Journal of Modern Physics, 2025, 16(5), 719-753
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Operational Calculus with Hermite Polynomials and a Quite General Gaussian Integral

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How to cite this paper: Wünsche, A. (2025) Operational Calculus with Hermite Polynomials and a Quite General Gaussian Integral. *Journal of Modern Physics*, **16**, 719-753.
https://doi.org/10.4236/jmp.2025.165040

Received: April 18, 2025
Accepted: May 27, 2025
Published: May 30, 2025

Abstract

Starting from two mutually alternative definitions of Hermite polynomials, we derive new relations between these polynomials for arbitrarily stretched arguments. Furthermore, some operational identities with Hermite polynomials are derived and proved by complete induction. The main result is a quite general integral with an arbitrary Gaussian distribution and the product of two Hermite polynomials with general linear arguments which generalize almost all known special integrals of such kind contained in the most comprehensive tables of integrals. Furthermore, some multi-dimensional integrals with Gaussian distributions are derived. Special representations are developed for two-dimensional integrals in complex and complex-conjugate variables which are important for the treatment of the phase space of harmonic oscillators.

Keywords

Legendre Polynomials, Jacobi Polynomials, Ultraspherical Polynomials, Laguerre 2D Polynomials, Operational Identities, Normal Ordering

1. Introduction

Integrals with Gaussian distributions (bell functions) together with Hermite and Laguerre polynomials play a great role in different branches of physics, in particular, in quantum optics, and also multi-dimensional integrals of such kind are important. The most comprehensive collections of integrals to our knowledge are that of Ryzhik and Gradshteyn [1] (1st ed. Ryzhik, 3rd ed. Gradshteyn, 4th ed. revised with participation of Geronimus and Tseitlin) and the 3-volume work of Prudnikov, Brychkov and Marichev from which the 2nd volume about Special functions [2] is of most interest for us in connection with the present article. To an essential part, they rest on the 2-volume set of Integral transforms of Bateman and Erdély (with participation of Magnus, Oberhettinger and Tricomi) from which we cite the 2nd volume [3] because it contains a chapter about Orthogonal polynomials (Chapter XVI) with integrals of the mentioned kind. Most of the integrals there with Hermite polynomials $\mathrm{H}_n(x)$ use them in the elder form $\mathrm{He}_n(x)$. Formulae for Hermite polynomials are to find in many works about Special functions from which we cite Bateman and Erdélyi [4], Szegö [5], Rainville [6] and Bell [7].

The basic aim was to calculate a quite general definite integral with a Gaussian distribution (bell function) and the product over two Hermite polynomials with an arbitrarily stretched and displaced argument. Most of the integrals with Gaussian distributions and products up to two Hermite polynomials in cited works but not all can be derived as special cases from this quite general formula which possesses an interesting structure and which we prove in the present article. Integrals with products of more than two Hermite polynomials are up to now, of lower interest and are seldom needed in applications since a main branch of using them is quantum optics of modes of a harmonic oscillator for which the phase space is a two-dimensional one. As preparation, we derive in Section 2 some formulae for Hermite polynomials, in particular, a relation between two Hermite polynomials with arbitrarily stretched arguments which we did not find in cited works. Similar formulae in comparably simple form should be possible, else only for Laguerre polynomials for which the Laguerre 2D polynomials are the most appropriate form [8]-[10]. For the Hermite polynomials themselves besides the well-known Rodrigues-type definition exists an up-to-now little known alternative definition [7] which sometimes is of advantage and which we discuss in Section 2 together with some other little known relations. Hermite polynomials also play an important role in operator ordering in quantum theory and this leads to so-called operational identities from which we discuss a few basic ones in Section 3. The Fourier transformation of Gaussian distributions (Section 4) leads to a favorable starting point for the evaluation of integrals with Hermite polynomials (Sections 4 and 5). Their main categories of special and limiting cases are considered in Section 6. In Section 7, we discuss the derivation of an alternative formula for the quite general integral of Section 5 in the form of a double sum. In Section 8, we discuss a multi-dimensional generalization of Gaussian integrals. In Section 9, we discuss a few such two-dimensional integrals. The peculiarity of this case is that we can use then a description by a pair of complex-conjugate variables, which is applicable to the two-dimensional phase space of a harmonic oscillator. To the **Appendices**, we sent some proofs of formulae with long calculations, in these cases, with complete inductions.

2. Alternative Definition of Hermite Polynomials and Some Basic Identities

Hermite polynomials $\mathrm{H}_n(x), (n = 0, 1, 2, \cdots)$ are usually introduced by the Rodrigues-type definition (e.g., [4])

$$
\begin{aligned}
\mathrm{H}_n(x) &= \exp\left(x^2\right)\left(-\frac{\partial}{\partial x}\right)^n \exp\left(-x^2\right) \\
&= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}, \quad \left(\left[\frac{n}{2}\right] \equiv \mathrm{Floor}\left[\frac{n}{2}\right]\right),
\end{aligned}
\tag{2.1}
$$

or as coefficients of their generating function $\exp\left(2tx - t^2\right)$ as follows

$$
\exp\left(2tx - t^2\right) = \sum_{n=0}^{\infty} \mathrm{H}_n(x)\frac{t^n}{n!}. \tag{2.2}
$$

In the early nineties, I found independently from others the following alternative definition of Hermite polynomials

$$
\begin{aligned}
\mathrm{H}_n(x) &\equiv \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!2^{2k}}\frac{\partial^{2k}}{\partial x^{2k}}(2x)^n \\
&= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k},
\end{aligned}
\tag{2.3}
$$

and extended it in published form to the whole complex of Hermite 2D and Laguerre 2D polynomials (see Section 9). Later it was seen that an equivalent relation was already derived from a generating function for Hermite polynomials by W. W. Bell [7] (pp. 159, 160)[1] and after acquaintance with Hong-yi Fan and now a friend of mine from China and when we became coauthors of a few common papers I realized that he used it also in some of his papers. Sometimes, this alternative definition (2.3) possesses advantages for derivations in comparison with (2.1).

From the explicit representations of the Hermite polynomials in (2.1) one derives in a simple way the differentiation formula

$$
\frac{\partial}{\partial x}\mathrm{H}_n(x) = 2n\mathrm{H}_{n-1}(x) \;\Rightarrow\; \frac{\partial^j}{\partial x^j}\mathrm{H}_n(x) = \frac{2^j n!}{(n-j)!}\mathrm{H}_{n-j}(x), \tag{2.4}
$$

and the recurrence relation

$$
\mathrm{H}_{n+1}(x) = 2x\mathrm{H}_n(x) - 2n\mathrm{H}_{n-1}(x) = \left(2x - \frac{\partial}{\partial x}\right)\mathrm{H}_n(x). \tag{2.5}
$$

The operators $\frac{\partial}{\partial x}$ and $2x - \frac{\partial}{\partial x}$ are the lowering and raising operators for the indices of Hermite polynomials, respectively.

The inversion of the alternative definition of Hermite polynomials (2.3) provides immediately

$$
\begin{aligned}
(2x)^n &= \exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\mathrm{H}_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!2^{2k}}\frac{\partial^{2k}}{\partial x^{2k}}\mathrm{H}_n(x) \\
&= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!(n-2k)!}\mathrm{H}_{n-2k}(x).
\end{aligned}
\tag{2.6}
$$

For the Hermite polynomials with the sum of two variables in the argument

[1]After the turn in GDR in 1990 we could privately order and buy books from Western countries.

$\mathrm{H}_n(x+y)$ one finds

$$\mathrm{H}_n(x+y)=\sum_{j=0}^{\infty}\frac{y^j}{j!}\frac{\partial^j}{\partial x^j}\mathrm{H}_n(x)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2y)^j\,\mathrm{H}_{n-j}(x). \tag{2.7}$$

From this formula, one derives for the Hermite polynomials with a stretching factor λ of the variable x in the argument $\mathrm{H}_n(\lambda x)$

$$\begin{aligned}\mathrm{H}_n(\lambda x)&\equiv\mathrm{H}_n\left(\left(1+\frac{y}{x}\right)x\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2y)^j\,\mathrm{H}_{n-j}(x)\\&=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2(\lambda-1)x)^j\,\mathrm{H}_{n-j}(x),\left(\lambda\equiv1+\frac{y}{x},y=(\lambda-1)x\right).\end{aligned} \tag{2.8}$$

Interestingly, one can also derive an essentially different formula for $\mathrm{H}_n(\lambda x)$ using the inversion of the alternative definition (2.6) of the Hermite polynomials in addition to the explicit definition as follows

$$\begin{aligned}\mathrm{H}_n(\lambda x)&=\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j n!}{j!(n-2j)!}\lambda^{n-2j}(2x)^{n-2j}\\&=\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j n!}{j!(n-2j)!}\lambda^{n-2j}\sum_{l=0}^{\left[\frac{n-2j}{2}\right]}\frac{(n-2j)!}{l!(n-2j-2l)!}\mathrm{H}_{n-2j-2l}(x)\quad|\,j-l\equiv k\\&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!(n-2k)!}\mathrm{H}_{n-2k}(x)\sum_{j=0}^{n}\frac{(-1)^j k!}{j!(k-j)!}\lambda^{n-2j},\end{aligned} \tag{2.9}$$

and using the binomial formula with the result

$$\begin{aligned}\mathrm{H}_n(\lambda x)&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!(n-2k)!}\left(\lambda^2-1\right)^k\lambda^{n-2k}\mathrm{H}_{n-2k}(x)\\&=\lambda^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!(n-2k)!}\left(1-\frac{1}{\lambda^2}\right)^k\mathrm{H}_{n-2k}(x).\end{aligned} \tag{2.10}$$

A generalization of this relation is easily obtained by substitutions, first $x\to\kappa x$ and then $\kappa\lambda\to\lambda$ and is[2]

$$\begin{aligned}\mathrm{H}_n(\lambda x)&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!(n-2k)!}\left(\frac{\lambda^2}{\kappa^2}-1\right)^k\left(\frac{\lambda}{\kappa}\right)^{n-2k}\mathrm{H}_{n-2k}(\kappa x)\\&=\left(\frac{\lambda}{\kappa}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!(n-2k)!}\left(1-\frac{\kappa^2}{\lambda^2}\right)^k\mathrm{H}_{n-2k}(\kappa x).\end{aligned} \tag{2.11}$$

In particular, for $\kappa=1$ and $\lambda=\frac{1}{\sqrt{2}}$

$$\mathrm{H}_n\left(\frac{x}{\sqrt{2}}\right)=\frac{1}{\left(\sqrt{2}\right)^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!(n-2k)!}\mathrm{H}_{n-2k}(x)\equiv\left(\sqrt{2}\right)^n\mathrm{He}_n(x), \tag{2.12}$$

[2]Analogous relations are likely possible for Laguerre polynomials, in the simplest form, for Laguerre 2D polynomials.

and for $\kappa = 1$ and $\lambda = \sqrt{2}$

$$\mathrm{H}_n\left(\sqrt{2}x\right) = \left(\sqrt{2}\right)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!(n-2k)!2^k} \mathrm{H}_{n-2k}(x) \equiv \left(\sqrt{2}\right)^n \mathrm{He}_n(2x), \tag{2.13}$$

where the connection to the sometimes defined "modified Hermite polynomials" $\mathrm{He}_n(x)$ is added. Thus, the modified Hermite polynomials can be alternatively defined by

$$\begin{aligned} \mathrm{He}_n(x) &= (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{\partial^n}{\partial x^n} \exp\left(-\frac{x^2}{2}\right) = \exp\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) x^n \\ &= \frac{1}{\left(\sqrt{2}\right)^n} \mathrm{H}_n\left(\frac{x}{\sqrt{2}}\right) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!} x^{n-2k}, \end{aligned} \tag{2.14}$$

with the orthonormality relations

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{x^2}{2}\right) \mathrm{He}_m(x) \mathrm{He}_n(x) = n!\delta_{m,n}. \tag{2.15}$$

The differentiation of the "modified Hermite polynomials" $\mathrm{He}_n(x)$ leads to

$$\frac{\partial}{\partial x} \mathrm{He}_n(x) = \mathrm{He}_{n-1}(x), \tag{2.16}$$

and the recurrence relations are

$$\mathrm{He}_{n+1}(x) = x\mathrm{He}_n(x) - n\mathrm{He}_{n-1}(x). \tag{2.17}$$

The "modified Hermite polynomials" $\mathrm{He}_n(x)$ are the most reduced form within the kinship of Hermite polynomials concerning possible integer factorizations and factors in the basic relations of differentiation and recurrence relations.

For the expansion of Hermite polynomials of imaginary argument into that of real argument, we set $\lambda = \mathrm{i}$ in (2.10) and find

$$\mathrm{H}_n(\mathrm{i}x) = \mathrm{i}^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!(n-2k)!} 2^k \mathrm{H}_{n-2k}(x). \tag{2.18}$$

If λ^2 is an integer (*i.e.*, $\lambda^2 \in \mathbb{Z}$) then the polynomials $\lambda^n \mathrm{H}_n(\lambda x)$ and $\lambda^n \mathrm{H}_n\left(\frac{x}{\lambda}\right), (\lambda \neq 0)$, respectively, related to $\mathrm{H}_n(x)$ possess also integer coefficients as one can see from (2.10).

