Advances in Pure Mathematics, 2025, 15(5), 339-349
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Results on Spectral Measures and an Application to a Spectral Theorem

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How to cite this paper: Bracken, P. (2025) Results on Spectral Measures and an Application to a Spectral Theorem. *Advances in Pure Mathematics*, **15**, 339-349.
https://doi.org/10.4236/apm.2025.155017

Received: May 3, 2025
Accepted: May 28, 2025
Published: May 31, 2025

Abstract

Properties of Hilbert spaces, projections and spectral measures are introduced and studied. Spectral integrals and the related operator are considered, and their relationship with the spectrum of an operator is given. The theorems that result yield a version of the spectral theorem which is suited to infinite dimensions.

Keywords

Projection, Adjoint, Hilbert Space, Spectral Measure, Operator, Spectrum, Idempotent

1. Introduction

One of the main aims of functional analysis is to study the spectra of various operators on Hilbert spaces [1]-[3]. Some basic mathematical properties of Hilbert spaces and spectral measures are developed and used to generalize a particular version of a spectral theorem in infinite dimensions. The study of the spectral properties of operators has always been of great interest since such results have applications in other areas of science [4] [5]. This subject has come up again in differential geometry, such as Ricci flow [6]-[10]. Moreover, there are numerous applications of these theorems and other ideas from functional analysis all through quantum mechanics as well.

Let $C(X)$ denote the set of complex-valued continuous functions on the space X and $C_0(X)$ the subset of functions of $C(X)$ which go to zero at infinity. Suppose X is a set with a specified Boolean σ-algebra S of subsets. A spectral measure is a function E whose domain is S and whose values are idempotent Hermitian operators usually called projections which satisfy $E(X)=1$ and $E\left(\bigcup_n M_n\right)=\sum_n E(M_n)$ whenever $\{M_n\}$ is a disjoint sequence of sets in S.

Such a set X and σ-algebra S of subsets is called a measurable space, (X,S). An example of a spectral measure is to take X a measure space with measure μ and Hilbert space $H = L^2(\mu)$ and write $E(M)f = \chi_M f$ where χ_M denotes the characteristic function of M with $M \in S$ and $f \in H$. If E is a spectral measure then $E(0) = 0$ and E is finitely additive.

It can be shown that a projection-valued function E on the class of measurable subsets of a measure space X is a spectral measure if and only if $E(X) = 1$, and moreover for each pair of vectors x and y, the complex-valued set function μ defined for any M in S by $\mu(M) = (E(M)x, y)$ is countably additive.

Some important theorems are required at the end in the proof of the spectral theorem and they are stated now so they may be used later. First, the *Stone-Weierstrass theorem* states that for a given compact set K, suppose $\mathcal{A}$ is an algebra of continuous functions $f : K \to \mathbb{C}$ closed under conjugation and separates points of K and there are no elements of K on which all functions in $\mathcal{A}$ vanish, then $\mathcal{A}$ is dense in $C(K)$. The *Riesz theorem* also appears: if X is a locally compact Hausdorff space and φ is a bounded linear functional on $C_0(X)$, there is a unique complex Borel measure μ such that $\varphi(f) = \int f \mathrm{d}\mu$ and $\|\varphi\| = |\mu(X)|$.

Let H be a Hilbert space such that for any v, w in H, the inner product is written (v, w). A star on the inner product or other complex object denotes complex conjugation whereas on an operator A it represents the adjoint A^*.

Lemma 1. If φ is a linear functional on H, then $\varphi(v) = (v, w)$ for a particular choice of w in H.

