}^{n}\left(\left(k-\frac{1}{2}\right)^2+y^2\right)\right)^{\frac{1}{2}},$$

$$\Gamma(1+iy)=\left[\pi y/\sinh(\pi y)\right]^{\frac{1}{2}}$$

These formula can be obtained by employing the following relations [21].

$$\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z) \text{ and } \Gamma\left(z+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-z\right)=\pi/\cos(\pi z)$$

$$\Gamma(z+1)=z\Gamma(z) \text{ and } \overline{\Gamma(z)}=\Gamma(\overline{z})$$

The reason why taking the absolute value of Γ-functions in Equation (55) is as follows.

Γ-function is defined by Euler as

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z \Big/ \left(1+\frac{z}{n}\right)\right]$$

When z is complex, we cannot specify $\Gamma(z)$ value. Thus we have to take the absolute value to specify $\Gamma(z)$ value in the case of complex z. Because distribution amplitude is absolute value of $F_3(|q|)$, we have to take the absolute value of $\Gamma(z)$ to evaluate it.

Although Equation (46) is obtained for large $|q|$ case, Expansion of Equation (46) becomes

$$\begin{aligned} I^{\gamma} &= \frac{i}{|q|}\left[1-\left(1-|q|\rho+\frac{1}{2}\left(|q|\rho\right)^{2}\right)\left(1+|q|\rho-\frac{1}{2}\left(|q|\rho\right)^{2}\right)\right] \\ &= \frac{i}{|q|}\left[\left(|q|\rho\right)^{2}+O\left(\left(|q|\rho\right)^{3}\right)\right]=i\rho^{2}|q| \end{aligned} \tag{56}$$

where $\rho=\frac{u}{\sqrt{2|\alpha|}}$.

Thus, the first term of I^{γ} is the same order of $|q|$ as that of Equation (55) so that we consider Equation (46) as an approximate form of distribution function.

Then, we obtain an approximate form of charged pion distribution amplitude in the range of $[0,\infty]$ as

$$F_3\left(|q|\right)=\frac{1}{|q|}\left[1-\exp\left(-|q|\rho\right)\left(\sin\left(|q|\rho\right)+\cos\left(|q|\rho\right)\right)\right] \tag{57}$$

In order to construct distribution amplitude in the range of $[0,1]$, we again set $|q|$ as $|q|=\frac{q_1}{1-q_1}$. The range of q_1 is $[0,1]$ and q_1 is considered as $\frac{\text{Momentum}}{\text{Maximum Momentum}}$.

Momentum can be written as Momentum = $\bar{x}$ ·(Max Momentum) where $\bar{x}\in[0,1]$.

Thus, Equation (57) is rewritten as

$$\left|F_3\left(\bar{x}\right)\right|=F_3\left(\bar{x}\right)=\frac{1-\bar{x}}{\bar{x}}\left[1-\mathrm{e}^{-\frac{\bar{x}}{1-\bar{x}}\rho}\left(\sin\left(\frac{\bar{x}}{1-\bar{x}}\rho\right)+\cos\left(\frac{\bar{x}}{1-\bar{x}}\rho\right)\right)\right] \tag{58}$$

The form of Equation (58) is very simple, however, when we compare this with other form of pion distribution amplitude such that $\phi_{\pi}=\frac{1}{B(m,n)}x^{m}(1-x)^{n}$ by Zhang [8] where $B(m,n)$ is Beta function, and $\phi_{\pi}=1.81\left[x(1-x)\right]^{0.31}\left[1-0.12C_2^{0.81}(2x-1)\right]$ by Chang [9], and the form of the distribution amplitude without considering pion radius effect by Raya as [7]

$$u^{P}=n_{P}x^{2}(1-x)^{2}\left[1+\rho_{P}x^{\frac{\alpha_{P}}{2}}(1-x)^{\frac{\beta_{P}}{2}}+\gamma_{P}x^{\alpha_{P}}(1-x)^{\beta_{P}}\right]^{2}$$

we can consider that the form of Equation (58) is sufficient to observe the basic behavior of our distribution amplitude.

We show behavior of normalized $F_3(x)$ based on Equation (58) for several ρ values case in **Figure 3**.

Note that we change the notation $\bar{x}$ to x from now on so that the range of this x is $[0,1]$.