3. Hermite Polynomials and Related Operational Identities

All identities derived in last section were identities between different representations of functions with Hermite polynomials. In present section, we derive identities between different representations of operators in the form of functions of the differential operator $\frac{\partial}{\partial x}$ and variable x which can be applied to arbitrary functions. The background for these formulae is the following general operator identity for arbitrary linear operators A and B acting onto vectors in a vector space and which may be considered as a precursor to the Baker-Campbell-Hausdorff theorem for Lie groups and is easily to prove (e.g., B. Hall[3] [11], p. 61)

$$\begin{aligned} \mathrm{e}^{A} B \mathrm{e}^{-A} &= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left[A,\left[A,\cdots\left[A\right.\right.\right.}_{n\times},B\underbrace{\left.\left.\left.\right]\cdots\right]\right]}_{n\times} \\ &= B + \left[A,B\right] + \frac{1}{2!}\left[A,\left[A,B\right]\right] + \frac{1}{3!}\left[A,\left[A,\left[A,B\right]\right]\right] + \cdots. \end{aligned} \tag{3.1}$$

If we specify in the theorem (3.1) the operators A and B by

$$A = x^2, \quad B = -\frac{\partial}{\partial x} \;\Rightarrow\; \left[A,B\right] = 2x, \quad \left[A,\left[A,B\right]\right] = 0, \quad \cdots, \tag{3.2}$$

the Rodrigues-type definition (2.1) of Hermite polynomials may be written

$$\begin{aligned} \mathrm{H}_n(x) &\equiv \left(\exp\left(x^2\right)\left(-\frac{\partial}{\partial x}\right)\exp\left(-x^2\right)\right)^n 1 \\ &= \left(\exp\left(x^2\right)\left(-\frac{\partial}{\partial x}\right)^n \exp\left(-x^2\right)\right) 1 = \left(2x - \frac{\partial}{\partial x}\right)^n 1, \end{aligned} \tag{3.3}$$

with the following possible definition of Hermite polynomials

$$\mathrm{H}_n(x) = \left(2x - \frac{\partial}{\partial x}\right)^n 1 = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}. \tag{3.4}$$

In the same way, if we specify in the theorem (3.1) the operators A and B by

$$A = -\frac{1}{4}\frac{\partial^2}{\partial x^2}, \quad B = 2x \;\Rightarrow\; \left[A,B\right] = -\frac{\partial}{\partial x}, \quad \left[A,\left[A,B\right]\right] = 0, \quad \cdots \tag{3.5}$$

we arrive at the alternative definition (2.3) of Hermite polynomials

$$\begin{aligned} \mathrm{H}_n(x) &\equiv \left(\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right) 2x \exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\right)^n 1 \\ &= \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n \underbrace{\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right) 1}_{=1} = \left(2x - \frac{\partial}{\partial x}\right)^n 1. \end{aligned} \tag{3.6}$$

that leads to (2.3). In both special cases, one has to take only the first two non-vanishing sum terms on the right-hand side of (3.1).

The operators x and $\frac{\partial}{\partial x}$ in $\left(2x - \frac{\partial}{\partial x}\right)^n$ are what we call entangled that means the partial operators x and $\frac{\partial}{\partial x}$ or $-\frac{1}{4}\frac{\partial^2}{\partial x^2}$ are not ordered in a certain way if we write down these operators in fully extended form using the binomial formula. A disentangled form of these operators is

$$\left(2x - \frac{\partial}{\partial x}\right)^n = \sum_{j=0}^{n} \frac{(-1)^j n!}{j!(n-j)!} \mathrm{H}_{n-j}(x) \frac{\partial^j}{\partial x^j}, \tag{3.7}$$

[3]Under the many possible citations which give this identity we choose [11] also for its profound investigation of the much more difficult genuine Baker-Campbell-Hausdorff theorem.

where all powers of the differentiation operators $\frac{\partial}{\partial x}$ stand behind the functions of variable x. We call this the normally ordered form of the operators on the left-hand side. The Formula (3.7) can be proved by complete induction where one has to use already known properties of Hermite polynomials derived in Section 2 that is presented in Appendix A. Other orderings may also be useful in some cases. The identity (3.7) can now be applied to arbitrary functions $f(x)$ according to

$$\left(2x-\frac{\partial}{\partial x}\right)^n f(x)=\sum_{j=0}^{n}\frac{(-1)^j n!}{j!(n-j)!}\mathrm{H}_{n-j}(x)\frac{\partial^j}{\partial x^j}f(x). \tag{3.8}$$

and therefore such forms of identities with differential operators will be called operational identities.

A kind of inversion of the Formula (3.6) is

$$\begin{aligned}(2x)^n &= \exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\mathrm{H}_n(x)\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)1\\ &=\mathrm{H}_n\left(\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)x\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\right)1\\ &=\mathrm{H}_n\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)1.\end{aligned} \tag{3.9}$$

Similarly to (3.6) the right-hand side is the application of an operator to a function. The operator in front of the function can be normally ordered and one obtains the operational identity

$$\mathrm{H}_n\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2x)^{n-j}\frac{\partial^j}{\partial x^j}. \tag{3.10}$$

Its right-hand side is normally ordered and it can be applied to arbitrary functions. In analogy to the proof of (3.6) given in **Appendix A** it can also be proved by complete induction that is made in **Appendix A**.

In most application of the Hermite polynomial in the following considerations we use them with a stretched argument. If one substitutes $x\to\frac{x}{a}$ then in operational identities with at once the variable x and the operator of derivative $\frac{\partial}{\partial x}$ one has to substitute it according to $\frac{\partial}{\partial x}\to a\frac{\partial}{\partial x}$. With this remark one may generalize (3.6) to the form

$$\mathrm{H}_n\left(\frac{x}{a}\right)\equiv\left(\exp\left(-\frac{a^2}{4}\frac{\partial^2}{\partial x^2}\right)\left(\frac{2x}{a}\right)^n\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial x^2}\right)\right)1=\left(\frac{2x}{a}-a\frac{\partial}{\partial x}\right)^n 1, \tag{3.11}$$

and, for example, the operational identity (3.7) to the more general form

$$\left(\frac{2x}{a}-a\frac{\partial}{\partial x}\right)^n=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\mathrm{H}_{n-j}\left(\frac{x}{a}\right)\left(-a\frac{\partial}{\partial x}\right)^j. \tag{3.12}$$

The operational identity (3.10) with this extension takes on the form

$$\mathrm{H}_n\left(\frac{x}{a}+\frac{a}{2}\frac{\partial}{\partial x}\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\frac{2x}{a}\right)^{n-j}\left(a\frac{\partial}{\partial x}\right)^{j}. \tag{3.13}$$

Using the considered extension from (3.7) to (3.12), it follows that also powers of an arbitrary linear combination of variable x and differential operator $\frac{\partial}{\partial x}$ with coefficients μ and ν can be represented by an operational identity normally ordered as follows

$$\begin{aligned}\left(\mu x-\nu\frac{\partial}{\partial x}\right)^n&=\left(\frac{\sqrt{2\mu\nu}}{2}\left(\frac{2\mu x}{\sqrt{2\mu\nu}}-\frac{\sqrt{2\mu\nu}}{\mu}\frac{\partial}{\partial x}\right)\right)^n \quad \Big| a\equiv\frac{\sqrt{2\mu\nu}}{\mu}\\&=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\frac{\sqrt{2\mu\nu}}{2}\right)^{n-j}\mathrm{H}_{n-j}\left(\frac{\mu x}{\sqrt{2\mu\nu}}\right)\left(-\nu\frac{\partial}{\partial x}\right)^{j},\end{aligned} \tag{3.14}$$

from which by substitution $\nu\to-\nu$ also follows

$$\left(\mu x+\nu\frac{\partial}{\partial x}\right)^n=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\frac{\sqrt{-2\mu\nu}}{2}\right)^{n-j}\mathrm{H}_{n-j}\left(\frac{\mu x}{\sqrt{-2\mu\nu}}\right)\left(\nu\frac{\partial}{\partial x}\right)^{j}. \tag{3.15}$$

In analogous way, one derives for the generalization of the operator (3.10) using (3.13)

$$\begin{aligned}\mathrm{H}_n\left(\mu x+\nu\frac{\partial}{\partial x}\right)&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!(n-2k)!}\left(2\left(\mu x+\nu\frac{\partial}{\partial x}\right)\right)^{n-2k}\\&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!(n-2k)!}\sum_{j=0}^{n-2k}\frac{2^{n-2k}(n-2k)!}{j!(n-2k-j)!}\left(\frac{\sqrt{-2\mu\nu}}{2}\right)^{n-2k-j}\mathrm{H}_{n-2k-j}\left(\frac{\mu x}{\sqrt{-2\mu\nu}}\right)\left(\nu\frac{\partial}{\partial x}\right)^{j}\\&=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\sqrt{-2\mu\nu}\right)^{n-j}\left(\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\frac{(n-j)!}{k!(n-j-2k)!}\left(\frac{1}{2\mu\nu}\right)^{k}\mathrm{H}_{n-j-2k}\left(\frac{\mu x}{\sqrt{-2\mu\nu}}\right)\right)\\&\quad\cdot\left(2\nu\frac{\partial}{\partial x}\right)^{j}.\end{aligned} \tag{3.16}$$

The sum in round brackets can be evaluated using (2.12) with $\kappa^2=-\frac{\mu^2}{2\mu\nu}$ and $\lambda^2=\frac{\mu^2}{1-2\mu\nu}$ resulting in

$$\mathrm{H}_n\left(\mu x+\nu\frac{\partial}{\partial x}\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\sqrt{1-2\mu\nu}\right)^{n-j}\mathrm{H}_{n-j}\left(\frac{\mu x}{\sqrt{1-2\mu\nu}}\right)\left(2\nu\frac{\partial}{\partial x}\right)^{j}. \tag{3.17}$$

This operational identity generalizes the identity (3.13) and thus also its special case (3.12). To show this, one has to set $\mu=\frac{1}{a},\nu=\frac{a}{2}\to 2\mu\nu=1$ and since $\varepsilon\equiv\sqrt{1-2\mu\nu}\to 0$ one has to consider the limiting case of Hermite polynomials

$$\lim_{\varepsilon\to0}\varepsilon^n\mathrm{H}_n\left(\frac{\mu x}{\varepsilon}\right)=\lim_{\varepsilon\to0}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!(n-2k)!}\varepsilon^{2k}(2\mu x)^{n-2k}=(2\mu x)^n, \tag{3.18}$$

applied to (3.17) leads to (3.13). Choosing $\mu=\frac{1}{a},\nu=-\frac{a}{2}$ leads to the operational identity

$$\mathrm{H}_n\left(\frac{x}{a}-\frac{a}{2}\frac{\partial}{\partial x}\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\left(\sqrt{2}\right)^{n-j}\mathrm{H}_{n-j}\left(\frac{\mu x}{\sqrt{2}}\right)\left(-a\frac{\partial}{\partial x}\right)^{j}. \tag{3.19}$$

An example for the application of this operational identity leads to the formula

$$\begin{aligned}\mathrm{H}_{m+n}(x)&=\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^{m+n}\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)1\\&=\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^{m}\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^{n}\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)1,\end{aligned} \tag{3.20}$$

with the result (Bateman and Erdélyi [4] Chapter 10.14.36)

$$\begin{aligned}\mathrm{H}_{m+n}(x)&=\sum_{j=0}^{m}\frac{(-1)^j m!}{j!(m-j)!}\mathrm{H}_{m-j}(x)\frac{\partial^j}{\partial x^j}\mathrm{H}_n(x)\\&=\sum_{j=0}^{\{m,n\}}\frac{(-2)^j m!n!}{j!(m-j)!(n-j)!}\mathrm{H}_{m-j}(x)\mathrm{H}_{n-j}(x),\end{aligned} \tag{3.21}$$

where we used the Formula (2.4) for the derivatives of the Hermite polynomials. Some kind of inversion of this relation is (Bateman and Erdélyi [4] Chapter 10.14.37)

$$\mathrm{H}_m(x)\mathrm{H}_n(x)=\sum_{j=0}^{\{m,n\}}\frac{2^j m!n!}{j!(m-j)!(n-j)!}\mathrm{H}_{m+n-2j}(x). \tag{3.22}$$

It is not in simple way to prove by the derived operational identities. Both relations (3.21) and (3.22) can be proved by complete induction.

Let us make an addition to the notion "operational identities" which is not directly connected with Hermite polynomials but which makes it clearer what it means. In all of the cited works to orthogonal polynomials [4]-[7]) one finds the following (Rodrigues) definition of Legendre polynomials $\mathrm{P}_n(x)$ (in [5] in the more general form for Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(x)$, Chapter IV, (4.3.1))

$$\mathrm{P}_n(x)=\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n \tag{3.23}$$

This is not fully consequent because on the left-hand side one has a function, the polynomial $\mathrm{P}_n(x)$, and on the right-hand side a differential operator $\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n$ which has to be applied onto a function to get a new function. This operator can be transformed to normal ordering by multiplying it with an arbitrary function $f(x)$ and with reordering of all the differential operators $\frac{\partial^k}{\partial x^k}$ in a way to bring them to the right-hand side of functions of x.

