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New Exact Solutions for the Coupled Gross-Pitaevskii Equation

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Abstract

The coupled Gross-Pitaevskii equations are the basic model describing the phenomenon of Bose-Einstein condensation. Therefore, the research on the coupled Gross-Pitaevskii equations is very important. One of the main tasks of studying the coupled Gross-Pitaevskii equations is to obtain the exact solutions. In this paper, the study of the exact solutions of the coupled Gross-Pitaevskii equations is mainly based by using the modified polynomial expansion method and the modified traveling wave solution transformation method, assisted by computer software. Firstly, the coupled Gross-Pitaevskii equations are changed into a nonlinear coupled ordinary differential system by a coupled traveling wave solution transformation. Secondly, by using the modified polynomial expansion method, we successfully obtain more new exact elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions for the coupled Gross-Pitaevskii equations. Finally, according to the special parameter values, we show the figures for some of the exact solutions.

Keywords

The Coupled Gross-Pitaevskii Equations, The Modified Hyperbolic Function Expanding Method, The Traveling Wave Solutions, Partial Differential Equations

1. Introduction

In recent decades, the coupled nonlinear partial differential equations (CNPDE) [1]-[15] have received much more attention in plasma physics [16], nonlinear optics [17]-[19], Bose-Einstein condensation [20]-[22], biophysics [23] [24], finance [25] [26], oceanography [27] and other fields. The equations provide mathematical models for studying complex systems and help us to better understand and control these systems. In addition, by solving the equations, we can better reveal the internal laws of many phenomena. Thus, the studying of the exact solutions of CNPDE is of great significance both in theory and practice. However, the studying of the exact solutions of CNPDE is difficult.

Now, nonlinear evolution equations have been a topic of deep theoretical research, hereby, a massive number of mathematicians, physicists and engineers have attempted to invent various approaches by which one can obtain the exact solutions of such equations. In the 1880s, after the French mathematician Darboux proposed the Darboux transformation method, many effective and reliable methods for nonlinear evolution equations were proposed. For example, the Hirota bilinear method [28], the Painlevé expansion method [29] [30], Bäcklund transformation [31] [32], the Riccati equation method [33] [34], unification and general unity [35], the Exp-function method [36], the homogeneous balance method [37], the $\frac{G'}{G}$-polynomial expansion method [38]-[40], the tanh-function method [41]-[43], the Jacobian elliptic function expansion method [43]-[45], the F-expansion method [46] [47], the auxiliary equation method [48], the first integration method [49] [50], etc. Due to the complexity and diversity of partial differential equations, the research methods for the exact solutions of partial differential equations are different until now, and the methods used for different equations are also different.

In 2010, Deng-Shan Wang *et al.* proposed a similarity transformation for solving the coupled Gross-Pitaevskii equations and derived two types of explicit exact solutions, where the coupled equations are denoted as [51]

$$\begin{aligned} i\frac{\partial \psi_1}{\partial t} &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\gamma^2}{2}x^2 + b_{11}\left|\psi_1\right|^2 + b_{12}\left|\psi_2\right|^2\right)\psi_1, \\ i\frac{\partial \psi_2}{\partial t} &= \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\gamma^2}{2}x^2 + b_{21}\left|\psi_1\right|^2 + b_{22}\left|\psi_2\right|^2\right)\psi_2. \end{aligned} \tag{1}$$

In 2012, Fei Jin-Xi *et al.* employed the improved homogeneous balance principle and mapping method to derive the periodic wave solutions and solitary wave solutions of the generalized (3 + 1)-dimensional Gross-Pitaevskii equation [52]

$$i\psi_t + \frac{1}{2}\rho(t)\nabla^2\psi - g(t)\left|\psi\right|^2\psi - V(t)r^2\psi = 0, \tag{2}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ and $r^2 = x^2 + y^2 + z^2$.