We can compare results of **Figure 3** to those of Raya *et al.* [7]. Especially, we can compare our results of $\rho=1.0$ and $\rho=0.5$ cases to Raya's results of $\Delta^2r^2=2$ and $\Delta^2r^2=5$ case, respectively. According to Raya, Δ is defined as $\Delta=p'-p$ where p' and p are the final, initial meson momenta in the defining scattering process and r is pion radius. For both cases, our peaks are almost same as theirs and maximum values are close but smaller than theirs. Most intriguing point is that, for less than $\rho = 1.0$ case, our peaks shift towards $x = 1$. This shift is also shown in Raya's results as $\Delta^2 r^2$ increases. In our case, the cause of peak shift is basically magnitude of a coupling constant g^2 because of the setting that $\rho = \frac{u}{\sqrt{2|\alpha|}} = \frac{2\sqrt{2}u}{\sqrt{g^2}}$. The behavior of $|F_3(x)|$ near $x = 1$ is that $|F_3(x)| \to \frac{1-x}{x}$ at $x \to 1$. This behavior is also true for the exact form of $|F_3(x)|$ case because the evaluation for very large $|q|$ case is sufficiently accurate.

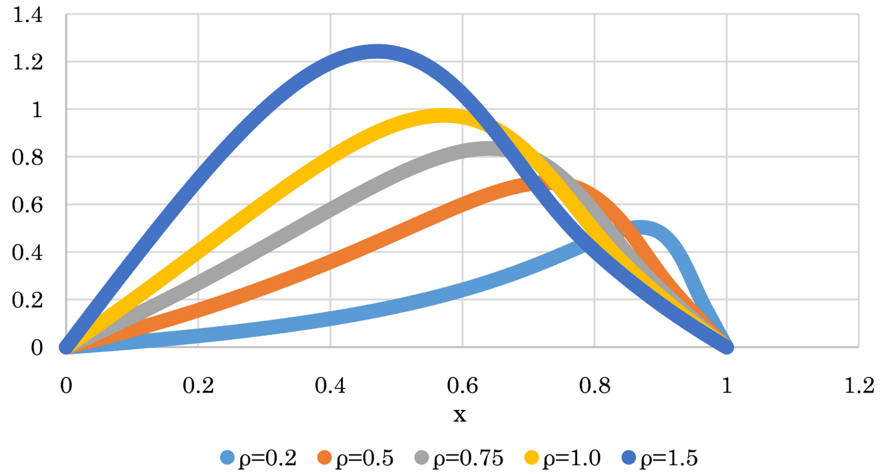

Figure 3. Distribution amplitude with various ρ values.

Important point is that Raya *et al.* consider that pion distribution amplitude is directly related to a valence u-quark distribution function of pion [7] and they employ one of their results of distribution amplitude to represent a valence u-quark distribution function of pion [6].

4. Conclusion and Discussion

Applying our solutions of the 't Hooft model to construct a pion distribution amplitude, we obtain a pion distribution amplitude $|F_3(x)|$ as the absolute value of Equation (41). By exploring its asymptotic form described as Equation (58), we find that our results show each peak shifts towards $x = 1$ caused by magnitude of a coupling constant g^2 and that $|F_3(x)| \to \frac{1-x}{x}$ as $x \to 1$. According to the correct Drell-Yan-West relation, for hadron H defined by $n+1$ valence spin $J = \frac{1}{2}$ partons, its leading elastic form factor scales as $\left(1/Q^2\right)^n$ [23]:

$$q_H(x) \sim (1-x)^\beta \text{ when } x \cong 1,$$

$$\beta = 2n - 1 + 2\Delta S_z \text{ where } \Delta S_z = \left|S_z^q - S_z^H\right|$$

Because $S_z^H = 0$ for pseudo-scalar meson case, $\Delta S_z = \frac{1}{2}$ so that $n = \frac{1}{2}$ for

$\beta = 1$ case. Thus, its leading elastic form factor scales as $\left(1/Q^2\right)^{\frac{1}{2}}$.

