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Variable-Order Fractional Derivatives in Financial Systems

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Abstract

Financial systems are inherently complex, exhibiting memory effects, nonlinearity, and evolving dynamics that cannot be adequately captured by traditional differential models. This study introduces a novel financial system modeled using variable-order fractional derivatives of the Caputo-Fabrizio type, allowing the system's memory to change dynamically over time. Three distinct memory structures-constant, periodic, and non-periodic (sigmoid-shaped)-are explored to simulate various economic regimes, such as stable markets, cyclical behaviors, and structural transitions. Through detailed numerical simulations and comparative analysis, the model demonstrates remarkable flexibility in capturing real-world financial behaviors, including oscillatory trends, amplification effects, and memory-driven regime shifts. The incorporation of variable-order dynamics provides a more adaptive and realistic framework for analyzing economic systems under uncertainty. Furthermore, the study outlines a path for integrating data-driven estimation techniques to learn the memory order $q(t)$ from empirical financial data, opening new directions for forecasting, control, and policy modeling. The proposed framework offers a significant advancement in fractional modeling, bridging theoretical innovation with practical financial relevance.

Keywords

Variable-Order Fractional Calculus, Caputo-Fabrizio Derivative, Financial Systems, Memory Effects, Nonlinear Dynamics, Fractional Modeling, Economic Simulation, Time-Varying Memory, Data-Driven Modeling

1. Introduction

Financial systems are inherently complex, often exhibiting memory-dependent behaviors that challenge the assumptions of classical differential models. Such systems respond not only to current inputs but also to their historical states, particularly during financial shocks, speculative bubbles, and long-term policy shifts [1] [2].

Fractional calculus, which generalizes classical differentiation to non-integer orders, has emerged as a powerful mathematical tool for modeling memory and hereditary properties in dynamical systems [3] [4]. In the context of finance, fractional derivatives have been successfully applied to capture anomalous diffusion, long-range dependence, and volatility clustering in asset prices [5] [6].

Traditionally, fractional-order models employ a *constant-order* derivative. However, this assumption can limit the adaptability of the model, especially in time-varying systems such as financial markets, where the memory effect is not static but evolves depending on factors like economic cycles, investor sentiment, or regulatory changes [7] [8].

To overcome this limitation, the concept of variable-order fractional derivatives has been introduced. In these models, the order of the derivative, denoted by $q(t)$, is allowed to vary with time. This added flexibility enables the system to dynamically adjust its memory strength, making it more suitable for modeling non-stationary financial processes such as sudden crashes or structural transitions [8] [9].

In this chapter, we propose a financial system model governed by a variable-order fractional derivative, specifically the Caputo-Fabrizio type, which uses a non-singular exponential kernel. We analyze the behavior of the system under different forms of the order function $q(t)$, including constant, periodic, and non-periodic cases. Our objective is to evaluate the effect of time-dependent memory on financial system dynamics and demonstrate the practical advantages of using variable-order modeling in finance [10] [11].

2. Preliminaries and Background

Fractional calculus allows the use of derivatives of non-integer order, making it well-suited for modeling systems with memory and hereditary effects [3] [4]. In financial systems, these properties are essential for capturing long-term dependencies and delayed market responses [5] [6].

The classical Caputo fractional derivative is commonly used but involves a singular kernel of the form $(t-\tau)^{-\alpha}$, which may lead to numerical instability and difficulties in computation [9]. To address this, Caputo and Fabrizio introduced a new definition with an exponential, non-singular kernel [10]:

$$ {}_{0}^{CF}D^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] f'(\tau)\, d\tau, \tag{1} $$

where $M(\alpha) = \dfrac{2}{2-\alpha}$.

This form improves numerical behavior and is more suitable for practical applications where memory decays gradually over time [12].

In real-world financial systems, memory effects often change over time due to market shocks or structural changes. To capture this, the fractional order α is replaced with a time-varying function $q(t)$, leading to the variable-order Caputo-Fabrizio derivative [8] [13]:

$$ {}_{0}^{CF}D^{q(t)} f(t) = \frac{(2-q(t))M(q(t))}{2(1-q(t))}\int_0^t \exp\left[-\frac{q(t)}{1-q(t)}(t-\tau)\right] f'(\tau)\,\mathrm{d}\tau. \qquad (2) $$

This variable-order model allows more flexibility in simulating financial behavior under changing market conditions.

3. Mathematical Model Formulation

We consider a nonlinear financial system described by variable-order Caputo-Fabrizio fractional derivatives. The model is adapted from the fractional-order system proposed by Malaikah and Al-Abdali [14], extended here to incorporate time-varying memory effects through a variable fractional order $q(t)$.