In 2017, Tao Xu *et al.* extended the single-component Gross-Pitaevskii equation to a two-component coupled case with damping terms, linear and parabolic density distributions. The coupled equations are given by

$$\begin{aligned} iq_{1t} + q_{1xx} + 2\mu^2\left(\left|q_1\right|^2 + \left|q_2\right|^2\right)q_1 + \left(i\beta - \alpha x + \beta^2 x^2\right)q_1 = 0, \\ iq_{2t} + q_{2xx} + 2\mu^2\left(\left|q_1\right|^2 + \left|q_2\right|^2\right)q_2 + \left(i\beta - \alpha x + \beta^2 x^2\right)q_2 = 0, \end{aligned} \tag{3}$$

where they derived the Lax pair for the two-component coupled Gross-Pitaevskii equations and obtained multi-nonautonomous solitons, a single breather soliton, and first-order rogue waves using the Darboux transformation [53].

In 2019, T. Kanna *et al.* obtained solutions for the coupled two-component and three-component non-autonomous Gross-Pitaevskii systems by introducing appropriate rotation and similarity transformations. The two-component system is given by

$$i\frac{\partial\psi_j}{\partial t}+\gamma(x)\frac{\partial^2\psi_j}{\partial x^2}+2\sum_{l=1}^{2}g_{jl}(x)|\psi_l|^2\psi_j+V(x,t)\psi_j-\sum_{l=1,l\neq j}^{2}\sigma\psi_l=0\ \ (j=1,2),\tag{4}$$

where $\gamma(x)$, $g_{jl}(x)$, and $V(x,t)$ are functions characterizing the non-autonomous features of the system, and σ is a coupling constant [54].

In 2020, H. Chaachoua Sameut *et al.* solved two one-dimensional coupled Gross-Pitaevskii equations with time-varying harmonic traps and found soliton solutions that can be effectively controlled by modulating the frequency of the external potential. The equations are given by

$$\begin{aligned}&i\psi_{1t}+\frac{1}{2}\psi_{1xx}+\left(V(x,t)+R_{11}|\psi_1|^2+R_{12}|\psi_2|^2\right)\psi_1=0,\\&i\psi_{2t}+\frac{1}{2}\psi_{2xx}+\left(V(x,t)+R_{21}|\psi_1|^2+R_{22}|\psi_2|^2\right)\psi_2=0,\end{aligned}\tag{5}$$

where $V(x,t)$ represents the time-varying harmonic trap potential, and R_{ij} are interaction coefficients [55].

In recent years, the properties of Bose-Einstein condensates (BEC) and nonlinear optics under the modulation of scattering length have attracted much attention. For example, the research of Fischbach resonance is well known and has led to the proposal of many new nonlinear phenomena. The Gross-Pitaevskii equation (GPE) is the basic model describing the phenomenon of BEC. The mathematical theoretical study of GPE is not only helpful to understand and analyze the BEC phenomena in physical experiments, but also to predict new BEC phenomena theoretically. This paper studies the following system of the coupled Gross-Pitaevskii equations (CGPE), without an external potential [56].

$$\begin{aligned}&i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-\left(\beta|u|^2+\gamma|v|^2\right)u=0,\\&i\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}-\left(\delta|v|^2+\gamma|u|^2\right)v=0,\end{aligned}\tag{6}$$

where, $i=\sqrt{-1}$, u,v are the function of t and x. In optics, t represents the propagation time variable, x represents the propagation space variable, u and v represent two unrelated beams of light, respectively. However, in order to better understand the nature and laws of the phenomena of CGPE (6), it is still necessary to study the exact solutions for the equations. In this paper, we transform (6) into a nonlinear coupled ordinary differential system by using coupled traveling wave solution transformation, and then obtain new exact solutions by using the F-expansion method.

The organization of this paper is as follows. Section 1 gives an introduction. Section 2 gives a brief description of the algorithm by using the F-expansion method. Section 3 gives the exact solutions of CGPE (6). Section 4 gives some numerical results and their figures to illustrate the solutions. Finally, the paper ends with a conclusion in Section 5.