This contradicts the fact that $n = 1$ for pseudo-scalar meson, however, we are interested in a leading elastic form factor when we know distribution function so that we used above argument. According to Raya *et al.* [7], distribution amplitude is directly related to distribution function so that our case indicates that a leading charged pion form factor scales as $\left(1/Q^2\right)^{\frac{1}{2}}$. This is slightly larger than experimental results as $1/Q^2$; however, Arriora *et al.* [24] demonstrate slightly larger space-like form factor using the argument of extrapolating from time-like region. Their upper bound result shows $\left(1/Q^2\right)^{\frac{1}{2}}$ behavior. Pasquini *et al.* [25] and Xie *et al.* [26] also show the distribution functions that have $(1-x)^1$ as $x \cong 1$ behavior (actuary exponent is close to 1 because both results fit E615 Drel Yan experiment [27]). Especially, Pasquini *et al.* show that their pion electromagnetic form factor explains well for experimental data up to $Q^2 = 10\left(\text{GeV}^2\right)$ although their distribution function behaves as $(1-x)^1$ as $x \cong 1$ because their form factor is evaluated by the following form:

$$F_{\pi,q\bar{q}}\left(Q^2\right) = \frac{1}{2}\int_0^1 \mathrm{d}x\,\phi_\pi(x)\exp\left(-a_{q\bar{q}}Q^2\frac{1-x}{2x}\right) \text{ where } a_{q\bar{q}} \text{ is a parameter.}$$

According to Xie *et al.*, Leading Neutron Deep Inelastic Scattering (LN-DIS) will reveal a structure of one dimensional mesons (1 + 1 dimensional meson) precisely in the range of $x \in [0.1, 0.9]$. Presently their simulation results are reliable only in the range of $x \geq 0.75$; however, these are good to see the behavior of distribution function near $x = 1$. This means that their results show the behavior of distribution function near $x = 1$ of one dimensional pion that corresponds to a meson of ‘t Hooft model. Thus, we can consider that the behavior of our result near $x = 1$ is consistent to that of their results.

Although we need to work out more detailed analysis of the absolute value of Equation (41) to confirm shifting property, we can claim that it is worthwhile to investigate pion distribution amplitude by using the exact solution of ‘t Hooft model. Here we would like to insist that the exact solution of ‘t Hooft model means not only the solution of ‘t Hooft singular integral equation but any solutions of large N limit two dimensional QCD massless quark case.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dlamini, M., *et al.* (2021) Deep Exclusive Electroproduction of π^0 at High Q^2 in the Quark Valence Regime. *Physical Review Letters*, **127**, Article ID: 152301.

[2] Huber, G.M. (2024) Measurement of the Charged Pion and Kaon Form Factors to High Q^2 at JLab and EIC. *CFNS Workshop on Elucidating the Structure of Nambu-Goldstone Bosons*, New York, 24-28 June 2024.

[3] Horm, T. (2024) Experimental Overview on Experiments for Pion/Kaon Structure. *CFNS Workshop on Elucidating the Structure of Nambu-Goldstone Bosons*, New York, 24-28 June 2024.

[4] Abramowicz, H., *et al.* (2015) Combination of Measurements of Inclusive Deep Inelastic $e^{\pm}p$ Scattering Cross Sections and QCD Analysis of HERA Data. *The European Physical Journal C*, **75**, Article No. 580.

[5] Aad, G., *et al.* (2022) Determination of the Parton Distribution Functions of the Proton Using Diverse ATLAS Data from pp Collisions at $\sqrt{S} = 7$, 8 and 13 TeV. *The European Physical Journal C*, **82**, Article No. 438.

[6] Lu, Y., Chang, L., Raya, K., Roberts, C.D. and Rodríguez-Quintero, J. (2022) Proton and Pion Distribution Functions in Counterpoint. *Physics Letters B*, **830**, Article ID: 137130. https://doi.org/10.1016/j.physletb.2022.137130

[7] Raya, K., Cui, Z., Chang, L., Morgado, J.M., Roberts, C.D. and Rodríguez-Quintero, J. (2022) Revealing Pion and Kaon Structure via Generalised Parton Distributions. *Chinese Physics C*, **46**, Article ID: 013105. https://doi.org/10.1088/1674-1137/ac3071

[8] Zhang, R., Honkala, C., Lin, H. and Chen, J. (2020) Pion and Kaon Distribution Amplitudes in the Continuum Limit. *Physical Review D*, **102**, Article ID: 094519. https://doi.org/10.1103/physrevd.102.094519

[9] Chang, L., Cloet, I.C., Cobos-Martinez, J.J., Roberts, C.D., Schmidt, S.M. and Tandy, P.C. (2013) Imaging Dynamical Chiral-Symmetry Breaking: Pion Wave Function on the Light Front. *Physics Letters B*, **110**, Article ID: 132001.