Proof. Let $\mathcal{K} = \mathrm{Ker}(\varphi)$ then it suffices to prove that $\mathcal{K}^\perp \neq \{0\}$, since if not, take $w = 0$. Then w is a nonzero vector in $\mathcal{K}^\perp$ and can be normalized by setting $v = w$ in φ, so $\varphi(w) = \|w\|^2$. Given any $v \in H$, set $v = v_1 + v_2$ where $v_1 = \left(\varphi(v)/\|w\|^2\right) w$ and so $v_2 = v - v_1$. Evaluating $\varphi(v)$, we get

$$\begin{aligned}\varphi(v_2) &= \varphi(v) - \varphi(v_1) = \varphi(v) - \varphi\left(\frac{\varphi(v)}{\|w\|^2} w\right) \\ &= \varphi(v) - \varphi(v)\frac{\varphi(w)}{\|w\|^2} = \varphi(v) - \varphi(v) = 0.\end{aligned}$$

This implies that $v \in \mathcal{K}$, and since $v \in \mathcal{K}^\perp$, it is the case that

$$(v, w) = (v_1 + v_2, w) = (v_1, w) + (v_2, w) = (v_1, w) = \frac{\varphi(v)}{\|w\|^2}(w, w) = \varphi(v).$$

□

Definition 1. A bounded linear functional is a map $\varphi : H \times H \to \mathbb{C}$ which is linear in the first term, conjugate linear in the second and such that there exists a non-negative constant denoted $\|\varphi\|$ which satisfies

$$|\varphi(v, w)| \leq \|\varphi\| \cdot \|v\| \cdot \|w\|. \tag{1}$$

Lemma 2. If φ is a bounded bilinear functional on H, then there exists a unique operator A on H such that $\varphi(v,w)=(Av,w)$ for all $v,w\in H$.

Proof. Let $v\in H$ be fixed and nonzero and let $\psi_v(w)=(\varphi(v,w))^*$. By Lemma 1, there exists a vector denoted Av such that $\psi_v(w)=(w,Av)=(Av,w)^*=\varphi(v,w)^*$. This implies that

$$\varphi(v,w)=(Av,w). \tag{2}$$

Since φ is a linear functional, A has to be a linear transformation. It suffices to verify that A is bounded. Set $w=Av$ in (2) to obtain

$$\|Av\|^2=|\varphi(v,Av)|\le\|\varphi\|\|v\|\|Av\|$$

or dividing out $\|Av\|\le\|\varphi\|\|v\|$. This implies that

$$\|A\|=\sup_{\|v\|=1}\|Av\|\le\|\varphi\|.$$

□

Lemma 2 is often referred to as the Riesz lemma.

2. Projections

Both the adjoint of an operator and projection operators as well play an important role here.

Theorem 1. Let A be an operator, then there exists a unique operator A^* called the adjoint of A which satisfies $(Av,w)=(v,A^*w)$ for all $v,w\in H$.

Proof. Set $\varphi(w,v)=(w,Av)$. Then φ is a bounded, bilinear functional Hence there exists a unique operator A^* such that $\varphi(w,v)=(A^*w,v)$. This must agree with the preceding expression which implies that $(Av,w)=(v,A^*w)$. □

Definition 2. Operator A is Hermitian if $A=A^*$, and it is normal if $\|Av\|=\|A^*v\|$.

Proposition 1. An operator A is normal if, and only if $AA^*=A^*A$.

Proof. Take an arbitrary vector $v\in H$, then by the definition of normal operator, $(Av,Av)=(A^*v,A^*v)$. This implies that $(A^*Av,v)=(AA^*v,v)$ from which we conclude that $AA^*=A^*A$. □

Riesz lemma 2 implies that, if φ is symmetric $\varphi(v,w)=\varphi(w,v)$, then the resulting operator will be Hermitean.

Definition 3. If M is a closed subspace of H, then Hilbert space theory states that every vector $v\in H$ has a unique decomposition $v=v_1+v_2$, where $v_1\in M$ and $v_2\in M^{\perp}$. The projection on M is defined to be the mapping $P:v\to v_1$. Note that P is necessarily an operator. If M is $\mathbb{C}$, denote P by 1 and if $M=\{0\}$, define P as 0. □

Definition 4. Let the set $\{P_i\}_{i\in I}$ be projections onto M_i respectively. These can be partially ordered by $P_i\le P_j$ if $M_i\subset M_j$. Further, the projection onto the space $\bigcup_{i\in I}M_i$ can be defined as $\sum_{i\in I}P_i$. □

Theorem 2. An operator P is a projection if and only if it is Hermitian and idempotent, so $P^2=P$.