The system is governed by the following equations:

$$ \begin{cases} {}_{0}^{CF}D^{q(t)}x(t) = z(t) + (y(t)-a)x(t) + u(t), \\ {}_{0}^{CF}D^{q(t)}y(t) = 2 - by(t) - x(t)^2, \\ {}_{0}^{CF}D^{q(t)}z(t) = x(t)y(t) - x(t) - cz(t), \\ {}_{0}^{CF}D^{q(t)}u(t) = -dx(t)y(t) - gu(t), \end{cases} \qquad (3) $$

where the variables represent key financial indicators:
- $x(t)$: interest rate,
- $y(t)$: investment demand,
- $z(t)$: price index,
- $u(t)$: average profit margin.

The system parameters are defined as follows:
- a: saving rate,
- b: cost per investment,
- c: market elasticity,
- $d, g > 0$: interaction and dissipation parameters.

By allowing the fractional order $q(t) \in (0,1)$ to vary with time, the system dynamically adjusts its memory depth. This makes the model more flexible and capable of capturing different financial regimes-such as stability, volatility, or transitions caused by economic shocks-more effectively than constant-order models [8] [10] [15].

4. Choice of Variable-Order Functions

In variable-order fractional models, the choice of the fractional order function $q(t)$ plays a central role in determining the system's memory behavior. Unlike constant-order models, where memory is fixed, variable-order functions allow the memory effect to evolve over time. This is particularly important in financial systems, where market memory can expand or contract in response to economic cycles, shocks, or structural changes [7] [8].

To explore the impact of different memory dynamics, we consider two representative forms of $q(t)$:

4.1. Periodic Order Function

A periodic form of $q(t)$ captures cyclical phenomena in financial systems such as business cycles, seasonal investment trends, or policy interventions. The functional form is:

$$q(t) = q_0 + A\cos\left(\frac{2\pi}{T}t\right), \tag{4}$$

where:

- q_0 is the base (average) order,
- A is the amplitude of oscillation,
- T is the period of the cycle.

In our simulations, we adopt:

$$q(t) = 0.9 + 0.1\cos\left(\frac{2\pi}{50}t\right),$$

which models memory that strengthens and weakens periodically-mimicking the expansion and contraction phases of the economic cycle [7] [15].

4.2. Non-Periodic (Sigmoid) Order Function

To represent irreversible or long-term transitions in financial memory (such as policy reforms, economic crises, or technological shifts), we use a sigmoid-shaped function:

$$q(t) = q_0 + \frac{A}{1 + \mathrm{e}^{-r(t-t_0)}}, \tag{5}$$

where:

- $q_0 = 0.85$: base memory level,
- $A = 0.15$: maximum increase in memory,
- $r = 0.1$: rate of transition,
- $t_0 = 100$: midpoint of the transition.

This function models a smooth but permanent change in memory, which is appropriate for capturing structural transitions in financial behavior [8] [13].

These two forms of $q(t)$ offer a useful contrast: periodic models account for short-term market fluctuations, while sigmoid forms capture long-term regime changes.

5. Simulation and Results

To analyze the effect of time-dependent memory on financial system behavior, we simulate the proposed variable-order fractional system under three different forms of the order function $q(t)$: constant, periodic, and non-periodic. The governing equations are solved numerically using a two-step predictor-corrector method tailored for the Caputo-Fabrizio derivative [11] [12].

The following parameter values and initial conditions are used, based on the settings in [14]:

- Parameters: $a = 0.3$, $b = 0.1$, $c = 1$, $d = 0.1$, $g = 0.1$.
- Initial conditions: $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = 0.3$, $u(0) = 0.4$.
- Time interval: $t \in [0,100]$, time step $h = 0.01$.

5.1. Case 1: Constant Order $q = 0.9$

In this baseline case, the memory level remains fixed. The system exhibits regular and smooth oscillations, reflecting stable long-term dependencies in all state variables. This suggests a stationary financial regime without external disruptions (**Figure 1**).