In the first step of this programme to bring to normal ordering the differential operator (3.23) one may use the Leibniz rule of differentiation of a product

$$\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n f(x)=\frac{1}{2^n n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{\partial^{n-k}}{\partial x^{n-k}}\left(x^2-1\right)^n\right)\frac{\partial^k}{\partial x^k}f(x)$$
$$=\frac{1}{2^n n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{\partial^{n-k}}{\partial x^{n-k}}(x-1)^n(x+1)^n\right)\frac{\partial^k}{\partial x^k}f(x). \tag{3.24}$$

In the second step, we apply again the Leibniz rule for differentiation of a product, in this case, to the product in big round brackets leading to a double sum

$$\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n f(x)$$
$$=\frac{(x-1)^n}{2^n}\left(\sum_{k=0}^{n}\frac{n!^2(x+1)^k}{k!(n-k)!(n+k)!}\sum_{l=0}^{n-k}\frac{(n-k)!(n+k)!}{l!(n-l)!(n-k-l)!(k+l)!}\left(\frac{x+1}{x-1}\right)^l\right)\frac{\partial^k}{\partial x^k}f(x)$$
$$=\sum_{k=0}^{n}\frac{n!^2}{k!(n-k)!(n+k)!}(x+1)^k\,\mathrm{P}_n^{(-k,k)}(x)\frac{\partial^k}{\partial x^k}f(x). \tag{3.25}$$

The inner sum could be represented by the Jacobi polynomial with lower index n and with integer upper indices $(-k,k)$. For Jacobi polynomials with, at least, one integer upper index (say the first $\alpha=-k$, ($k\le n$)), the following transformation formula is true and easily to prove from the explicit representations (see also Szegö [5], Chapter IV, (4.22.2))

$$\mathrm{P}_n^{(-k,\beta)}(x)=\frac{(n-k)!(n+\beta)!}{n!(n+\beta-k)!}\left(\frac{x-1}{2}\right)^k\mathrm{P}_{n-k}^{(k,\beta)}(x),\quad (k=\cdots,-1,0,1,\cdots,n), \tag{3.26}$$

in addition to the general symmetry

$$\mathrm{P}_n^{(\alpha,\beta)}(x)=(-1)^n\,\mathrm{P}_n^{(\beta,\alpha)}(x), \tag{3.27}$$

making possible further transformations. Applying (3.26) to (3.25) with $\beta=k$ one obtains the identity of functions

$$\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n f(x)=\sum_{k=0}^{n}\frac{1}{2^k k!}\left(x^2-1\right)^k\mathrm{P}_{n-k}^{(k,k)}(x)\frac{\partial^k}{\partial x^k}f(x). \tag{3.28}$$

This relation becomes the operational identity normally ordered under omission of $f(x)$ on both sides. It seems that this interesting and somehow beautiful relation is also provable by complete induction, however, not in simpler way. We checked it by computer for some low numbers n.

Applying relation (3.28) to the function $f(x)=1$ one has only to take sum term to $k=0$ as non-vanishing and finds

$$\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n 1=\mathrm{P}_n^{(0,0)}(x)\equiv\mathrm{P}_n(x). \tag{3.29}$$

If one applies the operational identity (3.28), for example, to the function $f(x)=x$ the first two sum terms are non-vanishing and one obtains

$$\frac{1}{2^n n!}\frac{\partial^n}{\partial x^n}\left(x^2-1\right)^n x=\mathrm{P}_n^{(0,0)}(x)\,x+\frac{1}{2}\left(x^2-1\right)\mathrm{P}_{n-1}^{(1,1)}(x). \tag{3.30}$$

Thus it would be more consequent to write the Rodrigues definition of Legendre polynomials instead of the form (3.23) similar to (3.29) to have functions on both sides.

4. Fourier Transformation of a Normalized Gaussian Function and Its Importance for Gaussian Integrals

We define the Fourier transform $\tilde{f}(u)$ of an arbitrary function $f(x)$ together with its inversion by

$$\tilde{f}(u)=\int_{-\infty}^{+\infty}\mathrm{d}x\, f(x)\mathrm{e}^{-\mathrm{i}ux},\quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}u\,\tilde{f}(u)\mathrm{e}^{\mathrm{i}ux}. \tag{4.1}$$

The Fourier transformation of a one-dimensional normalized Gaussian (distribution or bell) function with parameter a as it is well known is described by the formulae

$$\begin{aligned} f(x)&\equiv\frac{1}{\sqrt{\pi a^2}}\exp\left(-\frac{x^2}{a^2}\right),\quad \int_{-\infty}^{+\infty}\mathrm{d}x\, f(x)=1,\\ \tilde{f}(u)&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}-\mathrm{i}xu\right)\\ &=\exp\left(-\frac{a^2u^2}{4}\right)\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{1}{a^2}\left(x+\mathrm{i}\frac{a^2}{2}u\right)^2\right)\\ &=\exp\left(-\frac{a^2u^2}{4}\right). \end{aligned}\tag{4.2}$$

Then an integral over the normalized Gaussian function multiplied by an arbitrary function $F(x)$ can be calculated by the following special approach

$$\begin{aligned} &\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)F(x)=\left\{\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)F(u+x)\right\}_{u=0}\\ &=\left\{\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}-\mathrm{i}x\left(\mathrm{i}\frac{\partial}{\partial u}\right)\right)F(u)\right\}_{u=0}\\ &=\left\{\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)F(u)\right\}_{u=0}, \end{aligned}\tag{4.3}$$

where was used the displacement operator $\exp\left(x\frac{\partial}{\partial u}\right)$ of the argument of an arbitrary function $F(u)$ according to $\exp\left(x\frac{\partial}{\partial u}\right)F(u)=F(u+x)$ and the explicit form (4.2) of the Fourier transform of the Gaussian function with the substitution $u\to\mathrm{i}\frac{\partial}{\partial u}$. That this relation is true can be verified by Taylor series expansion of the function $F(x)$ according to

$$\begin{aligned} &\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)\sum_{n=0}^{\infty}\frac{x^n}{n!}F^{(n)}(0)\\ &=\sum_{m=0}^{\infty}\frac{F^{(2m)}(0)}{(2m)!}\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)x^{2m}\\ &=\sum_{m=0}^{\infty}\frac{F^{(2m)}(0)}{(2m)!}\frac{(2m)!}{m!}\left(\frac{a}{2}\right)^{2m}=\sum_{m=0}^{\infty}\frac{1}{m!}\frac{a^{2m}}{2^{2m}}F^{(2m)}(0), \end{aligned}\tag{4.4}$$

where the order of integration and summation is changed and where one has

$$\left\{\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)F(u)\right\}_{u=0} = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{a^{2m}}{2^{2m}}F^{(2m)}(0). \tag{4.5}$$

The operator $\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial x^2}\right)$ in (4.3) formed from the Fourier transform of the normalized Gaussian function is of the same kind as the operator in the alternative definition of the Hermite polynomials in (3.11) but with different signs in exponent. This can be used for the calculation of the following definite integral

$$\begin{aligned}
&\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x-x_1}{b}\right)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\\
&=\left\{\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)\mathrm{H}_m\left(\frac{u}{b}\right)\right\}_{u=x_0-x_1}\\
&=\left\{\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)\exp\left(-\frac{b^2}{4}\frac{\partial^2}{\partial u^2}\right)\left(\frac{2u}{b}\right)^m\exp\left(\frac{b^2}{4}\frac{\partial^2}{\partial u^2}\right)1\exp\left(-\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)\right\}_{u=x_0-x_1}\\
&=\left(\frac{\sqrt{b^2-a^2}}{b}\right)^m\left\{\exp\left(-\frac{b^2-a^2}{4}\frac{\partial^2}{\partial u^2}\right)\left(\frac{2u}{\sqrt{b^2-a^2}}\right)^m\exp\left(\frac{b^2-a^2}{4}\frac{\partial^2}{\partial u^2}\right)1\right\}_{u=x_0-x_1}\\
&=\left(\frac{\sqrt{b^2-a^2}}{b}\right)^m\mathrm{H}_m\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right).
\end{aligned} \tag{4.6}$$

This calculation shows by comparison with the proof of the formula by complete induction in Appendix B that it obviously provides the correct result. We do not stay for a long discussion at this formula because it is well known and is for us only an intermediate step to a more general formula which is the proper aim.

5. A Quite General Integral Formula over a Gaussian Function Multiplied by Two Different Hermite Polynomials

In this section, we discuss the calculation of the following integral over a Gaussian distribution (or bell function) multiplied by two different Hermite polynomials with displaced arguments

$$\begin{aligned}
&f(m,n;a,b,c;x_0,x_1,x_2)\\
&\equiv\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x-x_1}{b}\right)\mathrm{H}_n\left(\frac{x-x_2}{c}\right)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\\
&=f(m,n;a,b,c;0,x_0-x_1,x_0-x_2).
\end{aligned} \tag{5.1}$$

It contains 2 discrete parameters (m,n), 3 continuous, in general, complex stretching parameters (a,b,c) and furthermore 3, in general, also complex shift parameters from coordinate origin of the arguments of the involved bell function and of the two Hermite polynomials $\left(x_0,x_1,x_2\right)$. The shift parameters are not fully independent on each other in the result according to (5.1) that reduces them to 2 independent parameters $\left(x_0-x_1,x_0-x_2\right)$. The parameters within each group $\left(a,b,c\right)$ and $\left(x_0,x_1,x_2\right)$ and of the two groups possess the same dimension. The integral comprises almost all special and limiting cases of definite integrals over the product of a Gaussian function with no more than two Hermite polynomials which one may find in tables of integrals [1] [2] (and in [3] in the special chapter XVI, 16.5 about Hermite polynomials in the modified form of $\mathrm{He}_n\left(x\right)$, see (2.12)).

The result of the calculation of the integral (5.1) in two different representations is

$$\begin{aligned}
&f\left(m,n;a,b,c;x_0,x_1,x_2\right)\\
&=\left(\frac{\sqrt{b^2-a^2}}{b}\right)^m\left(\frac{\sqrt{c^2-a^2}}{c}\right)^n\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2a^2}{\sqrt{b^2-a^2}\sqrt{c^2-a^2}}\right)^j\\
&\cdot\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\mathrm{H}_{n-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)\\
&=\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2a^2}{bc}\right)^j\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\\
&\cdot\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n-j}\mathrm{H}_{n-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right),\quad\left(\{m,n\}\equiv\min\{m,n\}\right),
\end{aligned}\tag{5.2}$$

that will be proved in Appendix C and is discussed in the following considerations. The integral is symmetric under interchange of variables in the Hermite polynomials as follows

$$f\left(m,n;a,b,c;x_0,x_1,x_2\right)=f\left(n,m;a,c,b;x_0,x_2,x_1\right),\tag{5.3}$$

and, therefore, also in its result where, however, not every possible representation of the result must show this symmetry in obvious way (Section 7).

All continuous parameters $\left(a,b,c;x_0,x_1,x_2\right)$ can be complex numbers but parameter a in restricted way due to necessary convergence of the integral. For this purpose a^{-2} should possess a squared Real part bigger than its squared Imaginary part[4]. Likely, in most applications a is a real parameter. However, by experience, when something (integral or sum) does not converge in "usual sense" it, nevertheless, has some importance in the sense of weak convergence of the theory of Generalized functions.

It is easy but cumbersome to write instead of the parameters $\left(a,b,c\right)$ equivalent reciprocal parameters $\left(\alpha,\beta,\gamma\right)$ into the numerators of the corresponding functions in the integral. The integral becomes then

[4]If one sets with separation of Real and Imaginary parts $a=a'+\mathrm{i}a''$ then $\frac{1}{a^2}=\frac{a'^2-a''^2-\mathrm{i}2a'a''}{\left(a'^2+a''^2\right)^2}$, that requires $a'^2>a''^2$, for the convergence of the integral (5.1).

$$
\begin{aligned}
& f'\left(m,n;\alpha,\beta,\gamma;x_0,x_1,x_2\right) \\
& =\sqrt{\frac{\alpha^2}{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\alpha^2\left(x-x_0\right)^2\right)\mathrm{H}_m\left(\beta\left(x-x_1\right)\right)\mathrm{H}_n\left(\gamma\left(x-x_2\right)\right) \\
& =\sqrt{\frac{\alpha^2}{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\alpha^2x'^2\right)\mathrm{H}_m\left(\beta\left(x'+x_0-x_1\right)\right)\mathrm{H}_n\left(\gamma\left(x'+x_0-x_2\right)\right) \\
& = f'\left(m,n;\alpha,\beta,\gamma;0,x_0-x_1,x_0-x_2\right),
\end{aligned}
\tag{5.4}
$$

with the result

$$
\begin{aligned}
& f'\left(m,n;\alpha,\beta,\gamma;x_0,x_1,x_2\right) \\
& =\left(\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right)^m\left(\frac{\sqrt{\alpha^2-\gamma^2}}{\alpha}\right)^n\sum_{j=0}^{n}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2\beta\gamma}{\sqrt{\alpha^2-\beta^2}\sqrt{\alpha^2-\gamma^2}}\right)^j \\
& \cdot\mathrm{H}_{m-j}\left(\frac{\alpha\beta\left(x_0-x_1\right)}{\sqrt{\alpha^2-\beta^2}}\right)\mathrm{H}_{n-j}\left(\frac{\alpha\gamma\left(x_0-x_2\right)}{\sqrt{\alpha^2-\gamma^2}}\right) \\
& =\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2\beta\gamma}{\alpha^2}\right)^j\left(\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{\alpha\beta\left(x_0-x_1\right)}{\sqrt{\alpha^2-\beta^2}}\right) \\
& \cdot\left(\frac{\sqrt{\alpha^2-\gamma^2}}{\alpha}\right)^{m-j}\mathrm{H}_{n-j}\left(\frac{\alpha\gamma\left(x_0-x_2\right)}{\sqrt{\alpha^2-\beta^2}}\right).
\end{aligned}
\tag{5.5}
$$

In similar form we found it in our elder calculations made by stepwise generalization from simpler integrals involving transformations of the variables and parameters that is also a lengthy way. Therefore, we prove the form (5.1) in Appendix C by complete induction $(m,n)\rightarrow(m,n+1)$ using that it is already proved in previous Section 4 and **Appendix B** for general m and special $n=0$ by complete induction $(m,0)\rightarrow(m+1,0)$. We wanted to derive the result (5.2) from the Formula (4.3) and found also some essential elements of the structure of this result but it was not fully successful and convincing compared with the proof by complete induction in **Appendix C**. Problem could be that the operator $\exp\left(\frac{a^2}{4}\frac{\partial^2}{\partial u^2}\right)$ in (5.2) acts on the whole function $F(u)$ whereas analogous operators in this function act only within itself.

We made intensive numerical checkups of the basic result (5.2)[5].

6. Special Cases of the Quite General Gauss-Hermite Integral

We consider now some special cases of the integral (5.1) where the general structure of the result (5.2) simplifies in nontrivial way. The special case $f\left(m,0;a,b,0;x_0,x_1,0\right)$ was already considered in Section 4 with proof in (C.1).