[10] ‘t Hooft, G. (1974) A Two-Dimensional Model for Mesons. *Nuclear Physics B*, **75**, 461-470. https://doi.org/10.1016/0550-3213(74)90088-1

[11] Litvinov, A. and Meshcheriakov, P. (2025) Meson Mass Spectrum in QCD2 ‘t Hooft’s Model. *Nuclear Physics B*, **1010**, Article ID: 116766. https://doi.org/10.1016/j.nuclphysb.2024.116766

[12] Suura, H. (1979) Equation of Motion for String Operators in Quantum Chromodynamics. *Physical Review D*, **20**, 1412-1419. https://doi.org/10.1103/physrevd.20.1412

[13] Kurai, T. (2018) Light Meson Mass Spectra and Pion Electromagnetic Form Factor as a Bound System in 3 + 1 Dimensional QCD. *Results in Physics*, **10**, 865-881. https://doi.org/10.1016/j.rinp.2018.07.034

[14] Kurai, T. (2021) Light Meson Mass Spectra with Massive Quarks. *Journal of Modern Physics*, **12**, 1545-1572. https://doi.org/10.4236/jmp.2021.1211093

[15] Kurai, T. (2024) Describing a Baryon as a Composition of Bound Stated and Unbound Stated Sea-Quarks. *Journal of Modern Physics*, **15**, 1586-1602. https://doi.org/10.4236/jmp.2024.1510067

[16] Kurai, T. (2014) The Meson as a Bound System in 2D Quantum Chromodynamics. *Progress of Theoretical and Experimental Physics*, **2014**, 053B01.

[17] Buchmüller, W., Love, S.T. and Peccei, R.D. (1982) Zero Mass States in QCD2. *Physics Letters B*, **108**, 426-430. https://doi.org/10.1016/0370-2693(82)91227-8

[18] Casher, A., Kogut, J. and Susskind, L. (1974) Vacuum Polarization and the Absence of Free Quarks. *Physical Review D*, **10**, 732-745. https://doi.org/10.1103/physrevd.10.732

[19] Schwinger, J. (1962) Gauge Invariance and Mass. II. *Physical Review*, **128**, 2425-2429. https://doi.org/10.1103/physrev.128.2425

[20] Gakhov, F.D. (1966) Boundary Value Problems. Pergamon.

[21] Moriguchi, S., Udagawa, K. and Hititsumatsu, S. (1975) Formula of Mathematics III:

Special Functions. Iwanami.

[22] Gradshteyn, I.S. and Ryzhic, M. (1980) Table of Integral, Series and Products. Academic Press.

[23] Arrington, J., *et al*. (2021) Revealing the Structure of Light Pseudoscalar Mesons at the Electron-Ion Collider. *Journal of Physics G*, *Nuclear and Particle Physics*, **48**, Article No. 075106.

[24] Arriora, E.R. and Sanchez-Puertas, P. (2024) Phase of the Electromagnetic form Factor of the Pion. *Physical Review D*, **110**, Article ID: 054003.

[25] Pasquini, B., Rodini, S. and Venturini, S. (2023) Valence Quark, Sea, and Gluon Content of the Pion from the Parton Distribution Functions and the Electromagnetic Form Factor. *Physical Review D*, **107**, Article ID: 114023.
https://doi.org/10.1103/physrevd.107.114023

[26] Xie, G., Li, M., Han, C., Wang, R. and Chen, X. (2021) Simulation of Neutron-Tagged Deep Inelastic Scattering at EICC. *Chinese Physics C*, **45**, Article ID: 053002.
https://doi.org/10.1088/1674-1137/abe8cf

[27] Conway, J.S., *et al*. (1989) Experimental Study of Muon Pairs Produced by 252-GeV Pions on Tungsten. *Physical Review D*, **39**, 92.