Proof. Suppose M is a subset of H and projection P projects to the subspace M and for $w \in M$ we have $Pw \in M$. Thus $Pv \in M$ so $P^2 v = Pv \in M$ which gives $P^2 = P$. Let $v = v_1 + v_2$ and $w = w_1 + w_2$ in H such that $Pv = v_1$ and $Pw = w_1$. It follows that, since $w_2 \in M^{\perp}$ and $v_1 \in M$,

$$(Pv, w) = (w_1, w) = (v_1, w_1) + (v_1, w_2) = (v_1, w_1) = (v_1, w_1) + (v_2, w_1) = (v, Pw).$$

This implies P is Hermitian. □

Lemma 3. Suppose that P is Hermitian and idempotent. Define the space $M = \{w \in H : Pw = w\}$. Then P is a projection onto M.

Proof. To show this, it suffices to prove that, for all $v \in H$, $(Pv, v - Pv) = 0$. Expanding the left-hand side then, since P is assumed Hermitian, we have

$$(Pv, v - Pv) = (Pv, v) - (Pv, Pv) = (Pv, v) - (P^2 v, v) = (Pv, v) - (Pv, v) = 0.$$

□

Corollary 1. If P is a projection, then for all $v \in H$, $\|Pv\|^2 = (Pv, v)$.

Proof.

$$\|Pv\|^2 = (Pv, Pv) = (P^2 v, v) = (Pv, v).$$

□

3. Spectral Measures

The final theorem is concerned with spectral measures, and so these are studied now. Let $B(\mathbb{C})$ be the set of Borel sets in $\mathbb{C}$ and $\mathcal{P}(H)$ the set of projections on the Hilbert space H.

Definition 5. A complex spectral measure is a function $E : B(\mathbb{C}) \to P(H)$ which has the following properties: 1) $E(\varnothing) = 0$ and $E(\mathbb{C}) = 1$, 2) If $\{B_n\}$ is a family of disjoint Borel sets, then $E\left(\bigcup_n B_n\right) = \sum_n E(B_n)$. □

It is the case that if $B_0 \subset B_1$, then $E(B_0) \le E(B_1)$. Spectral measures also have the property that $E(B_0 \cup B_1) + E(B_0 \cup B_1) = E(B_0) + E(B_1)$. Act on both sides of this with $E(B_0)$ and use the corrolary and observation

$E(B_0) E(B_1 \cup B_1) = E(B_0)$ to arrive at

$E(B_0) E(B_0 \cup B_1) + E(B_0) E(B_0 \cap B_1) = E(B_0) + E(B_0) E(B_1)$ or

$E(B_0) + E(B_0 \cap B_1) = E(B_0) + E(B_0) E(B_1)$, which simplifies to

$E(B_0 \cap B_1) = E(B_0) E(B_1)$.

Proposition 2. Suppose $\mathbf{E} : B(X) \to \mathcal{P}$ is any operator such that for all $v, w \in H$, the related function satisfies $E(B) = (E(B) v, w)$,

$E\left(\bigcup_n B_n\right) = \sum_n E(B_n)$ and $E(X) = 1$. Then E is a spectral measure.

Proof. Suppose $\{B_n\}$ is a disjoint family of Borel sets in $\mathbb{C}$, then

$$\begin{aligned}\sum_n \|E(B_n) v\|^2 &= \sum_n (E(B_n) v, E(B_n) v) = \sum_n \left(E(B_n)^2 v, v\right) \\ &= \left(\sum_n E(B_n) v, v\right) = \left(E\left(\bigcup_n B_n\right) v, v\right) = \left\|E\left(\bigcup_n B_n\right) v\right\|^2.\end{aligned}$$

The sequence $v_n = E(B_n) v$ is therefore summable. For all disjoint Borel sets B, C and vectors $v, w \in H$,

$$\left(E(B)v+E(C)v,w\right)=\left(E(B)v,w\right)+\left(E(C)v,w\right)=\left(E(B\cup C)v,w\right).$$

The statement follows simply by examining the partial sums of $\sum_n E(B_n)$. □

Definition 6. (X,S) represents an arbitrary fixed measurable space and $\mathcal{B}$ the class of all complex-valued bounded measurable functions on X. $M(f)=\sup\{|f(\lambda)|:\lambda\in X\}$ when $f\in\mathcal{B}$. The expression spectral measure always refers to a spectral measure in X.