2. Algorithm of the Modified Polynomial Expansion Method

In this section, we describe the algorithm of polynomial expansion method by modified polynomial expansion method for finding the exact solutions of the coupled nonlinear partial differential equations. Suppose that the coupled nonlinear partial differential equations, which have independent space variable x and time variable t, are given by

$$\begin{cases} P\left(u,v,u_x,v_x,u_t,v_t,u_xv,uv_x,u_tv,uv_t,u_{xx},v_{xx},u_{xt},v_{xt},u_{tt},\cdots\right)=0, \\ Q\left(u,v,u_x,v_x,u_t,v_t,u_xv,uv_x,u_tv,uv_t,u_{xx},v_{xx},u_{xt},v_{xt},u_{tt},\cdots\right)=0, \end{cases} \tag{7}$$

where $u=u(x,t),v=v(x,t)$ are unknown functions of the coupled system (7), P,Q are polynomials of $u(x,t),v(x,t)$ and their partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved and the subscripts stand for the partial derivatives.

Next, we will discuss the specific steps of the modified polynomial expansion method.

(i) Transforming the coupled nonlinear partial differential system (7) into the coupled ordinary differential equations (ODE). Suppose that

$$\begin{cases} u=u(x-ct)=\phi(\xi), \\ v=v(x-ct)=\varphi(\xi), \end{cases} \tag{8}$$

where $\xi=x-ct$ and $c\in\left(R-\{0\}\right)$ is any real number. Substituting (8) into the coupled system (7), we can obtain

$$\begin{cases} P\left(\phi,\varphi,\phi',\varphi',\phi'',\varphi'',\phi'\varphi',\phi''',\cdots\right)=0, \\ Q\left(\phi,\varphi,\phi',\varphi',\phi'',\varphi'',\phi'\varphi',\phi''',\cdots\right)=0, \end{cases} \tag{9}$$

where the prime is the derivative with respect to ξ.

(ii) Assume that the expressions of the exact solutions of the system (9) are

$$\begin{cases} \phi(\xi)=\sum_{i=0}^{M}a_i\left(F(\xi)\right)^i, \\ \varphi(\xi)=\sum_{i=0}^{N}b_i\left(F(\xi)\right)^i, \end{cases} \tag{10}$$

where $F(\xi)$ satisfies the auxiliary differential equation

$$F'=\sqrt{c_0+c_2F^2+c_4F^4}, \tag{11}$$

where $F=F(\xi)$, c_0,c_2, and c_4 are real constants related to the elliptic modulus of elliptic Jacobian function $F(\xi)$.

(iii) Confirming the value of M and N in (10). To confirm the parameters M and N, we use the balance coefficient method by balancing the highest-order derivative term and the highest-degree nonlinear term in the coupled system (9).

(iv) Substituting the value of M,N and the solutions F of Equation (11) into the coupled solutions (9), with the help of the software Maple, we can obtain all the values of $a_i,(i=1,2,\cdots,M)$ and the values $b_i,(i=1,2,\cdots,N)$, by setting the coefficients of all terms with the same power exponent about $\phi(\xi)$ and $\varphi(\xi)$ in (9) to zeros.

(v) Substituting the values of $a_i,(i=1,2,\cdots,M)$ and the values $b_i,(i=1,2,\cdots,N)$ into (8), we get the exact solutions of the coupled system (7).

3. The Exact Solutions for the Coupled Gross-Pitaevskii Equations

Next, we give the exact solutions for CGPE (6). Firstly, we need to transform CGPE (6) into a coupled ordinary differential system.

Suppose the modified coupled traveling wave transformation is as follows

$$\begin{cases} u=u(x-ct)\exp\left(i(n_1x-\omega_1t)\right)=\phi(\xi)\exp\left(i(n_1x-\omega_1t)\right), \\ v=v(x-ct)\exp\left(ic(n_2x-\omega_2t)\right)=\varphi(\xi)\exp\left(i(n_2x-\omega_2t)\right), \end{cases} \tag{12}$$

where $\phi(\xi)=u(x-ct),\varphi(\xi)=v(x-ct)$ are real-valued functions about $\xi=x-ct$, $c\in(R-\{0\})$, n_1,n_2 are the wave numbers, and ω_1,ω_2 are the frequencies, respectively.