Appendix

Appendix A. $S(r)=\frac{1}{2\pi}Pr\frac{1}{r}$

The Green's function is defined as

$$G(x,x')=\langle 0|\Psi(x)\Psi^{\dagger}(x)|0\rangle \tag{A1}$$

where Ψ is a fermion field (quark field), and x and x' have finite separation.

This Green's function satisfies the following equation:

$$\left(\alpha^{\mu}\partial_{\mu}-i\alpha^{\mu}gA_{\mu}\right)G(x,x')=1\delta(x-x') \tag{A2}$$

where 1 denotes unit matrix.

Note that the form of Equation (A2) is taken by γ^{0} multiplying Dirac equation in the case of massless field. Because Dirac equation with massive case, in our choice of γ-matrices, should be

$$\left(\alpha^{\mu}D_{\mu}+\gamma^{0}m\right)\phi(x)=0 \tag{A3}$$

Dirac equation with massless quarks case becomes

$$i\alpha^{\mu}D_{\mu}q(x)=0 \tag{A4}$$

where $D_{\mu}=\partial_{\mu}-igA_{\mu}^{a}\frac{\lambda_{a}}{2}$.

The solution can be written in the form

$$G(x,x')=G_{0}(x,x')\exp\left(\varphi(x)-\varphi(x')\right) \tag{A5}$$

where

$$\alpha^{\mu}\partial_{\mu}G_{0}(x,x')=1\delta(x-x') \tag{A6}$$

$$\alpha^{\mu}\partial_{\mu}\varphi(x)=\alpha^{\mu}igA_{\mu}^{a}(x)\frac{\lambda_{a}}{2} \tag{A7}$$

Equation (A7) means as

$$\partial_{\mu}\varphi(x)=igA_{\mu}^{a}(x)\frac{\lambda_{a}}{2} \tag{A8}$$

Now we will find the solution of $G_{0}(x,x')$ that is time-independent. Starting equation is following:

$$\partial_{x}\bar{G}_{0}(x,x')=\delta(x-x') \tag{A9}$$

where $G_{0}(x,x')=\alpha^{1}\bar{G}_{0}(x,x')$.

Then $\bar{G}_{0}(x,x')$ can be obtained as follows:

$$\begin{aligned}\bar{G}_{0}(x,x')&=\int_{-\infty}^{x}\mathrm{d}\bar{x}\,\delta(\bar{x}-x')=\int_{-\infty}^{x}\mathrm{d}\bar{x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip(\bar{x}-x')}\\&=\int_{-\infty}^{x}\mathrm{d}\bar{x}\frac{1}{2\pi}\left[\int_{-\infty}^{0}\mathrm{d}p\,e^{ip(\bar{x}-x')}+\int_{0}^{\infty}\mathrm{d}p\,e^{ip(\bar{x}-x')}\right]\end{aligned} \tag{A10}$$

In the second line, we use the description of δ-function as $\delta(x-x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\exp\left(ip(x-x')\right)$.

For the first integral term in parenthesis, changing variable p to $-P'$, then

$$\int_{-\infty}^{0}\mathrm{d}p\,\mathrm{e}^{ip(\overline{x}-x')}=\int_{\infty}^{0}-\mathrm{d}p'\,\mathrm{e}^{-ip'(\overline{x}-x')}=\int_{0}^{\infty}\mathrm{d}p'\,\mathrm{e}^{-ip'(\overline{x}-x')} \tag{A11}$$

Combining the last line to the second integral in parenthesis, parenthesis terms become

$$\begin{aligned}\text{Parenthesis} &= 2\int_{0}^{\infty}\mathrm{d}p\cos\left(p\left(\overline{x}-x'\right)\right)=2\lim_{\epsilon\to0}\int_{\epsilon}^{\infty}\mathrm{d}p\cos\left(p\left(\overline{x}-x'\right)\right)\\ &=2\lim_{\epsilon\to0}\left[-\frac{\sin\left(\epsilon\left(\overline{x}-x'\right)\right)}{\overline{x}-x'}\right]=2\lim_{\epsilon\to0}\frac{-1}{2i}\frac{\mathrm{e}^{i\epsilon(\overline{x}-x')}-\mathrm{e}^{-i\epsilon(\overline{x}-x')}}{\overline{x}-x'}\end{aligned} \tag{A12}$$