Definition 7. Let E be a spectral measure then the spectral measure with respect to vectors $v,w\in H$ of a measurable function f is defined to be the Lebesgue-Stieltjes integral given by

$$\int f(\lambda)\mathrm{d}\left(E(\lambda)v,w\right). \tag{3}$$

The spectrum of a spectral measure E is denoted by $\Lambda(E)=X\setminus\bigcup_i U_i$. The union is taken over all open sets U_i which satisfy the condition $E(U_i)=0$. If the set $\Lambda(E)$ is compact, we say E is compact. Since $E(0)=0$, when λ is changing, $E(M\cap\lambda)$ means λ must overlap M to yield a nonzero contribution. □

Theorem 3. Let f and g be complex valued, bounded, measurable functions on X and E a spectral measure, then

$$\left(\int f\mathrm{d}E\right)\cdot\left(\int g\mathrm{d}E\right)=\int fg\mathrm{d}E. \tag{4}$$

Proof. Suppose we denote the integrals as

$$P=\int f\mathrm{d}E(\lambda),\ \ Q=\int g\mathrm{d}E(\lambda).$$

If the complex measure μ in X is defined for every set M in S by writing $\mu(M)=\left(E(M)Qx,y\right)$ with x,y fixed vectors in H, for every set M in S we have

$$\begin{aligned}\mu(M)&=\left(Qx,E(M)y\right)\\&=\int g(\lambda)\mathrm{d}\left(E(\lambda)x,E(M)y\right)\\&=\int g(\lambda)\mathrm{d}\left(E(M)E(\lambda)x,y\right)\\&=\int g(\lambda)\mathrm{d}\left(E(M\cap\lambda)x,y\right)\\&=\int_M g(\lambda)\mathrm{d}\left(E(\lambda)x,y\right).\end{aligned}$$

It follows that,

$$\begin{aligned}(P\cdot Qx,y)&=\left(Qx,P^*y\right)=\left(P^*y,Qx\right)^*=\left(\int f(\lambda)^*\mathrm{d}E(\lambda)y,Qx\right)^*\\&=\left(\int f(\lambda)^*\mathrm{d}\left(y,E(\lambda)Qx\right)\right)^*=\int f(\lambda)\mathrm{d}\left(E(\lambda)Qx,y\right)\\&=\int f(\lambda)\mathrm{d}\mu(\lambda)=\int f(\lambda)g(\lambda)\mathrm{d}\left(E(\lambda)x,y\right).\end{aligned}$$

The conclusion is that $P\cdot Q$ has to be equal to the integral $\int fg\mathrm{d}E(\lambda)$. □

The next theorem makes an important connection between the spectral integral and a unique operator A which is normal.

Theorem 4. Suppose E is a compact spectral measure, then there is a unique normal operator A such that for all vectors $v,w\in H$

$$\int \lambda \mathrm{d}\left(E(\lambda) v, w\right) = (Av, w). \tag{5}$$

This is often expressed by writing $A = \int \lambda \mathrm{d}E(\lambda)$.

Proof. Let $\varphi(v,w) = \int \lambda \mathrm{d}\left(E(\lambda)v, w\right)$ which is finite for all (v,w) as the set $\Lambda(E)$ is compact. Moreover, φ is a bilinear functional and $\varphi(v,w)$ is bounded as shown by working out

$$|\varphi(v,w)| \le \int |\lambda| \mathrm{d}\left(\|E(\lambda)v\|^2\right) \le \sup\left(|\lambda| : \lambda \in \Lambda(E)\right) \cdot \|v\|^2 = \beta \cdot \|v\|^2.$$

where $\beta = \sup\{|\lambda| : \lambda \in \Lambda(E)\}$. Applying the parallelogram law, the following bound is obtained

$$|\varphi(v,w)| \le \frac{1}{2}\beta\left(\|v+w\|^2 + \|v-w\|^2 + \|v+iw\|^2 + \|v-iw\|^2\right) \le \beta\left(\|v\|^2 + \|w\|^2\right).$$

Thus $\|\varphi\|$ can be computed by evaluating the supremum of this under the constraint $\|v\| = \|w\| = 1$. This implies that $\|\varphi\| \le 2 \cdot \beta$, so φ is bounded. By the Riesz lemma, a unique operator A must exist.