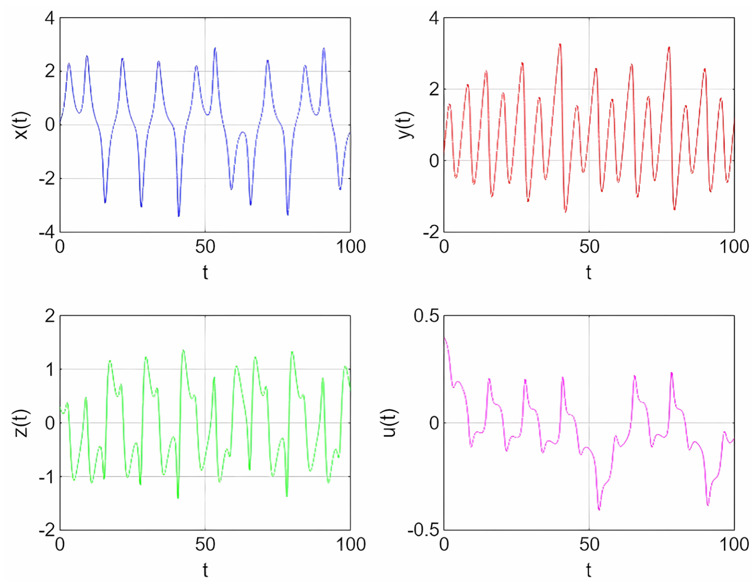

Figure 1. State trajectories under constant fractional order $q = 0.9$.

5.2. Case 2: Periodic Order $q(t) = 0.9 + 0.1\cos\left(\frac{2\pi}{50}t\right)$

This setting captures oscillating memory, simulating business cycles. As $q(t)$ increases and decreases periodically, we observe alternating behaviors of amplification and damping in the system (**Figure 2**). Such dynamics may reflect expansion and contraction phases in macroeconomic activity [7].

5.3. Case 3: Non-Periodic Order $q(t) = 0.85 + \frac{0.15}{1 + e^{-0.1(t-100)}}$

This sigmoid-shaped function models a structural shift in memory, such as a financial crisis or policy transition. The system starts with a lower memory effect and gradually moves toward a higher one (**Figure 3**). As shown in the trajectories, such memory adaptation alters both the amplitude and frequency of oscillations, reflecting long-term adjustment mechanisms [8] [13].

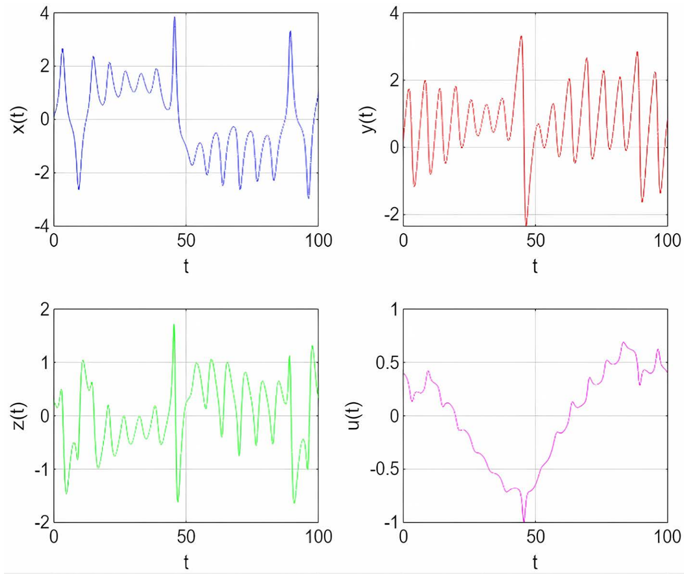

Figure 2. State trajectories under periodic $q(t)$.

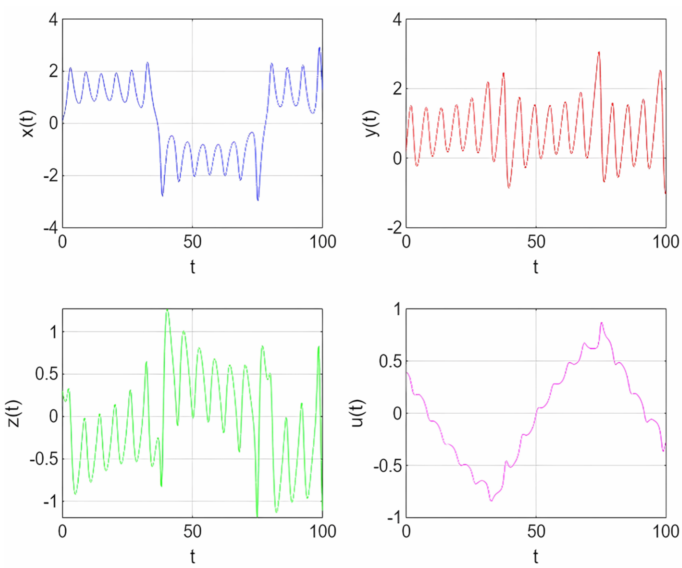

Figure 3. State trajectories under non-periodic $q(t)$.