[5]In the checkups one has to write equal square roots in the same way since computer chooses a certain sign of the roots and products, for example, of the signs of $\sqrt{b^2-a^2}\sqrt{c^2-a^2}$ and of $\sqrt{\left(b^2-a^2\right)\left(c^2-a^2\right)}$ for numbers (a,b,c) can be differently chosen by computer. We met such cases.

6.1. Special Case $x_0 \neq 0, x_1 = x_2 = 0$ and General (a,b,c)

If the shift parameters of the Hermite polynomials x_1 and x_2 are vanishing the general structure of the integral (5.1) does not simplify essentially

$$
\begin{aligned}
& f(m,n;a,b,c;x_0,0,0) \\
& \equiv \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{(x-x_0)^2}{a^2}\right) \mathrm{H}_m\left(\frac{x}{b}\right) \mathrm{H}_n\left(\frac{x}{c}\right) \\
& = \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\frac{2a^2}{bc}\right)^j \left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j} \mathrm{H}_{m-j}\left(\frac{x_0}{\sqrt{b^2-a^2}}\right) \\
& \cdot \left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n-j} \mathrm{H}_{n-j}\left(\frac{x_0}{\sqrt{c^2-a^2}}\right).
\end{aligned} \tag{6.1}
$$

However, if in addition the shift parameter x_0 of the Gaussian function vanishes the structure of the result is changed radically.

6.2. Special Case $x_0 = x_1 = x_2 = 0$ and General (a,b,c)

If all argument of the Hermite polynomials are vanishing then only the Hermite polynomials with even indices are non-vanishing in this case according to

$$
\mathrm{H}_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad \mathrm{H}_{2n+1}(0) = 0, \quad (n = 0,1,2,\cdots). \tag{6.2}
$$

Due to necessary symmetry of the integrand under interchanging the signs of its variable x it has to be symmetric and one must distinguish 3 partial cases of the indices (m,n) of the Hermite polynomials: 1) both indices are even $(m \to 2m, n \to 2n)$, 2) both indices are odd $(m \to 2m+1, n \to 2n+1)$, one of the indices is even and the other is odd, say $(m \to 2m, n \to 2n+1)$.

1) $m \to 2m,\ n \to 2n$:

$$
\begin{aligned}
& f(2m,2n;a,b,c;0,0,0) \\
& = \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{x^2}{a^2}\right) \mathrm{H}_{2m}\left(\frac{x}{b}\right) \mathrm{H}_{2n}\left(\frac{x}{c}\right) \\
& = \sum_{j=0}^{\{m,n\}} \frac{(-1)^{m+n}(2m)!(2n)!}{(2j)!(m-j)!(n-j)!} \left(\frac{2a^2}{bc}\right)^{2j} \left(\frac{b^2-a^2}{b^2}\right)^{m-j} \left(\frac{c^2-a^2}{c^2}\right)^{n-j},
\end{aligned} \tag{6.3}
$$

2) $m \to 2m+1,\ n \to 2n+1$:

$$
\begin{aligned}
& f(2m+1,2n+1;a,b,c;0,0,0) \\
& = \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{x^2}{a^2}\right) \mathrm{H}_{2m+1}\left(\frac{x}{b}\right) \mathrm{H}_{2n+1}\left(\frac{y}{c}\right) \\
& = \sum_{j=0}^{\{m,n\}} \frac{(-1)^{m+n}(2m+1)!(2n+1)!}{(2j+1)!(m-j)!(n-j)} \left(\frac{2a^2}{bc}\right)^{2j+1} \left(\frac{b^2-a^2}{b^2}\right)^{m-j} \left(\frac{c^2-a^2}{c^2}\right)^{n-j},
\end{aligned} \tag{6.4}
$$

3) $m \to 2m,\ n \to 2n+1$:

$$f(2m,2n+1;a,b,c;0,0,0) = \frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)\mathrm{H}_{2m}\left(\frac{x}{b}\right)\mathrm{H}_{2n+1}\left(\frac{x}{c}\right)=0. \tag{6.5}$$

This last result is trivial and follows from antisymmetry of the integrand under interchanging $x\to -x$.

Setting $a=b=c$ in the above relations one finds the orthonormality relations for Hermite polynomials specialized in other way and discussed below in (6.23). Integrals of the kind (6.3) and (6.4) in other representations one finds in [2] (from p. 502 on).

6.3. Special Case $m=n$ and $x_0=x_1=x_2=0$ with General (a,b,c)

Since in considered special case all shift parameters (x_0,x_1,x_2) are vanishing in the result (5.2) of the integral the arguments of Hermite polynomials are also vanishing and only the Hermite polynomials with even indices contribute to this integral. Therefore, one may set in (5.2)

$$m-j=n-j=2k,\quad \left(k=0,1,\cdots,\left[\frac{n-j}{2}\right]\right), \tag{6.6}$$

and have to apply (6.2) with $n\to k$. Then from (5.2) follows

$$\begin{aligned} &f(n,n;a,b,c;0,0,0) \\ &= \frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)\mathrm{H}_n\left(\frac{x}{b}\right)\mathrm{H}_n\left(\frac{x}{c}\right) \\ &= \left(\frac{2a^2}{bc}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!^2}{k!^2(n-2k)!}\left(\frac{(b^2-a^2)(c^2-a^2)}{4a^4}\right)^k \\ &= \left(\frac{\sqrt{(b^2-a^2)(c^2-a^2)}}{bc}\right)^n\sum_{n=0}^{\left[\frac{n}{2}\right]}\frac{n!^2}{k!^2(n-2k)!}\left(\frac{2a^2}{\sqrt{(b^2-a^2)(c^2-a^2)}}\right)^{n-2k}. \end{aligned} \tag{6.7}$$

Using now the following representation of the Legendre polynomials $\mathrm{P}_n(z)$ [6]

$$\mathrm{P}_n(z)=\left(\frac{\sqrt{z^2-1}}{2}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{k!^2(n-2k)!}\left(\frac{2z}{\sqrt{z^2-1}}\right)^{n-2k}, \tag{6.8}$$

one may represent the result (6.7) in the following form

[6]I do not know whether or not it is known. Apparently it follows from one of the many representations of the Legendre polynomials by the Hypergeometric function and their transformations given in [4] (Chapter 10.10.). Furthermore, I published already the narrowly related formula found in my records $\left(2\sqrt{1+z^2}\right)^n\mathrm{P}_n\left(\frac{z}{\sqrt{1+z^2}}\right)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!^2(n-2k)!}(2z)^{n-2k}$. A similar remark concerns also the generalization of (6.8) to (6.13) given below.

$$
\begin{aligned}
&f\left(n,n;a,b,c;0,0,0\right)\\
&=n!\left(\frac{2\sqrt{a^2\left(b^2+c^2\right)-b^2c^2}}{bc}\right)^n \mathrm{P}_n\left(\frac{a^2}{\sqrt{a^2\left(b^2+c^2\right)-b^2c^2}}\right).
\end{aligned}
\tag{6.9}
$$

With the reciprocal parameters (α,β,γ) instead of (a,b,c) the result for the integral takes on the form (see (5.4) and (5.5))

$$
\begin{aligned}
f'\left(n,n;\alpha,\beta,\gamma,0,0,0\right)&=\sqrt{\frac{\alpha^2}{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\alpha^2x^2\right)\mathrm{H}_n\left(\beta x\right)\mathrm{H}_n\left(\gamma x\right)\\
&=n!\left(\frac{2\sqrt{\beta^2+\gamma^2-\alpha^2}}{\alpha}\right)^n \mathrm{P}_n\left(\frac{\beta\gamma}{\alpha\sqrt{\beta^2+\gamma^2-\alpha^2}}\right).
\end{aligned}
\tag{6.10}
$$

In similar form one finds a closely related integral in [2] (p. 502, Chapter 2.20.16 (2)) which, however, is not yet contained in [1].

6.4. Special Case $m\neq n$ in General and $x_0=x_1=x_2=0$ with General (a,b,c)

The case $m\neq n$ is a generalization of the case $m=n=0$ of last Subsection and instead of (6.6) the result involves two different Hermite polynomials with argument zero and for non-vanishing of the integral one has to set their two different indices equal to an even number

$$
\begin{gathered}
m-j=2l,\ n-j=2k\ \Rightarrow\ m-n=2\left(l-k\right)\equiv 2i,\\
m=n+2i,\ l=k+i,\ \ j=n-2k.
\end{gathered}
\tag{6.11}
$$

Therefore, for non-vanishing of the integral the difference $m-n$ has to be a (positive or negative) even number which is written as $2i,\left(i=0,\mp1,\mp2,\cdots\right)$. Then from the general result (5.2) follows specialized to the considered case

$$
\begin{aligned}
&f\left(n+2i,n;a,b,c;0,0,0\right)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)\mathrm{H}_{n+2i}\left(\frac{x}{b}\right)\mathrm{H}_n\left(\frac{x}{c}\right)\\
&=\left(-\frac{b^2-a^2}{b^2}\right)^i\left(\frac{2a^2}{bc}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!(n+2i)!}{k!(k+\alpha)!(n-2k)!}\left(\frac{\left(b^2-a^2\right)\left(c^2-a^2\right)}{4a^4}\right)^k\\
&=\frac{(n+2i)!}{(n+i)!}\left(-\frac{b^2-a^2}{b^2}\right)^i\left(\frac{\sqrt{\left(b^2-a^2\right)\left(c^2-a^2\right)}}{bc}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!(n+i)!}{k!(k+i)!(n-2k)!}\\
&\quad\cdot\left(\frac{2a^2}{\sqrt{\left(b^2-a^2\right)\left(c^2-a^2\right)}}\right)^{n-2k}.
\end{aligned}
\tag{6.12}
$$

In special case $i=0$ it makes the transition into (6.7). Using now the Ultraspherical polynomials $\mathrm{P}_n^{\alpha,\alpha}(z)$ which among others possess the following explicit representation

$$\mathrm{P}_n^{(\alpha,\alpha)}(z)=\left(\frac{\sqrt{z^2-1}}{2}\right)^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n+\alpha)!}{k!(k+\alpha)!(n-2k)!}\left(\frac{2z}{\sqrt{z^2-1}}\right)^{n-2k}, \tag{6.13}$$

one finds the following representation of the considered integral by the Ultraspherical polynomials

$$\begin{aligned}&f(n+2i,n;a,b,c;0,0,0)\\&=\frac{n!(n+2i)!}{(n+i)!}\left(-\frac{b^2-a^2}{b^2}\right)^i\left(\frac{2\sqrt{a^2\left(b^2+c^2\right)-b^2c^2}}{bc}\right)^n\mathrm{P}_n^{(i,i)}\left(\frac{a^2}{\sqrt{a^2\left(b^2+c^2\right)-b^2c^2}}\right).\end{aligned} \tag{6.14}$$

This result generalizes the integral (6.9) and seems to be new.

The Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(z)$ are the special case $\alpha=\beta$ of the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(z)$. The Legendre polynomials $\mathrm{P}_n(z)$ according to

$$\begin{aligned}\mathrm{P}_n(z)&=\mathrm{P}_n^{(0,0)}(z)\\&=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(n-k-\frac{1}{2}\right)!}{k!(n-2k)!\left(-\frac{1}{2}\right)!}(2z)^{n-2k}\\&=(-1)^n\mathrm{P}_n(-z),\end{aligned} \tag{6.15}$$

are the special case $\alpha=0$ of the Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(z)$ [4]-[7]. The Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(z)$ possess a one-to-one correspondence to the Gegenbauer polynomials $\mathrm{C}_n^{\nu}(z)$ as follows

$$\begin{aligned}\mathrm{P}_n^{(\alpha,\alpha)}(z)&=\frac{(n+\alpha)!(2\alpha)!}{(n+2\alpha)!\alpha!}\mathrm{C}_n^{\alpha+\frac{1}{2}}(z)\\&=\frac{2^{2\alpha}(n+\alpha)!}{(n+2\alpha)!}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(n+\alpha-k-\frac{1}{2}\right)!}{k!(n-2k)!\left(-\frac{1}{2}\right)!}(2z)^{n-2k}\\&=(-1)^n\mathrm{P}_n^{(\alpha,\alpha)}(-z).\end{aligned} \tag{6.16}$$

The Chebyshev polynomials of first and second kind $\mathrm{T}_n(z)$ and $\mathrm{U}_n(z)$ are other special cases of the Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(z)$ but with semi-integer $\alpha=-\frac{1}{2}$ and $\alpha=+\frac{1}{2}$, respectively. They are not involved in Formula (6.14) because there are contained only integer $\alpha=i=0,\pm1,\pm2,\cdots$ plus proportionality factors.

Such as in Subsection 6.3, we give here also the representation of Formula (6.14) by reciprocal parameters (α,β,γ) to parameters (a,b,c)

$$f'(n+2i,n;\alpha,\beta,\gamma;0,0,0)=\sqrt{\frac{\alpha^2}{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\alpha^2x^2\right)\mathrm{H}_{n+2i}(\beta x)\mathrm{H}_n(\gamma x)$$
$$=\frac{n!(n+2i)!}{(n+i)!}\left(-\frac{\alpha^2-\beta^2}{\alpha^2}\right)^i\left(\frac{2\sqrt{\beta^2+\gamma^2-\alpha^2}}{\alpha}\right)^n\mathrm{P}_n^{(i,i)}\left(\frac{\beta\gamma}{\alpha\sqrt{\beta^2+\gamma^2-\alpha^2}}\right). \tag{6.17}$$

All other integrals $f'(m,n;a,b,c;0,0,0)$ with integer (m,n) are vanishing.

The specialized integrals in Subsections 6.3 and 6.4. possess a symmetric or an antisymmetric integrand with respect to transformations $x\to -x$ depending on the product of the two unshifted Hermite polynomials and only that with symmetric integrand are non-vanishing. For these integrals with symmetric integrand the value of the integrals over the positive (or negative) semi-axis is half the calculated value of the corresponding given integrals over the full real axis.