It remains to show that A is a normal operator. Construct an operator called $\tilde{A}$ along similar lines by means of the integral

$$\tilde{A} = \int \lambda^* \mathrm{d}E(\lambda). \tag{6}$$

It follows that

$$\begin{aligned}(v, \tilde{A}w) &= (\tilde{A}w, v)^* = \left(\int \lambda^* \mathrm{d}\left(E(\lambda)w, v\right)\right)^* = \int \lambda d\left(v, E(\lambda)w\right) \\ &= \int \lambda \mathrm{d}\left(E(\lambda)v, w\right) = (Av, w).\end{aligned}$$

Since the adjoint is unique, it must be that $\tilde{A} = A^*$.

Suppose a Borel set is taken, then we can calculate, for any elements $v, w \in H$,

$$\begin{aligned}\left(A^* v, E(B)w\right) &= \int \lambda^* \mathrm{d}\left(E(\lambda)v, E(B)w\right) = \int \lambda^* \mathrm{d}\left(E(B)E(\lambda)v, w\right) \\ &= \int \lambda^* \mathrm{d}\left(E(B \cap \lambda)v, w\right) = \int_B \lambda^* \mathrm{d}\left(E(\lambda)v, w\right).\end{aligned}$$

This implies that

$$\begin{aligned}(AA^* v, w) &= (A^* v, A^* w) = (A^* w, A^* v)^* \\ &= \left(\int \lambda^* \mathrm{d}\left(E(\lambda)w, A^* v\right)\right)^* = \int \lambda \mathrm{d}\left(A^* v, E(\lambda)w\right) \\ &= \int \lambda \mathrm{d}\left(E(\lambda)A^* v, w\right) = \int \lambda\lambda^* \mathrm{d}\left(E(\lambda)v, w\right).\end{aligned} \tag{7}$$

In a similar manner, we calculate

$$\begin{aligned}(A^* Av, w) &= (Av, Aw) = (Aw, Av)^* \\ &= \left(\int \lambda \mathrm{d}\left(E(\lambda)w, Av\right)\right)^* = \int \lambda^* \mathrm{d}\left(Av, E(\lambda)w\right) \\ &= \int \lambda^* \lambda \mathrm{d}\left(E(\lambda)^2 v, w\right) = \int \lambda^* \lambda \mathrm{d}\left(E(\lambda)v, w\right).\end{aligned} \tag{8}$$

Comparing the two results (7) and (8), it follows that $(AA^* v, w) = (A^* Av, w)$. Since v, w are arbitrary, this implies that $AA^* = A^* A$, so A is a normal operator. □

Theorem 4 is related to a theorem which is just stated and not proved here: If E is a spectral measure and if $f \in B$, then there exists a unique operator A such that $(Ax, y) = \int f(y) \mathrm{d}(E(\lambda)x, y)$ for every pair of vectors x and y in the space.

4. Operators and Spectra

Let us define what is meant by the spectrum of an operator. To get to the form of the theorem we want to prove, it is necessary to generalize the concept of eigenvalue. This often comes up very often in the study of both finite dimensional vector spaces as well as infinite.

Definition 8. The spectrum of an operator A is defined to be the set of numbers $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not invertible. □

In order to use the definition, a characterization of invertible and non-invertible operators is required, and leads into the following theorem.

Theorem 5. An operator A on H is invertible if the image of A is dense in H and as well there exists an $\alpha > 0$ such that for all $v \in H$, A is bounded from below

$$\|Av\| > \alpha \|v\|. \tag{9}$$

Proof. First suppose that A is invertible. The image of A is all of H, which is dense in H. Let us define $\alpha = \left\|A^{-1}\right\|^{-1}$, and so for all $v \in H$,

$$\|v\| = \left\|A^{-1}Av\right\| \leq \left\|A^{-1}\right\| \|Av\|.$$

Dividing this on both sides by $\left\|A^{-1}\right\|$ gives $\|Av\| \geq \alpha \|v\|$.