Then $\overline{G}_0$ becomes

$$\overline{G}_0\left(x,x'\right)=\int_{-\infty}^{x}\mathrm{d}\overline{x}\frac{1}{2\pi}2\left(\frac{-1}{2i}\right)\lim_{\epsilon\to0}\frac{\mathrm{e}^{i\epsilon(\overline{x}-x')}-\mathrm{e}^{-i\epsilon(\overline{x}-x')}}{\overline{x}-x} \tag{A13}$$

Changing variable as $\epsilon\left(\overline{x}-x'\right)=\overline{\overline{x}}$, integral becomes

$$\overline{G}_0\left(x,x'\right)=\lim_{\epsilon\to0}\int_{-\infty}^{\epsilon(x-x')}\mathrm{d}\overline{\overline{x}}\frac{1}{2\pi}2\left(\frac{-1}{2i}\right)\frac{\mathrm{e}^{i\overline{\overline{x}}}-\mathrm{e}^{-i\overline{\overline{x}}}}{\overline{\overline{x}}}=\left(\frac{-1}{2\pi i}\right)\int_{-\infty}^{0}\mathrm{d}\overline{\overline{x}}\frac{\mathrm{e}^{i\overline{\overline{x}}}-\mathrm{e}^{-i\overline{\overline{x}}}}{\overline{\overline{x}}} \tag{A14}$$

Note that the lower limit of integral is taken as $-\infty$ because we can always take $-\infty$ as $-\frac{1}{\epsilon}\infty$.

For the second term of the last line, changing variable as $-\overline{\overline{x}}=\overline{\overline{x}}'$, second integral term becomes

$$-\int_{-\infty}^{0}-\mathrm{d}\overline{\overline{x}}'\frac{\mathrm{e}^{i\overline{\overline{x}}'}}{-\overline{\overline{x}}'}=\int_{0}^{\infty}\mathrm{d}\overline{\overline{x}}\frac{\mathrm{e}^{i\overline{\overline{x}}}}{\overline{\overline{x}}}$$

Combining this to the first term of the last line, we obtain the form of $G_0\left(x,x'\right)$ as

$$\overline{G}_0\left(x,x'\right)=\left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty}\mathrm{d}\overline{\overline{x}}\frac{\mathrm{e}^{i\overline{\overline{x}}}}{\overline{\overline{x}}}=\left(\frac{i}{2\pi}\right)\int_{-\infty}^{\infty}\mathrm{d}x\frac{\mathrm{e}^{i(x-x')}}{x-x'}=\left(\frac{i}{2\pi}\right)Pr\left(\frac{1}{x-x'}\right) \tag{A15}$$

For the second line, we change the variable $\overline{\overline{x}}$ to $x-x'$.

Therefore $G_0\left(x,x'\right)$ is obtained as

$$G_0\left(x,x'\right)=\left(i\alpha^1\right)\frac{1}{2\pi}Pr\left(\frac{1}{x-x'}\right)=\left(i\sigma_3\right)\frac{1}{2\pi}Pr\left(\frac{1}{x-x'}\right) \tag{A16}$$

Now we will find a solution of Equation (A8). Because $G\left(x,x'\right)$ is time independent, Equation (A8) becomes

$$\partial_x\varphi\left(x\right)=igA^a\left(x\right)\frac{\lambda_a}{2} \tag{A17}$$

where $A^a\left(x\right)\frac{\lambda_a}{2}=A_1^a\left(x\right)\frac{\lambda_a}{2}$.

$$\varphi\left(x\right)=ig\int_{-\infty}^{x}\mathrm{d}z\,A^a\left(z\right)\frac{\lambda_a}{2} \tag{A18}$$

Then we can describe the Green's function $G\left(x,x'\right)$ as follows:

$$G\left(x,x'\right)=\left(i\sigma_3\right)\frac{1}{2\pi}Pr\left(\frac{1}{x-x'}\right)\exp\left(ig\int_{x'}^{x}\mathrm{d}z\,A^a\left(z\right)\frac{\lambda_a}{2}\right) \tag{A19}$$

The vacuum expectation value of our amplitude is given as follows.