6.5. Special Case $a=b=c$ and General (x_0,x_1,x_2)

In the special case

$$a=b=c\;\Rightarrow\;\sqrt{b^2-a^2}=\sqrt{c^2-a^2}=0, \tag{6.18}$$

the arguments of the Hermite polynomials in (5.1) become infinite and the factors in front of them become equal to zero and one has to consider this as the limiting case (compare (3.18))

$$\lim_{\varepsilon\to 0}\varepsilon^n\mathrm{H}_n\left(\frac{x}{\varepsilon}\right)=\lim_{\varepsilon\to 0}\varepsilon^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!}{k!(n-2k)!}\left(\frac{2x}{\varepsilon}\right)^{n-2k}=(2x)^2. \tag{6.19}$$

The result of the specialization of (5.1) is then

$$f(m,n;a,a,a;x_0,x_1,x_2)=f(m,n;a,a,a;0,x_0-x_1,x_0-x_2)$$
$$\equiv\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x-x_1}{a}\right)\mathrm{H}_n\left(\frac{x-x_2}{a}\right) \tag{6.20}$$
$$=a^{m+n}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2}{a^2}\right)^{m+n-j}(x_0-x_1)^{m-j}(x_0-x_2)^{n-j}.$$

This can be further specialized.

If in addition to $a=b=c$ the shift parameters $x_1=x_2=0$ are vanishing but $x_0\neq 0$ then one finds from (6.20)

$$f(m,n;a,a,a;x_0,0,0)=f(m,n;a,a,a;0,x_0,x_0)$$
$$\equiv\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x}{a}\right)\mathrm{H}_n\left(\frac{x}{a}\right) \tag{6.21}$$
$$=\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}2^{m+n-j}\left(\frac{x_0}{a}\right)^{m+n-2j}.$$

If, for example, the shift parameters $x_0=x_1$ are equal then only the sum term to $j=m$ is non-vanishing and one finds from (6.20)

$$
\begin{aligned}
& f\left(m,n;a,a,a;x_0,x_0,x_2\right)=f\left(m,n;a,a,a;0,0,x_0-x_2\right) \\
& \equiv \frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{\left(x-x_0\right)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x-x_0}{a}\right)\mathrm{H}_n\left(\frac{x-x_2}{a}\right) \\
& =a^{m+n}\frac{n!}{(n-m)!}\left(\frac{2}{a^2}\right)^n\left(x_0-x_2\right)^{n-m}.
\end{aligned}
\tag{6.22}
$$

If all shift parameters are the same $x_0=x_1=x_2$ then the integral is non-vanishing only for $m=n$ and one obtains from (6.22) and also from (6.21)

$$
\begin{aligned}
& f\left(m,n;a,a,a;x_0,x_0,x_0\right)=f\left(m,n;a,a,a;0,0,0\right) \\
& \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{a^2}}\exp\left(-\frac{x^2}{a^2}\right)\mathrm{H}_m\left(\frac{x}{a}\right)\mathrm{H}_n\left(\frac{x}{a}\right)=2^n n!\delta_{m,n}.
\end{aligned}
\tag{6.23}
$$

These are the well-known orthonormality relations for Hermite polynomials.

From these orthonormality relations setting $a=1$ and interchanging the order of summation and integration in the following relation

$$
\begin{aligned}
\mathrm{H}_m(y) &= \sum_{n=0}^{\infty}\delta_{m,n}\mathrm{H}_n(y) \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-x^2\right)\mathrm{H}_m(x)\mathrm{H}_n(x)\right)\mathrm{H}_n(y) \\
&= \int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{H}_m(x)\left(\frac{1}{\sqrt{\pi}}\exp\left(-x^2\right)\sum_{n=0}^{\infty}\frac{1}{2^n n!}\mathrm{H}_n(x)\mathrm{H}_n(y)\right) \\
&= \int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{H}_m(x)\delta(x-y),
\end{aligned}
\tag{6.24}
$$

and with introduction of the Hermite functions $\mathrm{h}_n(x)$ according to

$$
\mathrm{h}_n(x)\equiv\frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}}\exp\left(-\frac{x^2}{2}\right)\mathrm{H}_n(x),\quad (n=0,1,2,\cdots),
\tag{6.25}
$$

follows the completeness relation[7]

$$
\sum_{n=0}^{\infty}\mathrm{h}_n(x)\mathrm{h}_n(y)=\delta(x-y).
\tag{6.26}
$$

Then the orthonormality relations (6.23) can be written

$$
\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{h}_m(x)\mathrm{h}_n(x)=\delta_{m,n}.
\tag{6.27}
$$

The completeness relation (6.26) can be also obtained from the sum formula Mehler [4] (Chapter 10.13. Equation (22))

$$
\sum_{n=0}^{\infty}\frac{t^n}{2^n n!}\mathrm{H}_n(x)\mathrm{H}_n(y)=\frac{1}{\sqrt{1-t^2}}\exp\left(\frac{2txy-t^2\left(x^2+y^2\right)}{1-t^2}\right),
\tag{6.28}
$$

[7]In analogous way follows, for example, from the well-known orthonormality relations for the Legendre polynomials $\mathrm{P}_n(x)$ $\int_{-1}^{+1}\mathrm{d}x\mathrm{p}_m(x)\mathrm{p}_n(x)=\delta_{m,n}$, $\mathrm{p}_n(x)\equiv\sqrt{n+\frac{1}{2}}\mathrm{P}_n(x)$, the completeness relation $\sum_{n=0}^{\infty}\mathrm{p}_n(x)\mathrm{p}_n(y)=\delta(x-y)$, $-1\le(x,y)\le 1$, in the interval from -1 to $+1$.

by the limiting transition $t \to +1$ which with the abbreviation $1-t^2 \equiv \varepsilon^2 \Rightarrow$ $t=\sqrt{1-\varepsilon^2} \approx 1-\frac{\varepsilon^2}{2}+o\left(\varepsilon^4\right)$ on the right-hand side leads to

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^n n!} \mathrm{H}_n(x) \mathrm{H}_n(y) &= \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon^2}} \exp\left(\frac{2\left(1-\frac{\varepsilon^2}{2}-\cdots\right)xy-\left(1-\varepsilon^2\right)\left(x^2+y^2\right)}{\varepsilon^2}\right) \\
&= \exp\left(-xy+x^2+y^2\right) \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon^2}} \exp\left(-\frac{(x-y)^2}{\varepsilon^2}\right) \\
&= \exp\left(\frac{x^2+y^2}{2}\right) \sqrt{\pi}\,\delta(x-y),
\end{aligned} \tag{6.29}$$

and where one has to bring the factor $\exp\left(\frac{x^2+y^2}{2}\right)\sqrt{\pi}$ to the left-hand side of the Equation (6.29) to obtain (6.26).

6.6. Limiting Case $a \equiv \varepsilon \to 0$

In the limiting case $a \equiv \varepsilon \to 0$ function $\frac{1}{\sqrt{\pi\varepsilon^2}} \exp\left(-\frac{x^2}{\varepsilon^2}\right)$ goes to a delta function according to

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi\varepsilon^2}} \left(-\frac{x^2}{\varepsilon^2}\right) = \delta(x). \tag{6.30}$$

For the integral (5.1) then follows

$$\begin{aligned}
&\lim_{\varepsilon \to 0} f\left(m,n;\varepsilon,b,c;x_0,x_1,x_2\right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi\varepsilon^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{\left(x-x_0\right)^2}{\varepsilon^2}\right) \mathrm{H}_m\left(\frac{x-x_1}{b}\right) \mathrm{H}_n\left(\frac{x-x_2}{c}\right) \\
&= \mathrm{H}_m\left(\frac{x_0-x_1}{b}\right) \mathrm{H}_n\left(\frac{x_0-x_2}{c}\right).
\end{aligned} \tag{6.31}$$

In connection with (6.30) we mention that the derivatives of the delta function may be represented by

$$\begin{aligned}
\delta^{(n)}(x) &= \frac{\partial^n}{\partial x^n} \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi\varepsilon^2}} \exp\left(-\frac{x^2}{\varepsilon^2}\right) = \lim_{\varepsilon \to 0} \frac{\partial^n}{\partial x^n} \frac{1}{\sqrt{\pi\varepsilon^2}} \exp\left(-\frac{x^2}{\varepsilon^2}\right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi\varepsilon^2}} \exp\left(-\frac{x^2}{\varepsilon^2}\right) \frac{(-1)^n}{\varepsilon^n} \mathrm{H}_n\left(\frac{x}{\varepsilon}\right),
\end{aligned} \tag{6.32}$$

where the Rodrigues-type definition (2.1) of the Hermite polynomials is used.

6.7. Extension of the Integral to Inclusion of Cosine or Sine Functions

One can make an extension of the quite general integral (5.1) by substituting the shift x_0 from the coordinate origin in the Gaussian distribution by $x_0 \rightarrow x_0 \pm \mathrm{i}y_0$ leading to

$$\begin{aligned}
&\exp\left(-\frac{y_0^2 \pm \mathrm{i}2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 \pm \mathrm{i}y_0,x_1,x_2\right) \\
&= \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{\left(x-x_0\right)^2}{a^2}\right) \exp\left(\pm \mathrm{i}\frac{2y_0x}{a^2}\right) \mathrm{H}_m\left(\frac{x-x_1}{b}\right) \mathrm{H}_n\left(\frac{x-x_2}{c}\right),
\end{aligned} \tag{6.33}$$

with the result

$$\begin{aligned}
&\exp\left(-\frac{y_0^2 \pm \mathrm{i}2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 \pm \mathrm{i}y_0,x_1,x_2\right) \\
&= \exp\left(-\frac{y_0^2 \pm \mathrm{i}2x_0y_0}{a^2}\right) \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\frac{2a^2}{bc}\right)^j \\
&\cdot \left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j} \mathrm{H}_{m-j}\left(\frac{x_0 \pm \mathrm{i}y_0 - x_1}{\sqrt{b^2-a^2}}\right) \left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n-j} \mathrm{H}_{n-j}\left(\frac{x_0 \pm \mathrm{i}y_0 - x_2}{\sqrt{c^2-a^2}}\right).
\end{aligned} \tag{6.34}$$

From this one finds

$$\begin{aligned}
&\frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{\left(x-x_0\right)^2}{a^2}\right) \left(\begin{matrix} \cos\left(\frac{2y_0x}{a^2}\right) \\ \sin\left(\frac{2y_0x}{a^2}\right) \end{matrix}\right) \mathrm{H}_m\left(\frac{x-x_1}{b}\right) \mathrm{H}_n\left(\frac{x-x_2}{c}\right) \\
&= \exp\left(-\frac{y_0^2}{a^2}\right) \left(\frac{1}{2}\left\{\exp\left(+\mathrm{i}\frac{2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 + \mathrm{i}y_0,x_1,x_2\right)\right.\right. \\
&\quad \left. + \exp\left(-\mathrm{i}\frac{2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 - \mathrm{i}y_0,x_1,x_2\right)\right\} \\
&\quad -\frac{\mathrm{i}}{2}\left\{\exp\left(+\mathrm{i}\frac{2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 + \mathrm{i}y_0,x_1,x_2\right)\right. \\
&\quad \left.\left. - \exp\left(-\mathrm{i}\frac{2x_0y_0}{a^2}\right) f\left(m,n;a,b,c;x_0 - \mathrm{i}y_0,x_1,x_2\right)\right\}\right),
\end{aligned} \tag{6.35}$$

where the functions $f\left(m,n;a,b,c;x_0 \pm \mathrm{i}y_0,x_1,x_2\right)$ must be taken from (5.2) and are not explicitly written down here within the big round brackets due to their lengths.

7. Alternative Formula and Its Derivation for the Quite General Gauss-Hermite Integral by a Double Sum

In this section we derive an alternative formula for the Gauss-Hermite integral (5.1) with an essentially different form of the solution in comparison to (5.2). The solution is in the form of a double sum.

Starting from the second representation of the integral in (5.1) and using the Formula (2.7) for the Hermite polynomials for the sum of two arguments as variable one finds

$$
\begin{aligned}
&f\left(m,n;a,b,c;x_0,x_1,x_2\right)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\\
&=\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{m!n!}{k!(m-k)!l!(n-l)!}\mathrm{H}_{m-k}\left(\frac{x_0-x_1}{b}\right)\mathrm{H}_{n-l}\left(\frac{x_0-x_2}{c}\right)\\
&\quad\cdot\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{x^2}{a^2}\right)\left(\frac{2x}{b}\right)^k\left(\frac{2x}{c}\right)^l\\
&=\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{m!n!}{k!(m-k)!l!(n-l)!}\mathrm{H}_{m-k}\left(\frac{x_0-x_1}{b}\right)\mathrm{H}_{n-l}\left(\frac{x_0-x_2}{c}\right)\\
&\quad\cdot\frac{1+(-1)^{k+l}}{2}\frac{\left(\frac{k+l-1}{2}\right)!}{\left(-\frac{1}{2}\right)!}\frac{(2a)^{k+l}}{b^kc^l},\quad\left(\left(-\frac{1}{2}\right)!=\sqrt{\pi}\right),
\end{aligned}
\tag{7.1}
$$

where a well-known integral was used for the evaluation. Due to factor $\frac{1}{2}\left(1+(-1)^{k+l}\right)$ which is equal to 1 for even $k+l$ and 0 for odd $k+l$ and is caused by the symmetry of the integral one may exclude the terms with odd $k+l$ by the substitution $k+l=2j$ and obtains from (7.1) the following result

$$
\begin{aligned}
&f\left(m,n;a,b,c;x_0,x_1,x_2\right)\\
&=\sum_{k=0}^{m}\frac{m!n!}{k!(m-k)!}\frac{1}{b^k}\mathrm{H}_{m-k}\left(\frac{x_0-x_1}{b}\right)\\
&\quad\cdot\sum_{j=\left[\frac{k}{2}\right]}^{\left[\frac{n+k}{2}\right]}\frac{\left(j-\frac{1}{2}\right)!}{(2j-k)!(n+k-2j)!\left(-\frac{1}{2}\right)!}\frac{\left(4a^2\right)^j}{c^{2j-k}}\mathrm{H}_{n+k-2j}\left(\frac{x_0-x_2}{c}\right).
\end{aligned}
\tag{7.2}
$$

This form of the solution for the integral is a more inconvenient double sum in comparison with the simple sum in (5.2). Furthermore, it does not show in obvious way the symmetry (5.3) of the integral and, therefore, is unattractive. An easy direct transformation of this double sum into the simple sum (5.2) seems to be hopeless. We have numerically checked the identity of (7.2) with (5.2).