5.4. Comparative Analysis

A comparison of the three cases yields the following observations:

- **Constant q**: Generates consistent oscillations; useful as a stable reference model but lacks adaptability.
- **Periodic $q(t)$**: Introduces alternating dynamical behavior, well-suited for modeling cycles in financial activity.
- **Non-periodic $q(t)$**: Reflects long-term memory adaptation, ideal for modeling structural transitions in financial systems.

These findings highlight the power of variable-order modeling in capturing a broader spectrum of financial dynamics compared to constant-order models.

6. Stability and Economic Interpretation

The stability of a fractional-order financial system depends not only on the system parameters but also on the nature of the memory function $q(t)$. Unlike integer-order systems, fractional systems exhibit memory-driven dynamics, where the influence of past states gradually fades or intensifies based on $q(t)$ [9].

Although a rigorous Lyapunov-based analysis for variable-order fractional systems remains mathematically challenging, we can gain qualitative insights by observing the numerical trajectories:

- In the **constant-order** case, the system remains oscillatory and bounded, indicating local asymptotic stability under fixed memory influence.
- In the **periodic-order** case, the cyclic memory shifts introduce alternating phases of amplification and damping, which reflect transitions between risk-taking and risk-averse economic behavior.
- In the **non-periodic (sigmoid)** case, the system exhibits gradual behavioral shifts, indicating long-term structural adaptation. This is analogous to a market adjusting to new regulatory frameworks or major policy changes.

From an economic standpoint:

- High memory (large $q(t)$) implies long-term influence of historical market behavior, often observed during financial uncertainty.
- Low memory (small $q(t)$) implies the system is more sensitive to recent shocks, modeling reactive markets or speculative bubbles.
- Transitions in $q(t)$ can be interpreted as responses to fiscal reforms, shifts in investor confidence, or global crises.

Thus, the variable-order framework not only provides richer mathematical dynamics but also aligns well with the heterogeneous and evolving nature of real financial systems [8] [13].

7. Estimating the Order Function from Real Financial Data

While the choice of $q(t)$ in this study is based on theoretical forms (constant, periodic, sigmoid), a critical step toward real-world application is estimating the order function $q(t)$ directly from empirical financial data.

In practice, the memory effect in financial markets-captured by $q(t)$-may evolve based on various observable variables such as:

- Volatility indices (e.g., VIX),
- Trading volume or liquidity,
- Macroeconomic indicators (e.g., inflation, GDP growth),
- Sentiment or uncertainty measures.

Several methods can be employed to estimate $q(t)$ from such data:

1) **Optimization-Based Estimation:** Calibrating the model by minimizing the difference between simulated trajectories and historical data [16].

2) **Machine Learning Approaches:** Using neural networks or regression models to learn $q(t)$ as a function of external variables [17].

3) **Time Series Inversion:** Fitting inverse models using observed output to re-

cover the best-fitting fractional order profile [18].

Incorporating data-driven $q(t)$ estimation transforms the fractional model into a data-adaptive forecasting tool, enhancing its predictive capability and relevance for economic policy design.

Future work may include constructing models where $q(t)$ is dynamically estimated online, allowing the system to adapt in real time to market fluctuations or shocks.

8. Challenges and Limitations

Despite the advantages of using variable-order fractional derivatives in modeling financial systems, several challenges remain. First, the mathematical complexity of variable-order operators-especially those with non-singular kernels-can hinder analytical tractability and complicate stability analysis.

Moreover, the choice of $q(t)$ in this study was predefined and idealized. In practical applications, determining the correct form of $q(t)$ from real-world financial data remains an open problem, particularly in the presence of noise, regime shifts, and non-stationary behavior.

9. Future Directions

Future research may focus on developing data-driven methods to estimate the fractional order function $q(t)$ dynamically using machine learning or optimization-based techniques. This would allow the system to adapt its memory structure in real time based on observed financial indicators.

Another promising direction is to integrate the variable-order framework into more complex financial systems such as multi-agent models, networked markets, or policy-regulated environments. These extensions would increase the model's realism and relevance for economic forecasting and decision-making.

10. Conclusions

This study presented a variable-order fractional financial model based on the Caputo-Fabrizio derivative, offering a flexible framework to capture time-varying memory effects in economic systems. Three types of memory dynamics-constant, periodic, and sigmoid-were investigated to demonstrate their impact on system behavior.

Simulation results confirmed that memory variability plays a crucial role in shaping financial dynamics. The proposed model offers a promising foundation for realistic and adaptive modeling of financial systems, especially in volatile or evolving market conditions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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