Substituting the modified coupled traveling wave transformation (12) into CGPE (6), we get $n_1=n_2=c$, for simplicity, letting $\omega_1=c\mu$ and $\omega_2=c\lambda$, then CGPE can be changed into the following coupled ordinary differential system,

$$\begin{cases} \frac{1}{2}\phi''+c\left(\mu-\frac{c}{2}\right)\phi-\beta\phi^3-\gamma\phi\varphi^2=0, \\ \frac{1}{2}\varphi''+c\left(\lambda-\frac{c}{2}\right)\varphi-\delta\varphi^3-\gamma\varphi\phi^2=0, \end{cases} \tag{13}$$

where $\phi''=\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2},\varphi''=\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}$, by balancing the order of the highest derivative term and the highest order nonlinear term of (13), we obtain that $M=1,N=1$.

Consequently, we yield the expression of the exact solutions for CGPE (6) as

$$\begin{cases} \phi(\xi)=a_1+a_2F(\xi), \\ \varphi(\xi)=b_1+b_2F(\xi). \end{cases} \tag{14}$$

Substituting (14) into (13), combining terms with the same power of $F(\xi)$ and setting the coefficient of the same power of $F(\xi)$ is to zero, we yield the system of algebraic equations about a_1,a_2,b_1,b_2 as shown below,

$$\begin{gathered} a_1=b_1=0,a_2=\pm\sqrt{\frac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}, \\ b_2=\pm\sqrt{\frac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}},c=c,\lambda=\mu=\frac{c^2-c_2}{2c}. \end{gathered} \tag{15}$$

Then, substituting the solution (15) about a_1,a_2,b_1,b_2 and Equation (14) into equation system (13), according to the auxiliary Equation (11), we can obtain the exact solutions for CGPE (6).

In order to obtain the solutions for CGPE (6), we need to give the main solutions $F(\xi)$ of the auxiliary differential Equation (11) according to the parameters

c_0, c_2, c_4 as follows [57] [58]: (Note, $F_i(\xi)(i = 1, 2, \cdots, 37)$ are the solutions for Equation (11).)

Case 1:

$$\text{If } c_0 = 1, c_2 = -k^2 - 1, c_4 = k^2, \text{then } F_1(\xi) = \text{sn}(\xi). \tag{16}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_1(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{sn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_1(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{sn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{17}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, β, δ, γ and c are all real constants.

Case 2:

$$\text{If } c_0 = k^2, c_2 = -1 - k^2, c_4 = 1, \text{then } F_2(\xi) = \text{ns}(\xi). \tag{18}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_2(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{ns}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_2(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{ns}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{19}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, β, δ, γ and c are all real constants.

Case 3:

$$\text{If } c_0 = k^2 - 1, c_2 = 2 - k^2, c_4 = -1, \text{then } F_3(\xi) = \text{dn}(\xi). \tag{20}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_3(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{dn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_3(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{dn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{21}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, β, δ, γ and c are all real constants.

Case 4:

$$\text{If } c_0 = 1 - k^2, c_2 = 2k^2 - 1, c_4 = -k^2, \text{then } F_4(\xi) = \text{cn}(\xi). \tag{22}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_4(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{cn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_4(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}}\text{cn}(x - ct)\exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{23}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 5:

$$\text{If } c_0=-k^2, c_2=-1+2k^2, c_4=1-k^2, \text{then } F_5(\xi)=\frac{1}{\text{cn}(\xi)}. \tag{24}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_5(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{1}{\text{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_5(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{1}{\text{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{25}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 6:

$$\text{If } c_0=-1, c_2=2-k^2, c_4=k^2-1, \text{then } F_6(\xi)=\frac{1}{\text{dn}(\xi)}. \tag{26}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_6(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{1}{\text{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_6(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{1}{\text{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{27}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 7:

$$\text{If } c_0=1-k^2, c_2=2-k^2, c_4=1, \text{then } F_7(\xi)=\frac{\text{cn}(\xi)}{\text{sn}(\xi)}. \tag{28}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_7(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\text{cn}(x-ct)}{\text{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_7(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\text{cn}(x-ct)}{\text{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{29}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 8:

$$\text{If } c_0=1, c_2=2-k^2, c_4=1-k^2, \text{then } F_8(\xi)=\frac{\text{sn}(\xi)}{\text{cn}(\xi)}. \tag{30}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_8(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_8(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{31}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 9:

$$\text{If } c_0 = 1, c_2 = 2k^2-1, c_4 = k^2\left(k^2-1\right), \text{then } F_9(\xi) = \frac{\operatorname{sn}(\xi)}{\operatorname{dn}(\xi)}. \tag{32}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_9(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_9(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{33}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 10:

$$\text{If } c_0 = k^2\left(k^2-1\right), c_2 = 2k^2-1, c_4 = 1, \text{then } F_{10}(\xi) = \frac{\operatorname{dn}(\xi)}{\operatorname{sn}(\xi)}. \tag{34}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{10}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{dn}(x-ct)}{\operatorname{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{10}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{dn}(x-ct)}{\operatorname{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{35}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 11:

$$\text{If } c_0 = \frac{1}{4}, c_2 = \frac{1-2k^2}{2}, c_4 = \frac{1}{4}, \text{then } F_{11}(\xi) = \operatorname{ns}(\xi) \pm \operatorname{cs}(\xi). \tag{36}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{11}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(\operatorname{ns}(x-ct)\pm\operatorname{cs}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{11}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(\operatorname{ns}(x-ct)\pm\operatorname{cs}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{37}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 12:

$$\text{If } c_0=\frac{1-k^2}{4}, c_2=\frac{1+k^2}{2}, c_4=\frac{1-k^2}{4}, \text{then } F_{12}(\xi)=\mathrm{nc}(\xi)\pm(\xi). \tag{38}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{12}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{nc}(x-ct)\pm\mathrm{sc}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{12}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{nc}(x-ct)\pm\mathrm{sc}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{39}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 13:

$$\text{If } c_0=\frac{k^2}{4}, c_2=\frac{k^2-2}{2}, c_4=\frac{1}{4}, \text{then } F_{13}(\xi)=\mathrm{ns}(\xi)+\mathrm{ds}(\xi). \tag{40}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{13}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{ns}(x-ct)+\mathrm{ds}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{13}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{ns}(x-ct)+\mathrm{ds}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{41}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 14:

$$\begin{aligned} &\text{If } c_0=\frac{k^2}{4}, c_2=\frac{k^2-2}{2}, c_4=\frac{k^2}{4}, \\ &\text{then } F_{14.1}(\xi)=\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi); \text{or } F_{14.2}(\xi)=\frac{\mathrm{dn}(\xi)}{i\sqrt{1-k^2}\,\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}. \end{aligned} \tag{42}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{14.1}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{sn}(x-ct)\pm i\,\mathrm{cn}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{14.1}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(\mathrm{sn}(x-ct)\pm i\,\mathrm{cn}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{43}$$

or

$$\begin{cases} u_{14.2}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{dn}(x-ct)}{i\sqrt{1-k^2}\,\mathrm{sn}(x-ct)\pm\mathrm{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{14.2}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{dn}(x-ct)}{i\sqrt{1-k^2}\,\mathrm{sn}(x-ct)\pm\mathrm{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{44}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 15:

$$\text{If } c_0 = 1, c_2 = 2 - 4k^2, c_4 = 1, \text{then } F_{15}(\xi) = \frac{\text{sn}(\xi)\text{dn}(\xi)}{\text{cn}(\xi)}. \tag{45}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{15}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{sn}(x-ct)\text{dn}(x-ct)}{\text{cn}(x-ct)} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_{15}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{sn}(x-ct)\text{dn}(x-ct)}{\text{cn}(x-ct)} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{46}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, β, δ, γ and c are all real constants.