8. Transition to Multi-Dimensional Integrals with Gaussian Functions

Integrals over Gaussian distributions play an important role in different branches of natural sciences, in particular, in quantum optics (e.g., squeezing of light modes in phase space). My scientific colleague Hong-yi Fan from China applied them in almost all of his papers up to the millennium turn and later in a method of integration which he abbreviated IWOP (Integration within ordered products) and which is roughly speaking the possibility of interchange of the order of operator ordering and of integration and was very successful with it [12]. For this purpose one needs a great supply of multi-dimensional integrals with Gaussian functions.

In the following we consider a few multi-dimensional generalizations of integrals with Gaussian distributions.

In Section 4 we defined the one-dimensional Fourier transformation $\tilde{f}(u)$ of functions $f(x)$ by (4.1). It is well known and easily to see that under this choice of the factors in the Fourier transformation the convolution of two functions $g(x)*h(x)$ makes the transition into the product of the Fourier transforms $\tilde{g}(u)\tilde{h}(u)$ of the two functions $g(x)$ and $h(x)$ that is

$$f(x) = g(x)*h(x) \equiv \int_{-\infty}^{+\infty} \mathrm{d}y\, g(x-y)h(y) \;\Leftrightarrow\; \tilde{f}(u) = \tilde{g}(u)\tilde{h}(u). \tag{8.1}$$

In special case of two normalized Gaussian functions

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{\pi a^2}}\exp\left(-\frac{x^2}{a^2}\right),\; h(x) = \frac{1}{\sqrt{\pi b^2}}\exp\left(-\frac{x^2}{b^2}\right) \\ \Leftrightarrow\; \tilde{g}(u) &= \exp\left(-\frac{a^2u^2}{4}\right),\; \tilde{h}(u) = \exp\left(-\frac{b^2u^2}{4}\right), \end{aligned} \tag{8.2}$$

this leads to (e.g., Vladimirov [13], p. 150)

$$\tilde{f}(u) = \exp\left(-\frac{(a^2+b^2)u^2}{4}\right) \;\Leftrightarrow\; f(x) = \frac{1}{\sqrt{\pi(a^2+b^2)}}\exp\left(-\frac{x^2}{a^2+b^2}\right). \tag{8.3}$$

In the following this will be generalized to n dimensions and to shift parameter in the Gaussian function.

The n-dimensional Fourier transform $\tilde{f}(\boldsymbol{k})$ of a function $f(\boldsymbol{r})$ we define in analogy to (4.1) by

$$\tilde{f}(\boldsymbol{k}) = \int_V \mathrm{d}^n r\, f(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{kr}),\; f(\boldsymbol{r}) = \frac{1}{(2\pi)^n}\int_{\tilde{V}} \mathrm{d}^n k\, \tilde{f}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{kr}). \tag{8.4}$$

The integrals go over the whole volume V of n-dimensional linear space of vectors $\boldsymbol{r}$ or volume $\tilde{V}$ of dual space of co-vectors $\boldsymbol{k}$, respectively (usually Euclidean spaces). As in one-dimensional case (8.1) the convolution of two functions $g(\boldsymbol{r})$ and $h(\boldsymbol{r})$ makes then the transition into the product of the Fourier transforms $\tilde{g}(\boldsymbol{k})$ and $\tilde{h}(\boldsymbol{k})$ of these functions

$$f(\boldsymbol{r}) = g(\boldsymbol{r})*h(\boldsymbol{r}) \equiv \int_V \mathrm{d}^n s\, g(\boldsymbol{r}-\boldsymbol{s})h(\boldsymbol{s}) \;\Leftrightarrow\; \tilde{f}(\boldsymbol{k}) = \tilde{g}(\boldsymbol{k})\tilde{h}(\boldsymbol{k}). \tag{8.5}$$

We consider a normalized n-dimensional Gaussian function shifted from the coordinate origin with vector parameter $\boldsymbol{r}_0$

$$f(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^n |\mathsf{A}|}}\exp\left(-(\boldsymbol{r}-\boldsymbol{r}_0)\mathsf{A}^{-1}(\boldsymbol{r}-\boldsymbol{r}_0)\right)\exp(\mathrm{i}\boldsymbol{kr}),\; \int_V \mathrm{d}^n r\, f(\boldsymbol{r}) = 1, \tag{8.6}$$

where A in an Euclidian space is a symmetric second-order tensor equivalent to an operator and $|\mathsf{A}|$ its determinant[8]. The value $\boldsymbol{r}_0$ of the shift vector does not possess an influence onto the value of the normalization integral. Its Fourier transform is (compare [13], p. Chapter II, & 9, p. 172)

[8]In coordinates: $\boldsymbol{r}\mathsf{A}\boldsymbol{r} \to r^k A_{kl} r^l$ with $\mathsf{A} = \mathsf{A}^{\mathrm{T}} \Rightarrow A_{kl} = A_{lk}$ a symmetric two-rank tensor to a quadratic form.

$$
\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{\pi^n |\mathsf{A}|}} \int_V \mathrm{d}^n r \exp\left(-(\boldsymbol{r}-\boldsymbol{r}_0)\mathsf{A}^{-1}(\boldsymbol{r}-\boldsymbol{r}_0)\right)\exp(-\mathrm{i}\boldsymbol{kr}) \\
&= \exp\left(-\frac{\boldsymbol{k}\mathsf{A}\boldsymbol{k}}{4}\right)\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_0) \\
&\quad \cdot \frac{1}{\sqrt{\pi^n |\mathsf{A}|}} \int_V \mathrm{d}^n r \exp\left\{-\left(\boldsymbol{r}-\boldsymbol{r}_0+\frac{\mathrm{i}}{2}\boldsymbol{k}\mathsf{A}\right)\mathsf{A}^{-1}\left(\boldsymbol{r}-\boldsymbol{r}_0+\frac{\mathrm{i}}{2}\mathsf{A}\boldsymbol{k}\right)\right\} \\
&= \exp\left(-\frac{\boldsymbol{k}\mathsf{A}\boldsymbol{k}}{4}-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_0\right).
\end{aligned}
\tag{8.7}
$$

Such as in one-dimensional case the Fourier transform of a normalized centralized Gaussian distribution ($\boldsymbol{r}_0=\boldsymbol{0}$) is again centralized Gaussian distribution and the presence of a shift vector $\boldsymbol{r}_0$ multiplies this Fourier transform by a phase factor.

We now consider two normalized n-dimensional Gaussian functions displaced from coordinate origin

$$
\begin{aligned}
g(\boldsymbol{r}) &= \frac{1}{\sqrt{\pi^n |\mathsf{A}|}} \exp\left(-(\boldsymbol{r}-\boldsymbol{r}_0)\mathsf{A}^{-1}(\boldsymbol{r}-\boldsymbol{r}_0)\right)\exp(\mathrm{i}\boldsymbol{kr}), \quad \int_V \mathrm{d}^n r\, g(\boldsymbol{r}) = 1, \\
h(\boldsymbol{r}) &= \frac{1}{\sqrt{\pi^n |\mathsf{B}|}} \exp\left(-(\boldsymbol{r}-\boldsymbol{s}_0)\mathsf{B}^{-1}(\boldsymbol{r}-\boldsymbol{s}_0)\right)\exp(\mathrm{i}\boldsymbol{kr}), \quad \int_V \mathrm{d}^n r\, g(\boldsymbol{r}) = 1.
\end{aligned}
\tag{8.8}
$$

The Fourier transforms of these functions according to (8.7) are

$$
\tilde{g}(\boldsymbol{k}) = \exp\left(-\frac{\boldsymbol{k}\mathsf{A}\boldsymbol{k}}{4}-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_0\right), \quad \tilde{h}(\boldsymbol{k}) = \exp\left(-\frac{\boldsymbol{k}\mathsf{B}\boldsymbol{k}}{4}-\mathrm{i}\boldsymbol{k}\boldsymbol{s}_0\right),
\tag{8.9}
$$

with their product

$$
\tilde{f}(\boldsymbol{k}) = \tilde{g}(\boldsymbol{k})\tilde{h}(\boldsymbol{k}) = \exp\left(-\frac{\boldsymbol{k}(\mathsf{A}+\mathsf{B})\boldsymbol{k}}{4}-\mathrm{i}\boldsymbol{k}(\boldsymbol{r}_0+\boldsymbol{s}_0)\right).
\tag{8.10}
$$

The back translation to the convolution of the two functions $g(\boldsymbol{r})$ and $h(\boldsymbol{r})$ is now very simple and provides the value of the integral as follows

$$
\begin{aligned}
f(\boldsymbol{r}) &= g(\boldsymbol{r}) * h(\boldsymbol{r}) \\
&= \frac{1}{\pi^n \sqrt{|\mathsf{A}||\mathsf{B}|}} \int_V \mathrm{d}^n s \exp\left(-(\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{s})\mathsf{A}^{-1}(\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{s})\right) \\
&\quad \cdot \exp\left(-(\boldsymbol{s}-\boldsymbol{s}_0)\mathsf{B}^{-1}(\boldsymbol{s}-\boldsymbol{s}_0)\right) \\
&= \frac{1}{\sqrt{\pi^n |\mathsf{A}+\mathsf{B}|}} \exp\left(-(\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{s}_0)(\mathsf{A}+\mathsf{B})^{-1}(\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{s}_0)\right),
\end{aligned}
\tag{8.11}
$$

and in special case without argument shifts of the Gaussian functions

$$
\begin{aligned}
&\frac{1}{\pi^n \sqrt{|\mathsf{A}||\mathsf{B}|}} \int_V \mathrm{d}^n s \exp\left(-(\boldsymbol{r}-\boldsymbol{s})\mathsf{A}^{-1}(\boldsymbol{r}-\boldsymbol{s})\right)\exp\left(-\boldsymbol{s}\mathsf{B}^{-1}\boldsymbol{s}\right) \\
&= \frac{1}{\sqrt{\pi^n |\mathsf{A}+\mathsf{B}|}} \exp\left(-\boldsymbol{r}(\mathsf{A}+\mathsf{B})^{-1}\boldsymbol{r}\right).
\end{aligned}
\tag{8.12}
$$

The expressions for the determinant and for the inverse of the sum of two operators A and B by the separate operators are specific for the dimension n and are in coordinate-invariant way:

1) Two-dimensional case: ($|\mathsf{A}| \rightarrow [\mathsf{A}]$ is determinant, I is two-dimensional identity operator, $\overline{\mathsf{A}}$ is complementary operator to A)

$$
\begin{gathered}
[\mathsf{A}+\mathsf{B}] = [\mathsf{A}] + \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}\rangle + [\mathsf{B}], \\
(\mathsf{A}+\mathsf{B})^{-1} = \frac{\overline{\mathsf{A}+\mathsf{B}}}{[\mathsf{A}+\mathsf{B}]} = \frac{(\langle\mathsf{A}\rangle+\langle\mathsf{B}\rangle)\mathsf{I}-(\mathsf{A}+\mathsf{B})}{[\mathsf{A}]+\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle-\langle\mathsf{AB}\rangle+[\mathsf{B}]}, \\
\overline{\mathsf{A}} = \langle\mathsf{A}\rangle\mathsf{I}-\mathsf{A},\ \langle\overline{\mathsf{A}}\rangle = \langle\mathsf{A}\rangle,\ [\overline{\mathsf{A}}] = [\mathsf{A}],
\end{gathered} \tag{8.13}
$$

2) Three-dimensional case: ($|\mathsf{A}|$ is determinant, I is three-dimensional identity operator, $\overline{\mathsf{A}}$ is complementary operator to A)

$$
\begin{aligned}
|\mathsf{A}+\mathsf{B}| &= |\mathsf{A}| + \langle\overline{\mathsf{A}}\mathsf{B}\rangle + \langle\mathsf{A}\overline{\mathsf{B}}\rangle + |\mathsf{B}|, \\
(\mathsf{A}+\mathsf{B})^{-1} &= \frac{\overline{\mathsf{A}+\mathsf{B}}}{|\mathsf{A}+\mathsf{B}|} \\
&= \frac{\overline{\mathsf{A}}+\mathsf{AB}+\mathsf{BA}+\overline{\mathsf{B}}-(\langle\mathsf{B}\rangle\mathsf{A}+\langle\mathsf{A}\rangle\mathsf{B})+(\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle-\langle\mathsf{AB}\rangle)\mathsf{I}}{|\mathsf{A}|+\langle\overline{\mathsf{A}}\mathsf{B}\rangle+\langle\mathsf{A}\overline{\mathsf{B}}\rangle+|\mathsf{B}|}, \\
\overline{\mathsf{A}} &= [\mathsf{A}]\mathsf{I}-\langle\mathsf{A}\rangle\mathsf{A}+\mathsf{A}^2,\ \langle\overline{\mathsf{A}}\rangle = [\mathsf{A}],\ [\overline{\mathsf{A}}] = |\mathsf{A}|\langle\mathsf{A}\rangle,\ |\overline{\mathsf{A}}| = |\mathsf{A}|^2.
\end{aligned} \tag{8.14}
$$