Suppose the range of A is dense in H and there exists a real number $\alpha > 0$ such that $\|Av\| \geq \alpha \|v\|$. It has to be shown that the range of A is H. It suffices to show that it is closed. Suppose that the set $\{v_n\}$ is a Cauchy sequence in the range of A. For all n, choose $\{w_n\}$ such that $Aw_n = v_n$, then since A is linear,

$$\|v_n - w_m\| = \|Aw_n - Aw_m\| = \|A(w_n - w_m)\| \geq \alpha \|w_n - w_m\|,$$

by hypothesis, dividing by α,

$$\|w_n - w_m\| \leq \frac{1}{\alpha} \|v_n - v_m\|.$$

This result implies that $\{w_n\}$ is a Cauchy sequence whenever $\{v_n\}$ is. Hence, $\{w_n\}$ must converge to a $w \in H$. By continuity, v_n has to approach Aw, which implies the range is closed.

To prove that A is injective, notice that if $Av_1 = Av_2$, then $0 = \|Av_1 - Av_2\| \geq \alpha \|v_1 - v_2\|$. From this it follows that $v_1 = v_2$, and so A is bijective. The inverse operator A^{-1} is also linear, so it suffices to show that A^{-1} is a bounded operator. This is just a consequence of the fact that

$$\|w\| = \|Av\| \geq \alpha \|v\| = \alpha \left\|A^{-1}v\right\|,$$

which implies that

$$\left\|A^{-1}w\right\| \le \alpha^{-1}\|w\|.$$

□

Proposition 3. If A is any operator which satisfies the condition $\|A-I\|<1$, then A is invertible.

Proof. Set $0<\alpha=1-\|A-I\|<1$. Then for any $v \in H$, we find

$$\begin{aligned}\|Av\| &= \|v-(v-Av)\| = \|v-(I-A)v\| \\ &\ge \|v\| - \|Av-v\| \ge (1-\|I-A\|)\|v\| = \alpha\|v\|.\end{aligned}$$

This implies that A is bounded from below.

Let $\mathcal{R}_A$ be the range of operator A in H. Define the constant $\epsilon = \inf_{\substack{v\in H \\ w\in\mathcal{R}_A}} \|v-w\|$. It suffices to prove that $\epsilon=0$. Suppose this is not the case. Since $0<1-\alpha<1$, there exists a $v \in H$ and $w \in \mathcal{R}_A$ such that

$$\|v-w\| < \frac{\epsilon}{1-\alpha}. \tag{10}$$

It follows then that since we have taken $\alpha = 1-\|A-I\|$, using (10), we estimate that

$$\epsilon \le \|v-w-A(v-w)\| \le \|A-I\|\cdot\|v-w\| = (1-\alpha)\cdot\|v-w\| < \epsilon. \tag{11}$$

Since ϵ is assumed to be nonzero, a contradiction has been obtained. □

Theorem 6. If A is an operator then $\Lambda(A)$ is compact and if $\lambda \in \Lambda(A)$, then $|\lambda| \le \|A\|$.

Proof. It is proved that $\Lambda(A)$ is closed so the first statement follows from the second. Suppose λ_0 is not an element of $\Lambda(A)$. Let $\delta>0$ be such that $\delta < \|A-\lambda_0 I\|$ and that $|\lambda-\lambda_0|<\delta$. Then we have

$$\begin{aligned}\left\|I-(A-\lambda_0 I)^{-1}(A-\lambda I)\right\| &= \left\|(A-\lambda_0 I)^{-1}((A-\lambda_0)-(A-\lambda I))\right\| \\ &\le \|A-\lambda_0 I\|^{-1}\cdot|\lambda-\lambda_0| < 1.\end{aligned}$$

By Proposition 3, the operator $A-\lambda I$ is invertible in a ball of radius δ about λ_0. It may be concluded that $\mathbb{C}\setminus\Lambda(A)$ is open.