Case 16:

$$\begin{gathered} \text{If } c_0 = \frac{(k-1)^2}{4D_1^2}, c_2 = \frac{1 + k^2 + 6k}{2}, c_4 = \frac{D_1^2 (k-1)^2}{4}, \\ \text{then } F_{16}(\xi) = \frac{\text{dn}(\xi)\text{cn}(\xi)}{D_1(1 + \text{sn}(\xi))(1 + k\text{sn}(\xi))}. \end{gathered} \tag{47}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{16}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{dn}(x-ct)\text{cn}(x-ct)}{D_1(1 + \text{sn}(x-ct))(1 + k\text{sn}(x-ct))} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right) \\ v_{16}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{dn}(x-ct)\text{cn}(x-ct)}{D_1(1 + \text{sn}(x-ct))(1 + k\text{sn}(x-ct))} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{48}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, β, δ, γ and c are all real constants.

Case 17:

$$\begin{gathered} \text{If } c_0 = \frac{(k+1)^2}{4D_1^2}, c_2 = \frac{1 + k^2 - 6k}{2}, c_4 = \frac{D_1^2 (k+1)^2}{4}, \\ \text{then } F_{17}(\xi) = \frac{\text{dn}(\xi)\text{cn}(\xi)}{D_1(1 + \text{sn}(\xi))(1 - k\text{sn}(\xi))}. \end{gathered} \tag{49}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{17}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{dn}(x-ct)\text{cn}(x-ct)}{D_1(1 + \text{sn}(x-ct))(1 - k\text{sn}(x-ct))} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \\ v_{17}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta - \gamma^2}} \dfrac{\text{dn}(x-ct)\text{cn}(x-ct)}{D_1(1 + \text{sn}(x-ct))(1 - k\text{sn}(x-ct))} \exp\left(ic\left(x - \dfrac{c^2 - c_2}{2c}t\right)\right), \end{cases} \tag{50}$$

where $0 < k < 1$ denotes the modulus of the Jacobian elliptic function, $\beta, \delta, \gamma, D_1$ and c are all real constants.

Case 18:

$$\begin{gathered} \text{If } c_0 = -2k^3 + k^4 + k^2, c_2 = 6k - k^2 - 1, c_4 = -\frac{4}{k}, \\ \text{then } F_{18}(\xi) = \frac{k\text{dn}(\xi)\text{cn}(\xi)}{1 + k\text{sn}^2(\xi)}. \end{gathered} \tag{51}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{18}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k\mathrm{dn}(x-ct)\mathrm{cn}(x-ct)}{1+k\mathrm{sn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{18}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k\mathrm{dn}(x-ct)\mathrm{cn}(x-ct)}{1+k\mathrm{sn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{52}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 19:

$$\begin{aligned} &\text{If } c_0 = 2k^3+k^4+k^2, c_2 = -6k-k^2-1, c_4 = \frac{4}{k}, \\ &\text{then } F_{19}(\xi) = \frac{k\mathrm{dn}(\xi)\mathrm{cn}(\xi)}{k\mathrm{sn}^2(\xi)-1}. \end{aligned} \tag{53}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{19}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k\mathrm{dn}(x-ct)\mathrm{cn}(x-ct)}{k\mathrm{sn}^2(x-ct)-1}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{19}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k\mathrm{dn}(x-ct)\mathrm{cn}(x-ct)}{k\mathrm{sn}^2(x-ct)-1}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{54}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 20:

$$\begin{aligned} &\text{If } c_0 = 2+2k_1-k^2, c_2 = 6k_1-k^2-2, c_4 = 4k_1, \\ &\text{then } F_{20}(\xi) = \frac{k^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{k_1-\mathrm{dn}^2(\xi)}; \end{aligned} \tag{55}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{20}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k^2\mathrm{sn}(x-ct)\mathrm{cn}(x-ct)}{k_1-\mathrm{dn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{20}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k^2\mathrm{sn}(x-ct)\mathrm{cn}(x-ct)}{k_1-\mathrm{dn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{56}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, $k_1=\sqrt{1-k^2}$, β,δ,γ and c are all real constants.