3) Four-dimensional case: ($|\mathsf{A}| \rightarrow \|\mathsf{A}\|$ is determinant, I is four-dimensional identity operator, $\overline{\mathsf{A}}$ is complementary operator to A)

$$
\begin{aligned}
\|\mathsf{A}+\mathsf{B}\| &= \|\mathsf{A}\| + \langle\overline{\mathsf{A}}\mathsf{B}\rangle + [\mathsf{A}][\mathsf{B}] - \left\langle\left([\mathsf{A}]\mathsf{I}-\langle\mathsf{A}\rangle\mathsf{A}+\mathsf{A}^2\right)\left([\mathsf{B}]\mathsf{I}-\langle\mathsf{B}\rangle\mathsf{B}+\mathsf{B}^2\right)\right\rangle \\
&\quad + [\mathsf{AB}] + \langle\mathsf{A}\overline{\mathsf{B}}\rangle + \|\mathsf{B}\|, \\
(\mathsf{A}+\mathsf{B})^{-1} &= \frac{\overline{\mathsf{A}+\mathsf{B}}}{\|\mathsf{A}+\mathsf{B}\|} \\
&= \frac{|\mathsf{A}+\mathsf{B}|\mathsf{I}-[\mathsf{A}+\mathsf{B}](\mathsf{A}+\mathsf{B})+\langle\mathsf{A}+\mathsf{B}\rangle(\mathsf{A}+\mathsf{B})^2-(\mathsf{A}+\mathsf{B})^3}{\|\mathsf{A}+\mathsf{B}\|}, \\
\overline{\mathsf{A}} &= |\mathsf{A}|\mathsf{I}-[\mathsf{A}]\mathsf{A}+\langle\mathsf{A}\rangle\mathsf{A}^2-\mathsf{A}^3,\quad \langle\overline{\mathsf{A}}\rangle = |\mathsf{A}|,\quad [\overline{\mathsf{A}}] = \|\mathsf{A}\|[\mathsf{A}], \\
&\qquad |\overline{\mathsf{A}}| = \|\mathsf{A}\|^2\langle\mathsf{A}\rangle,\quad \|\overline{\mathsf{A}}\| = \|\mathsf{A}\|^3.
\end{aligned} \tag{8.15}
$$

The formula for the complementary operator $\overline{\mathsf{A}+\mathsf{B}}$ to the sum of two general operators $\mathsf{A}+\mathsf{B}$ written more in detail in four-dimensional case and in coordinate-invariant form is already quite complicated but we write it down to give an impression. It takes on 4 lines as follows

$$
\begin{aligned}
\overline{\mathsf{A}+\mathsf{B}} = \overline{\mathsf{A}} &+ \left\langle\left([\mathsf{A}]\mathsf{I}-\langle\mathsf{A}\rangle\mathsf{A}+\mathsf{A}^2\right)\mathsf{B}\right\rangle - \left(\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle-\langle\mathsf{AB}\rangle\right)\mathsf{A} - [\mathsf{A}]\mathsf{B} \\
&+ \langle\mathsf{B}\rangle\mathsf{A}^2 + \langle\mathsf{A}\rangle(\mathsf{AB}+\mathsf{BA}) - \left(\mathsf{A}^2\mathsf{B}+\mathsf{ABA}+\mathsf{BA}^2\right) \\
&- \left(\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle-\langle\mathsf{AB}\rangle\right)\mathsf{B} - [\mathsf{B}]\mathsf{A} + \langle\mathsf{A}\rangle\mathsf{B}^2 + \langle\mathsf{B}\rangle(\mathsf{BA}+\mathsf{AB}) \\
&- \left(\mathsf{B}^2\mathsf{A}+\mathsf{BAB}+\mathsf{AB}^2\right) + \left\langle\mathsf{A}\left([\mathsf{B}]\mathsf{I}-\langle\mathsf{B}\rangle\mathsf{B}+\mathsf{B}^2\right)\right\rangle + \overline{\mathsf{B}},
\end{aligned} \tag{8.16}
$$

and the danger to make errors becomes fastly increasing with the dimension. From the four-dimensional case on the coordinate-invariant calculations depending on the problem can become already immense and, moreover, corresponding coordinate calculations become almost impossible. The way out of the dilemma is then only the computer which, however, up to now calculates only with coordinate representations of the operators.

The invariants of operators are formally independent from the dimension and reflect the cycle structure of permutations of the symmetric group S_n of the $n!$ elements together with their signs

$$
\begin{aligned}
\langle \mathsf{A} \rangle &\equiv \mathrm{trace}(\mathsf{A}) = A_i^i, \\
[\mathsf{A}] &= \frac{1}{2!}\left(\langle \mathsf{A} \rangle^2 - \langle \mathsf{A}^2 \rangle\right), \\
|\mathsf{A}| &= \frac{1}{3!}\left(\langle \mathsf{A} \rangle^3 - 3\langle \mathsf{A} \rangle\langle \mathsf{A}^2 \rangle + 2\langle \mathsf{A}^3 \rangle\right), \\
\|\mathsf{A}\| &= \frac{1}{4!}\left(\langle \mathsf{A} \rangle^4 - 6\langle \mathsf{A} \rangle^2\langle \mathsf{A}^2 \rangle + 3\langle \mathsf{A}^2 \rangle^2 + 8\langle \mathsf{A} \rangle\langle \mathsf{A}^3 \rangle - 6\langle \mathsf{A}^4 \rangle\right),
\end{aligned}
\tag{8.17}
$$

but the operators themselves with their invariants become more complicated with increasing dimension and the operators $\mathsf{I} \equiv \mathsf{A}^0$ are the identity operators in the considered dimension. Though last formulae are already hardly complicate this, nevertheless, is much simpler than to write them down in coordinate representation.

9. Two-Dimensional Case of Integrals with Gaussian Distributions in Representation by Complex Conjugated Variables

The two-dimensional case of integrals over Gaussian distributions plays a role in quantum optics of phase-space variables of one mode of a harmonic oscillator. Its treatment possesses some specifics and it is favorable to deal with them by introduction of a pair of complex-conjugate variables. The extension to multi-modes of n harmonic oscillators is then possible by introduction of n pairs of complex-conjugate variables. Instead of Hermite polynomials in one-dimensional case a main role play here Laguerre polynomials which are best introduced as Laguerre 2D polynomials.

Special Laguerre-2D polynomials were introduced as follows (e.g., [8]-[10])

$$
\begin{aligned}
\mathrm{L}_{m,n}\left(z,z^*\right) &\equiv (-1)^{m+n}\exp\left(zz^*\right)\frac{\partial^{m+n}}{\partial z^{*m}\partial z^n}\exp\left(-zz^*\right) \\
&= \exp\left(-\frac{\partial^2}{\partial z\partial z^*}\right)z^m z^{*n} \\
&= \sum_{j=0}^{\{m,n\}}\frac{(-1)^j m!n!}{j!(m-j)!(n-j)!}z^{m-j}z^{*n-j}.
\end{aligned}
\tag{9.1}
$$

They are related to usual generalized Laguerre polynomials $\mathrm{L}_n^{\nu}(z)$ by

$$\begin{aligned} \mathrm{L}_{m,n}\left(z,z^*\right) &= \left(-1\right)^n n!z^{m-n}\mathrm{L}_n^{m-n}\left(zz^*\right) \\ &= \left(-1\right)^m m!z^{*n-m}\mathrm{L}_m^{n-m}\left(zz^*\right). \end{aligned} \tag{9.2}$$

The first definition in (9.1) corresponds to the Rodrigues-type definition (2.1) and the second to the alternative definition (2.3) of Hermite polynomials.

A first basic integral over a two-dimensional Gaussian distribution and powers of the complex conjugate variables $\left(z,z^*\right)$ is

$$\begin{aligned} &\frac{1}{\pi a^2}\int \frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^* \exp\left(-\frac{\left(z-w\right)\left(z^*-w^*\right)}{a^2}\right)z^m z^{*n} \\ &= \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} a^{2j} w^{m-j} w^{*n-j} \\ &= \left(-\mathrm{i}a\right)^{m+n} \mathrm{L}_{m,n}\left(\mathrm{i}\frac{w}{a},\mathrm{i}\frac{w^*}{a}\right), \end{aligned} \tag{9.3}$$

where $\frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^*$ is the area element over the 2-dimensional complex space

$$z = x+\mathrm{i}y,\;\; z^* = x-\mathrm{i}y,\;\; \frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^* = \frac{\mathrm{i}}{2}\left(\mathrm{d}x+\mathrm{i}\mathrm{d}y\right)\wedge\left(\mathrm{d}x-\mathrm{i}\mathrm{d}y\right) = \mathrm{d}x\wedge \mathrm{d}y. \tag{9.4}$$

A more general two-dimensional integral in complex coordinates in comparison to (9.3) is

$$\begin{aligned} &\frac{1}{\pi a^2}\int \frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^* \exp\left(-\frac{\left(z-w\right)\left(z^*-w^*\right)}{a^2}\right)\mathrm{L}_{m,n}\left(\frac{z}{b},\frac{z^*}{b}\right) \\ &= \left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m+n} \mathrm{L}_{m,n}\left(\frac{w}{\sqrt{b^2-a^2}},\frac{w^*}{\sqrt{b^2-a^2}}\right). \end{aligned} \tag{9.5}$$

The proof of this integral can be made as follows

$$\begin{aligned} &\frac{1}{\pi a^2}\int \frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^* \exp\left(-\frac{\left(z-w\right)\left(z^*-w^*\right)}{a^2}\right)\mathrm{L}_{m,n}\left(\frac{z}{b},\frac{z^*}{b}\right) \\ &= \frac{1}{\pi a^2}\int \frac{\mathrm{i}}{2}\mathrm{d}z\wedge \mathrm{d}z^* \exp\left(-\frac{\left(w-z\right)\left(w^*-z^*\right)}{a^2}\right)\mathrm{L}_{m,n}\left(\frac{z}{b},\frac{z^*}{b}\right) \\ &= \exp\left(a^2\frac{\partial^2}{\partial w\partial w^*}\right)\mathrm{L}_{m,n}\left(\frac{w}{b},\frac{w^*}{b}\right) \\ &= \frac{1}{b^{m+n}}\exp\left(-\left(b^2-a^2\right)\frac{\partial^2}{\partial w\partial w^*}\right)w^m w^{*n} \\ &= \left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m+n} \mathrm{L}_{m,n}\left(\frac{w}{\sqrt{b^2-a^2}},\frac{w^*}{\sqrt{b^2-a^2}}\right). \end{aligned} \tag{9.6}$$

The second line is written down to show that this integral may be considered as a convolution.

10. Conclusion

Starting from two alternative definitions of Hermite polynomials, we derived a new relation which connects these polynomials with arbitrarily stretched arguments. Then there are derived some operational identities with the variable and its differential operator in Hermite polynomials polynomials disentangled to their normally ordered form. The main purpose was the evaluation of a quite general integral (5.1) with a generally shifted Gaussian distribution and the product of two Hermite polynomials with general linear arguments with the result (5.2). It generalizes almost all special cases of such integrals considered in the best-known tables of integrals. The further sections of this paper deal with multi-dimensional integrals with Gaussian distributions and partially with Hermite or Laguerre 2D polynomials. The proofs are mostly made by complete induction and are placed in the Appendices. The two-dimensional case of integrals over Gaussian distributions multiplied by Laguerre 2D polynomials is not nearly as general as it was achieved with the Hermite polynomials and one has to think about a good representation of these last cases.

It is intended to use a special case of the derived integral with Hermite polynomials in a paper of quantum optics in preparation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: Proofs of Two Operational Identities

In this Appendix, we prove first the operational identity

$$\left(2x-\frac{\partial}{\partial x}\right)^{n}=\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\mathrm{H}_{n-j}(x)\frac{\partial^{j}}{\partial x^{j}}. \tag{A.1}$$

by complete induction using the differentiation Formula (2.4) and the recurrence relation (2.5) for Hermite polynomials.

The theorem (A.1) is true for $n=0$ (and obviously $n=1$) as initial term of the complete induction. Assuming that it is true for arbitrary n one proves that it is also true for $n+1$ as follows

$$\begin{aligned}
\left(2x-\frac{\partial}{\partial x}\right)^{n+1}&=\left(2x-\frac{\partial}{\partial x}\right)\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\mathrm{H}_{n-j}(x)\frac{\partial^{j}}{\partial x^{j}}\\
&=\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\left(2x\mathrm{H}_{n-j}(x)-\frac{\partial}{\partial x}\mathrm{H}_{n-j}(x)\right)\frac{\partial^{j}}{\partial x^{j}}\\
&=\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\left(2x\mathrm{H}_{n-j}(x)-2(n-j)\mathrm{H}_{n-1-j}(x)-\mathrm{H}_{n-j}(x)\frac{\partial}{\partial x}\right)\frac{\partial^{j}}{\partial x^{j}}\\
&=\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\mathrm{H}_{n+1-j}(x)\frac{\partial^{j}}{\partial x^{j}}-\sum_{j=0}^{n}\frac{(-1)^{j}\,n!}{j!(n-j)!}\mathrm{H}_{n-j}(x)\frac{\partial^{j+1}}{\partial x^{j+1}}\quad |\, j\to j-1\\
&=\sum_{j=0}^{n+1}\frac{(-1)^{j}\,n!(n+1-j)}{j!(n+1-j)!}\mathrm{H}_{n+1-j}(x)\frac{\partial^{j}}{\partial x^{j}}+\sum_{j=0}^{n+1}\frac{(-1)^{j}\,n!\,j}{j!(n+1-j)!}\mathrm{H}_{n+1-j}(x)\frac{\partial^{j}}{\partial x^{j}}\\
&=\sum_{j=0}^{n+1}\frac{(-1)^{j}(n+1)!}{j!(n+1-j)!}\mathrm{H}_{n+1-j}(x)\frac{\partial^{j}}{\partial x^{j}}.
\end{aligned} \tag{A.2}$$

In one sum term is made a substitution of the summation index j such as shown.