$$S(x,x') = \langle 0|q(x)q^{\dagger}(x')\exp\left(\int_x^{x'} \mathrm{d}z igA^a(z)\frac{\lambda_a}{2}\right)|0\rangle \tag{A20}$$

Inserting $|0\rangle\langle 0|$ between $q^{\dagger}$ and exponent in Equation (A20), we obtain

$$S(x,x') = G(x,x')\langle 0|\exp\int_x^{x'} \mathrm{d}z igA^a(z)\frac{\lambda_a}{2}|0\rangle \tag{A21}$$

$G(x,x')$ is given in Equation (A19) and then we notice that the phase factor is cancelled out.

Thus we obtain

$$S(x,x') = (i\sigma_3)\frac{1}{2\pi}Pr\left(\frac{1}{x-x'}\right)$$

Therefore using the following expression is justified:

$$S(r) = (i\sigma_3)\frac{1}{2\pi}Pr\left(\frac{1}{r}\right) \tag{A22}$$

Appendix B. $\int_{\infty}^{x} \mathrm{d}x'\delta(x')f(x') = 0$

We use the definition of the δ-function as

$$\delta(z) = \begin{cases} \frac{1}{2\varepsilon} & -\varepsilon \le z \le \varepsilon \\ 0 & \text{otherwise} \end{cases} \quad \text{when } \varepsilon \text{ approaches 0.}$$

We may conclude that

$$\int_x^{\infty} \mathrm{d}z\delta(z)f(z) = 0$$

under setting the condition that x is always larger than ε.

We can always set this condition except exact $x = 0$ point.

At $x = 0$, we go back to original equation such as $\partial_x(\delta(x)f(x)) = 0$.

At $x = 0$, $f(0) = \text{const}$, and in our definition of $\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}$, δ-function is quite large, but just number. Thus, $\delta(x)f(x)$ is just constant. Thus, derivative of constant with respect to x is 0. Thus at $x = 0$, $\partial_x(\delta(x)f(x)) = 0$ is satisfied with.

Because integral of 0 is always 0, we can consider that $\int_x^{\infty} \mathrm{d}z\delta(z)f(z) = 0$ for $x \ge 0$.

Appendix C. Derivation of $\phi_-^{(\mp)}$

Equation (18) is expressed as follows;

$$\frac{\mathrm{d}^2\phi_-^{(+)}}{\mathrm{d}x^2} + (\beta - \alpha x)\frac{\mathrm{d}\phi_-^{(+)}}{\mathrm{d}x} + \left(-\left(\frac{g^2}{8\pi}\right) - \alpha\right)\phi_-^{(+)} = 0 \tag{A23}$$

where $\alpha = \frac{g^2}{8}\frac{1}{2i}, \beta = \frac{W_0}{2i}$.

After first taking

$$\phi_-^{(+)}(x) = e^{\frac{1}{4}\alpha x^2} u(x)$$

Equation (A23) becomes

$$\frac{d^2u}{dx^2} + \beta\frac{du}{dx} + \left[-\frac{1}{2}\alpha - \left(\frac{g^2}{8\pi}\right) - \frac{1}{4}(\alpha x - \beta)^2 + \frac{1}{4}\beta^2\right] \tag{A24}$$

Next taking $u(x) = e^{-\frac{1}{2}\beta x} v(x)$.

Equation (A24) becomes

$$\frac{d^2v}{dx^2} + \left[-\frac{1}{2}\alpha - \left(\frac{g^2}{8\pi}\right) - \frac{1}{4}(\alpha x - \beta)^2\right] v(x) = 0 \tag{A25}$$

Changing a variable as $\frac{1}{2}(\alpha x - \beta) = z$, Equation (A25) becomes

$$\frac{d^2v}{dz^2} + \left[\frac{-\frac{1}{2}\alpha - \left(\frac{g^2}{8\pi}\right)}{\frac{1}{4}\alpha^2} - \frac{z^2}{\frac{1}{4}\alpha^2}\right] v(z) = 0 \tag{A26}$$

Taking $\gamma = \dfrac{-\frac{1}{2}\alpha - \left(\frac{g^2}{8\pi}\right)}{\frac{1}{4}\alpha^2}$ and changing a variable as $t = \dfrac{2}{\sqrt{\alpha}} z$,

Equation (A26) becomes

$$\frac{d^2v}{dt^2} + \left[\frac{\alpha\gamma}{4} - \frac{t^2}{4}\right] v(t) = 0 \tag{A27}$$

By comparing Equation (A27) with the standard form of the parabolic cylinder function (Weber function) [19], we find $\frac{\alpha\gamma}{4} = \bar{\lambda} + \frac{1}{2}$.