Case 21:

$$\begin{aligned} &\text{If } c_0 = 2-2k_1-k^2, c_2 = -6k_1-k^2+2, c_4 = -4k_1, \\ &\text{then } F_{21}(\xi) = -\frac{k^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{k_1+\mathrm{dn}^2(\xi)}. \end{aligned} \tag{57}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{21}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{k^2\mathrm{sn}(x-ct)\mathrm{cn}(x-ct)}{k_1+\mathrm{dn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{21}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} -\dfrac{k^2\mathrm{sn}(x-ct)\mathrm{cn}(x-ct)}{k_1+\mathrm{dn}^2(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{58}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, $k_1=\sqrt{1-k^2}$, β,δ,γ and c are all real constants.

Case 22:

$$\begin{aligned}&\text{If } c_0=\frac{k^2-1}{4\left(D_3^2k^2-D_2^2\right)},c_2=\frac{k^2+1}{2},c_4=\frac{\left(D_3^2k^2-D_2^2\right)\left(k^2-1\right)}{4},\\&\text{then } F_{22}(\xi)=\frac{\sqrt{\dfrac{D_2^2-D_3^2}{D_2^2-D_3^2k^2}}+\operatorname{sn}(\xi)}{D_2\operatorname{cn}(\xi)+D_3\operatorname{dn}(\xi)}.\end{aligned}\tag{59}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases}u_{22}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{\dfrac{D_2^2-D_3^2}{D_2^2-D_3^2k^2}}+\operatorname{sn}(x-ct)}{D_2\operatorname{cn}(x-ct)+D_3\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right),\\v_{22}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{\dfrac{D_2^2-D_3^2}{D_2^2-D_3^2k^2}}+\operatorname{sn}(x-ct)}{D_2\operatorname{cn}(x-ct)+D_3\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right),\end{cases}\tag{60}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, $\beta,\delta,\gamma,D_2,D_3$ and c are all real constants.

Case 23:

$$\begin{aligned}&\text{If } c_0=\frac{k^4}{4\left(D_3^2+D_2^2\right)},c_2=\frac{k^2-2}{2},c_4=\frac{D_3^2+D_2^2}{4},\\&\text{then } F_{23}(\xi)=\frac{\sqrt{\dfrac{D_2^2+D_3^2-D_3^2k^2}{D_2^2+D_3^2}}+\operatorname{dn}(\xi)}{D_2\operatorname{sn}(\xi)+D_3\operatorname{cn}(\xi)}.\end{aligned}\tag{61}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases}u_{23}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{\dfrac{D_2^2+D_3^2-D_3^2k^2}{D_2^2+D_3^2}}+\operatorname{dn}(x-ct)}{D_2\operatorname{sn}(x-ct)+D_3\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right),\\v_{23}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{\dfrac{D_2^2+D_3^2-D_3^2k^2}{D_2^2+D_3^2}}+\operatorname{dn}(x-ct)}{D_2\operatorname{sn}(x-ct)+D_3\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right),\end{cases}\tag{62}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, $\beta,\delta,\gamma,D_2,D_3$ and c are all real constants.

Cas 24:

$$\begin{aligned}&\text{If } c_0=\frac{2k-k^2-1}{D_2^2},c_2=2k^2+2,c_4=-D_2^2k^2-D_2^2-2D_2^2k,\\&\text{then } F_{24}(\xi)=\frac{k\operatorname{sn}^2(\xi)-1}{D_2\left(k\operatorname{sn}^2(\xi)+1\right)}.\end{aligned}\tag{63}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{24}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{k\text{sn}^2(x-ct)-1}{D_2\left(k\text{sn}^2(x-ct)+1\right)} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{24}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{k\text{sn}^2(x-ct)-1}{D_2\left(k\text{sn}^2(x-ct)+1\right)} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{64}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ,D_2 and c are all real constants.