Now suppose that $|\lambda|>\|A\|$ so we have $\|A/\lambda\|<1$ which, after adding and subtracting I is equivalent to $\|I-(I-A/\lambda)\|<1$. This implies that $I-A/\lambda$ is invertible. Multiply this operator by the scalar λ and it follows that $\lambda I - A$ is invertible and λ is not an element of $\Lambda(A)$. □

Theorem 7. If E is a compact spectral measure and $A=\int \lambda \mathrm{d}E(\lambda)$, then $\Lambda(E)=\Lambda(A)$.

Proof. Suppose that $\lambda_0 \in \mathbb{C}\setminus\Lambda(E)$, so $\Lambda(E)$ is open by definition. Hence there exists $\delta>0$ such that $B=B(\lambda_0,\delta)\subset\mathbb{C}\setminus\Lambda(E)$ with $E(B)=0$. Since E is a spectral measure, $E(\mathbb{C})=1$ and so for any $v\in H$,

$$\begin{aligned}\|Av-\lambda_0 v\|^2 &= \int_{\mathbb{C}} |\lambda-\lambda_0|^2 \,\mathrm{d}(E(\lambda)v,v) = \int_{\mathbb{C}/B} |\lambda_0-\lambda|^2 \,\mathrm{d}(E(\lambda)v,v) \\ &\ge \int_{\mathbb{C}/B} \delta^2 \mathrm{d}(E(\lambda)v,v) = \delta^2\|v\|^2.\end{aligned}$$

Consequently, the operator $A-\lambda_0 I$ is bounded from below.

It remains to prove that the image of the operator $A_0 = A - \lambda_0 I$ is dense in H. Suppose that A_0 is any normal operator which is bounded from below and has range R_{A_0}. It suffices to show that $R_{A_0}^{\perp} = 0$. Suppose $R_{A_0}^{\perp}$ is nonempty set, so there is a $w \in H$ such that $w \in_{A_0}^{\perp}$. Then for all $v \in H$, it follows that $(A_0 v, w) = 0$ and hence

$$0 = (A_0 v, w) = (v, A_0^* w).$$

This implies that $A_0^* w = 0$. Moreover, since operator A_0 is bounded below, there exists $\alpha > 0$ such that

$$0 = \|A_0^* w\| = \|A_0 w\| \geq \alpha \|w\|.$$

This means $w = 0$, hence the operator $A - \lambda_0 I$ is invertible so $\lambda_0 \notin \Lambda(A)$ and $\Lambda(A) \subset \Lambda(E)$.

Suppose now that $\lambda_0 \in \Lambda(E)$. Choose a $\delta > 0$ such that $E(B(\lambda_0, \delta)) \neq 0$. For any $v \in E(B)$,

$$\|Av - \lambda_0 v\|^2 = \int_B |\lambda - \lambda_0|^2 \,\mathrm{d}(E(\lambda)v, v) \leq \delta^2 \cdot \|v\|^2.$$

However, δ is arbitrary, so $A - \lambda_0 I$ is not bounded from below. Hence it cannot be invertible, which means $\lambda \in \Lambda(A)$. It follows that $\Lambda(E) \subset \Lambda(A)$. Combining these two results, $\Lambda(A) \subset \Lambda(E)$ and $\Lambda(E) \subset \Lambda(A)$, it is concluded that $\Lambda(E) = \Lambda(A)$. □

5. Spectral Theorem

Let $C(X)$ be the set of complex-valued continuous functions on X and $C_0(X)$ the subset of $C(X)$ of functions which approach zero at infinity. A spectral theorem is now developed from what has been proved so far. In the process, the Stone-Weierstrass and Riesz theorems are used.