We prove now by complete induction a second operational identity (3.12)

$$\mathrm{H}_{n}\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2x)^{n-j}\frac{\partial^{j}}{\partial x^{j}}. \tag{A.3}$$

It is easily to see that it is true for $n=0,1,2$. Assuming that it is correct up to a certain n one obtains for $n+1$

$$\begin{aligned}
\mathrm{H}_{n+1}\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)&=2\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)\mathrm{H}_{n}\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)-2n\mathrm{H}_{n-1}\left(x+\frac{1}{2}\frac{\partial}{\partial x}\right)\\
&=2x\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2x)^{n-j}\frac{\partial^{j}}{\partial x^{j}}+\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\frac{\partial}{\partial x}(2x)^{n-j}\frac{\partial^{j}}{\partial x^{j}}\\
&\quad-2n\sum_{j=0}^{n-1}\frac{(n-1)!}{j!(n-1-j)!}(2x)^{n-1-j}\frac{\partial^{j}}{\partial x^{j}}\\
&=\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2x)^{n+1-j}\frac{\partial^{j}}{\partial x^{j}}+2\sum_{j=0}^{n-1}\frac{n!}{j!(n-1-j)!}(2x)^{n-1-j}\frac{\partial^{j}}{\partial x^{j}}\\
&\quad+\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(2x)^{n-j}\frac{\partial^{j+1}}{\partial x^{j+1}}-2n\sum_{j=0}^{n-1}\frac{(n-1)!}{j!(n-1-j)!}(2x)^{n-1-j}\frac{\partial^{j}}{\partial x^{j}}\quad |\, j\to j-1
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{j=0}^{n+1}\frac{n!(n+1-j)}{j!(n+1-j)!}(2x)^{n+1-j}\frac{\partial^j}{\partial x^j}+\sum_{j=0}^{n+1}\frac{n!\,j}{j!(n+1-j)!}(2x)^{n+1-j}\frac{\partial^j}{\partial x^j}\\
&=\sum_{j=0}^{n+1}\frac{(n+1)!}{j!(n+1-j)!}(2x)^{n+1-j}\frac{\partial^j}{\partial x^j}.
\end{aligned}
\tag{A.4}
$$

Thus (A.3) is proved for general n.

Appendix B: Proof of a Definite Gauss-Hermite Integral by Complete Induction

In this Appendix we prove the integral

$$
\begin{aligned}
f(m,0;a,b,0;x_0,x_1,0)&\equiv\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x\exp\left(-\frac{(x-x_0)^2}{a^2}\right)\mathrm{H}_m\left(\frac{x-x_1}{b}\right)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x+x_0-x_1}{b}\right)\\
&=\left(\frac{\sqrt{b^2-a^2}}{b}\right)^m\mathrm{H}_m\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right).
\end{aligned}
\tag{B.1}
$$

by complete induction from $m\to m+1$. In view of its extension in Section 5 with proof in **Appendix C** we abbreviated it by $f(m,0;a,b,0;x_0,x_1,0)$ as special case of the more general integral $f(m,n;a,b,c;x_0,x_1,x_2)$.

Obviously, the given solution of the integral is true for $m=0,1$. Assuming that it is true for arbitrary m one has to prove that it is also true for $m+1$. Using the recurrence relation (2.5) for Hermite polynomials one derives as first step

$$
\begin{aligned}
&f(m+1,0;a,b,0;x_0,x_1,0)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\left\{2\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\right.\\
&\left.-2m\mathrm{H}_{m-1}\left(\frac{x'+x_0-x_1}{b}\right)\right\}.
\end{aligned}
\tag{B.2}
$$

Since we do not have or did not introduce the first part of the sum of two integrals in (B.1) we try to transform it by partial integration to a form of an integral with a part which vanishes as follows

$$
\begin{aligned}
&f(m+1,0;a,b,0;x_0,x_1,0)\\
&=\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\left\{-\frac{a^2}{b}\left(\frac{\partial}{\partial x'}\exp\left(-\frac{x'^2}{a^2}\right)\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\right.\\
&\left.+\exp\left(-\frac{x'^2}{a^2}\right)\left(2\left(\frac{x_0-x_1}{b}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)-2m\mathrm{H}_{m-1}\left(\frac{x'+x_0-x_1}{b}\right)\right)\right\}\\
&=-\frac{a^2}{b}\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\frac{\partial}{\partial x'}\left\{\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\right\}\\
&+\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\left\{2\left(\frac{x_0-x_1}{b}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\right.\\
&\left.-2m\left(1-\frac{a^2}{b^2}\right)\mathrm{H}_{m-1}\left(\frac{x'+x_0-x_1}{b}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x' \exp\left(-\frac{x'^2}{a^2}\right) \left\{ 2\left(\frac{x_0 - x_1}{b}\right) \mathrm{H}_m\left(\frac{x' + x_0 - x_1}{b}\right) \right. \\
&\left. -2m\left(1 - \frac{a^2}{b^2}\right) \mathrm{H}_{m-1}\left(\frac{x' + x_0 - x_1}{b}\right) \right\},
\end{aligned} \tag{B.3}
$$

where the integral of the form $\int_{-\infty}^{+\infty} \mathrm{d}x' \frac{\partial}{\partial x'} \exp\left(-\frac{x'^2}{a^2}\right) P_m(x')$ with a polynomial $P_m(x')$ of degree m vanishes at the boundaries $\pm$ infinity. In the second final step one has to apply to (B.3) the Formula (B.1) for indices m and $m-1$. Taking in addition into account

$$
\mathrm{H}_{m+1}\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) = 2\frac{x_0 - x_1}{\sqrt{b^2 - a^2}} \mathrm{H}_m\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) - 2m\mathrm{H}_{m-1}\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right), \tag{B.4}
$$

from (B.3 follows)

$$
\begin{aligned}
& f(m+1, 0; a, b, 0; x_0, x_1, 0) \\
&= \left(\frac{\sqrt{b^2 - a^2}}{b}\right)^{m+1} \left\{ 2\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) \mathrm{H}_m\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) - 2m\mathrm{H}_{m-1}\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) \right\} \\
&= \left(\frac{\sqrt{b^2 - a^2}}{b}\right)^{m+1} \mathrm{H}_{m+1}\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right).
\end{aligned} \tag{B.5}
$$

Thus it is proved that (B.1) is true also for $m+1$ supposed that it is true for a certain m.

Appendix C: Proof of a Quite General Definite Gauss-Hermite Integral by Complete Induction

In this Appendix we prove the quite general Gauss-Hermite integral (C.1)

$$
\begin{aligned}
& f(m, n; a, b, c; x_0, x_1, x_2) \\
&\equiv \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{(x - x_0)^2}{a^2}\right) \mathrm{H}_m\left(\frac{x - x_1}{b}\right) \mathrm{H}_n\left(\frac{x - x_2}{c}\right) \\
&= \frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \mathrm{d}x' \exp\left(-\frac{x'^2}{a^2}\right) \mathrm{H}_m\left(\frac{x' + x_0 - x_1}{b}\right) \mathrm{H}_n\left(\frac{x' + x_0 - x_2}{c}\right) \\
&= \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\frac{2a^2}{bc}\right)^j \left(\frac{\sqrt{b^2 - a^2}}{b}\right)^{m-j} \mathrm{H}_{m-j}\left(\frac{x_0 - x_1}{\sqrt{b^2 - a^2}}\right) \\
&\cdot \left(\frac{\sqrt{c^2 - a^2}}{c}\right)^{n-j} \mathrm{H}_{n-j}\left(\frac{x_0 - x_2}{\sqrt{c^2 - a^2}}\right),
\end{aligned} \tag{C.1}
$$

by complete induction from n to $n+1$. Due to prove of (B.1) in last Appendix B it is proved for initial value $n = 0$. Supposed it is proved for an arbitrary n one has to prove that it is also true for $n = 1$. As in Appendix B we first derive an equation for the integral $f(m, n+1; a, b, c; x_0, x_1, x_2)$ which may serve as starting point for the proper complete induction. Using the recurrence relation (2.5) for

Hermite polynomials one finds

$$
\begin{aligned}
& f\left(m,n+1;a,b,c;x_0,x_1,x_2\right) \\
& =\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_{n+1}\left(\frac{x'+x_0-x_2}{c}\right) \\
& =\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right) \\
& \quad\cdot\left\{\frac{2\left(x'+x_0-x_2\right)}{c}\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)-2n\mathrm{H}_{n-1}\left(\frac{x'+x_0-x_2}{c}\right)\right\} \\
& =-\frac{a^2}{c}\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\frac{\partial}{\partial x'}\left\{\exp\left(-\frac{x'^2}{a^2}\right)\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\right\} \\
& \quad+\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\left\{\frac{a^2}{c}\frac{\partial}{\partial x'}\left(\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\right)\right. \\
& \quad\left.+\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\left(\frac{2\left(x_0-x_2\right)}{c}\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)-2n\mathrm{H}_{n-1}\left(\frac{x'+x_0-x_2}{c}\right)\right)\right\}.
\end{aligned}
\tag{C.2}
$$

The first integral of this intermediary result vanishes at the boundaries in infinity and can be omitted. In the remaining integral one has to perform the differentiation of the product of two Hermite polynomials and then to collect analogous terms that lead to

$$
\begin{aligned}
& f\left(m,n+1;a,b,c;x_0,x_1,x_2\right) \\
& =\frac{1}{\sqrt{\pi a^2}}\int_{-\infty}^{+\infty}\mathrm{d}x'\exp\left(-\frac{x'^2}{a^2}\right)\left\{\frac{a^2}{bc}2m\mathrm{H}_{m-1}\left(\frac{x'+x_0-x_1}{b}\right)\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\right. \\
& \quad+\mathrm{H}_m\left(\frac{x'+x_0-x_1}{b}\right)\left(\frac{2\left(x_0-x_2\right)}{c}\mathrm{H}_n\left(\frac{x'+x_0-x_2}{c}\right)\right. \\
& \quad\left.\left.+\left(\frac{a^2}{c^2}-1\right)2n\mathrm{H}_{n-1}\left(\frac{x'+x_0-x_2}{c}\right)\right)\right\}.
\end{aligned}
\tag{C.3}
$$

This relation may serve as starting point for the proper complete induction. Inserting the conjectured result (C.1) into (C.3) one obtains after substitution of the summation index $j'\rightarrow j-1$

$$
\begin{aligned}
& f\left(m,n+1;a,b,c;x_0,x_1,x_2\right) \\
& =\sum_{j'=0}^{\{m-1,n\}}\frac{m!n!}{j'!(m-1-j')!(n-j')!}\left(\frac{2a^2}{bc}\right)^{j'+1} \\
& \quad\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-1-j'}\mathrm{H}_{m-1-j'}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n-j'}\mathrm{H}_{n-j'}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right) \\
& \quad+\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(\frac{2a^2}{bc}\right)^{j} \\
& \quad\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\frac{2\left(x_0-x_2\right)}{\sqrt{c^2-a^2}}\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\mathrm{H}_{n-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{j=0}^{\{m,n-1\}}\frac{m!n!}{j!(m-j)!(n-1-j)!}\left(\frac{2a^2}{bc}\right)^j\\
&\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\left(\frac{a^2}{c^2}-1\right)\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n-1-j}\mathrm{H}_{n-1-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)\\
&=\sum_{j=0}^{\{m,n\}}\frac{m!n!j}{j!(m-j)!(n+1-j)!}\left(\frac{2a^2}{bc}\right)^j\\
&\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\mathrm{H}_{n+1-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)\\
&+\sum_{j=0}^{\{m,n\}}\frac{m!n!(n+1-j)}{j!(m-j)!(n+1-j)!}\left(\frac{2a^2}{bc}\right)^j\\
&\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\frac{2(x_0-x_2)}{\sqrt{c^2-a^2}}\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\mathrm{H}_{n-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)\\
&-2\sum_{j=0}^{\{m,n-1\}}\frac{m!n!(n-j)(n+1-j)}{j!(m-j)!(n+1-j)!}\left(\frac{2a^2}{bc}\right)^j\\
&\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\mathrm{H}_{n-1-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right).
\end{aligned}
\tag{C.4}
$$

The 3 sum terms with products of two Hermite polynomials with complicated arguments take on here two lines each that makes it so difficult to write down the proof by complete induction due to its length. In principle, it is simple. By further collection of analogous sum terms in (C.4) this simplifies to

$$
\begin{aligned}
&f\left(m,n+1;a,b,c;x_0,x_1,x_2\right)\\
&=\sum_{j=0}^{\{m,n\}}\frac{m!n!(n+1)}{j!(m-j)!(n+1-j)!}\left(\frac{2a^2}{bc}\right)^j\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\\
&\cdot\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\left\{\frac{2(x_0-x_2)}{\sqrt{c^2-a^2}}\mathrm{H}_{n-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)-2(n-j)\mathrm{H}_{n-1-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right)\right\}\\
&=\sum_{j=0}^{\{m,n+1\}}\frac{m!(n+1)!}{j!(m-j)!(n+1-j)!}\left(\frac{2a^2}{bc}\right)^j\\
&\cdot\left(\frac{\sqrt{b^2-a^2}}{b}\right)^{m-j}\mathrm{H}_{m-j}\left(\frac{x_0-x_1}{\sqrt{b^2-a^2}}\right)\left(\frac{\sqrt{c^2-a^2}}{c}\right)^{n+1-j}\mathrm{H}_{n+1-j}\left(\frac{x_0-x_2}{\sqrt{c^2-a^2}}\right).
\end{aligned}
\tag{C.5}
$$

The recurrence relation for Hermite polynomials (2.5) is applied in this last transformation.

Thus, the basic result (5.2) for the evaluation of the integral (5.1) expressing its symmetry is proved. We intend to apply a special case in a following paper to quantum optics.