Recalling the definition of α, we obtain $\bar{\lambda} = -1 - \frac{2i}{\pi}$.

The solution of Equation (A27) is then

$$v(t) = D_{\bar{\lambda}}(t) = 2^{\frac{\bar{\lambda}}{2} + \frac{1}{4}} t^{-\frac{1}{2}} W_{\frac{\bar{\lambda}}{2} + \frac{1}{4}, -\frac{1}{4}}\left(\frac{t^2}{2}\right) \tag{A28}$$

Substituting the value obtained for $\bar{\lambda}$ into Equation (A28), we find

$$v(t) = 2^{-\frac{1}{4} - \frac{i}{\pi}} t^{-\frac{1}{2}} W_{-\frac{1}{4} - \frac{i}{\pi}, -\frac{1}{4}}\left(\frac{t^2}{2}\right) \tag{A29}$$

For $\phi_-^{(-)}$, Equation (19) is represented by α and β such that

$$\frac{d^2\phi_-^{(-)}}{dx^2} + (\beta + \alpha x)\frac{d\phi_-^{(-)}}{dx} + \left(-\frac{g^2}{8\pi} + \alpha\right)\phi_-^{(-)} = 0 \tag{A30}$$

By using the same argument applied previously, we obtain

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}+\left[\frac{\alpha\gamma'}{4}-\frac{t^2}{4}\right]v(t)=0 \tag{A31}$$

where

$$\gamma'=\frac{\frac{1}{2}\alpha-\left(\frac{g^2}{8\pi}\right)}{\frac{1}{4}\alpha^2}$$

In this case, $\bar{\lambda}=-\frac{2i}{\pi}$.

Because the solution of the parabolic cylinder function equation are $D_{\bar{\lambda}}(t)$ and $D_{-\bar{\lambda}-1}(it)$ [19], we select $D_{-\bar{\lambda}-1}(it)$ in this case. The solution is then

$$v(t)=D_{-\bar{\lambda}-1}(it)=2^{-\frac{1}{4}+\frac{i}{\pi}}(it)^{-\frac{1}{2}}W_{-\frac{1}{4}+\frac{i}{\pi},-\frac{1}{4}}\left(-\frac{t^2}{2}\right) \tag{A32}$$

Appendix D

The reason why I_2 integral of both Equation (38) and Equation (39) vanish is following.

On the contour of I_2, we can set $z=Re^{i\theta}$. Then Real part of exponents becomes

$$\mathrm{Real}(\exp)=-\frac{|q|}{\sqrt{2|\alpha|}}(R\cos\theta+R\sin\theta)\le 0\ \left(0\le\theta\le\frac{3\pi}{4}\right)\ \text{for Equation (38)}$$

$$\mathrm{Real}(\exp)=-\frac{|q|}{\sqrt{2|\alpha|}}(R\cos\theta-R\sin\theta)\le 0\ \left(0\le\theta\le\frac{\pi}{4}\right)\ \text{for Equation (39)}$$

Thus, exponent term is less than 1 on this contour.

Recalling asymptotic behavior of Whittaker function $W_{\kappa,\mu}(z)$ in Equation (43), asymptotic behavior of $\mathrm{e}^{\frac{1}{4}z^2}\frac{1}{\sqrt{z}}W_{\kappa,\mu}\left(\frac{1}{2}z^2\right)=T$ ($|z|$ is large) becomes

$$T\sim\frac{1}{R}\left[1+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{16}-\left(\frac{3}{4}\pm\frac{i}{\pi}\right)^2\right)\cdots\left(\frac{1}{16}-\left(\frac{1}{4}\mp\frac{i}{\pi}-n\right)^2\right)}{n!R^{2n}}\right]\to 0\ (\text{as } R\to\infty)$$

Therefore, I_2 integral vanish for both cases.