Theorem 8. Let A be a Hermitean operator so $A = A^*$. Then there exists a spectral measure E such that for all $v, w \in H$.

$$(Av, w) = \int_{\Lambda(A)} \lambda \,\mathrm{d}(E(\lambda)v, w). \tag{12}$$

Proof. Let v and w be two fixed vectors in H and let p be a given polynomial which is used to define a functional ϑ,

$$\vartheta(p) = (p(A)v, w). \tag{13}$$

This functional is bounded above,

$$|\vartheta(p)| = |(p(A)v, w)| \leq \xi \cdot \|v\| \|w\|,$$

where $\xi = \sup_{\lambda \in \Lambda(A)} |p(\lambda)|$. Since $\Lambda(A)$ is compact, the Stone-Weierstrass theorem can be used to show these polynomials are dense in $C(\Lambda(A))$. Therefore, ϑ defines a bounded linear functional on all of $C(\Lambda(A)) = C_0(\Lambda(A))$. Hence there exists a unique complex measure $\mu_{v,w}$ such that, by the Riesz theorem,

$$(p(A)v, w) = \int p(\lambda) \,\mathrm{d}\mu_{(v,w)}(\lambda). \tag{14}$$

Take a Borel set B and use it to define $\mu_B(v, w) = \mu_{(v,w)}(B)$, so μ_B is a bounded symmetric bilinear form. By the Riesz lemma, there exists a unique Hermitean operator $E(B)$ such that

$$\mu_B(v,w) = (E(B)v,w). \tag{15}$$

So from (14), we have

$$(p(A)v,w) = \int p(\lambda)\,\mathrm{d}\mu_B(v,w) = \int p(\lambda)\,\mathrm{d}(E(B)v,w). \tag{16}$$

Now let $B = \Lambda(A)$ and set $p(\lambda) = 1$ in (16),

$$(v,w) = \int \mathrm{d}(E(\Lambda(A))v,w).$$

This implies that $E(\Lambda(A)) = 1$. Next set $p(\lambda) = \lambda$ and substitute it into (16) to obtain the important result valid for all $v,w \in H$

$$\int \lambda\,\mathrm{d}(E(B)v,w) = (Av,w). \tag{17}$$

This can be expressed in an equivalent way,

$$\int \lambda\,\mathrm{d}(E(\lambda)v,w) = (Av,w). \tag{18}$$

It is sufficient to check that E is projection valued so it can be said to be a spectral measure. It is known that E is Hermitian, so it suffices to verify that it is an idempotent operator. To this end, let p and q be arbitrary polynomials and define a measure ν as

$$\nu(B) = \int_B p(\lambda)\,\mathrm{d}(E(\lambda)v,w). \tag{19}$$

Then based on the construction of E it follows that

$$\begin{aligned}\int q(\lambda)\,\mathrm{d}\nu(\lambda) &= \int q(\lambda)p(\lambda)\,\mathrm{d}(E(\lambda)v,w) = (q(A)p(A)v,w)\\ &= (p(A)q(A)v,w) = (q(A)v,p(A)w)\\ &= \int q(\lambda)\,\mathrm{d}(E(\lambda)v,p(A)w).\end{aligned}$$

Since q was chosen arbitrarily, the Stone-Weierstrass theorem and combined with the fact that compactly supported continuous functions are dense in L^1, replacing q by the characteristic function χ_B, leads to the conclusion that for all Borel sets B,

$$\int \chi_B(\lambda)p(\lambda)\,\mathrm{d}(E(\lambda)v,w) = \nu(B) = (E(B)v,p(A)w).$$

The Stone-Weierstrass theorem can be employed once again, but with respect to p. So using the fact E is idempotent, it follows that

$$\begin{aligned}(E(B)v,w) &= \int \chi_B(\lambda)\,\mathrm{d}(E(\lambda)v,w) = \int \chi_B(\lambda)^2\,\mathrm{d}(E(\lambda)v,w)\\ &= (E(B)v,E(B)w) = (E(B)v,w).\end{aligned}$$

Since v,w are arbitrary elements of H, result (12) of the theorem follows. □

6. Conclusions

The spectral theorem has become a major part of functional analysis and not without reason, as it has many applications in science such as in quantum mechanics.

It comes in different versions depending on such things as compactness of the operator and dimension of the space. Some introductory theorems from functional analysis have been proved related to functional, projectors, adjoints and spectra. Some new proofs have been given concerning spectral measure and spectra which results in the second last section in a proof of a particular form of a spectral theorem.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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