Case 25:

$$\begin{aligned} &\text{If } c_0 = -\frac{2k+k^2+1}{D_2^2}, c_2 = 2k^+2, c_4 = -D_2^2k^2 - D_2^2 - 2D_2^2k, \\ &\text{then } F_{25}(\xi) = \frac{k\text{sn}^2(\xi)+1}{D_2\left(k\text{sn}^2(\xi)-1\right)}. \end{aligned} \tag{65}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{25}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{k\text{sn}^2(x-ct)+1}{D_2\left(k\text{sn}^2(x-ct)-1\right)} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{25}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{k\text{sn}^2(x-ct)+1}{D_2\left(k\text{sn}^2(x-ct)-1\right)} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{66}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ,D_2 and c are all real constants.

Case 26:

$$\begin{aligned} &\text{If } c_0 = \frac{1}{4}, c_2 = \frac{1-2k^2}{2}, c_4 = \frac{1}{4}, \\ &\text{then } F_{26.1}(\xi) = k\text{sn}(\xi) \pm i\text{dn}(\xi); \\ &F_{26.2}(\xi) = \frac{\text{dn}(\xi)}{k\text{cn}(\xi) \pm i\sqrt{1-k^2}}; \\ &F_{26.3}(\xi) = k\text{ns}(\xi) \pm \text{cs}(\xi); \\ &F_{26.4}(\xi) = \frac{\text{sn}(\xi)}{1 \pm \text{cn}(\xi)}; \\ &\text{or } F_{26.5}(\xi) = \frac{\text{cn}(\xi)}{\sqrt{1-k^2}\text{sn}(\xi) \pm \text{dn}(\xi)}. \end{aligned} \tag{67}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{26.1}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \left(k\text{sn}(x-ct) \pm i\text{dn}(x-ct)\right) \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{26.1}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \left(k\text{sn}(x-ct) \pm i\text{dn}(x-ct)\right) \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{68}$$

$$\begin{cases} u_{26.2}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{\text{dn}(x-ct)}{k\text{cn}(x-ct) \pm i\sqrt{1-k^2}} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{26.2}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}} \dfrac{\text{dn}(x-ct)}{k\text{cn}(x-ct) \pm i\sqrt{1-k^2}} \exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{69}$$

$$\begin{cases} u_{26.3}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(k\mathrm{ns}(x-ct)\pm\mathrm{cs}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{26.3}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(k\mathrm{ns}(x-ct)\pm\mathrm{cs}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{70}$$

$$\begin{cases} u_{26.4}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{sn}(x-ct)}{1\pm\mathrm{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{26.4}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{sn}(x-ct)}{1\pm\mathrm{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{71}$$

or

$$\begin{cases} u_{26.5}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{cn}(x-ct)}{\sqrt{1-k^2}\mathrm{sn}(x-ct)\pm\mathrm{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{26.5}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{cn}(x-ct)}{\sqrt{1-k^2}\mathrm{sn}(x-ct)\pm\mathrm{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{72}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 27:

$$\begin{aligned} &\text{If } c_0=\frac{k^2-1}{4}, c_2=\frac{k^2+1}{2}, c_4=\frac{k^2-1}{4}, \\ &\text{then } F_{27.1}(\xi)=\frac{\mathrm{dn}(\xi)}{1\pm k\mathrm{sn}(\xi)}; \\ &\text{or } F_{27.2}(\xi)=k\mathrm{sd}(\xi)\pm\mathrm{nd}(\xi). \end{aligned} \tag{73}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{27.1}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{dn}(x-ct)}{1\pm k\mathrm{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{27.1}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\mathrm{dn}(x-ct)}{1\pm k\mathrm{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{74}$$

or

$$\begin{cases} u_{27.2}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(k\mathrm{sd}(x-ct)\pm\mathrm{nd}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{27.2}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\left(k\mathrm{sd}(x-ct)\pm\mathrm{nd}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{75}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 28:

$$\begin{aligned} &\text{If } c_0=\frac{1-k^2}{4}, c_2=\frac{k^2+1}{2}, c_4=\frac{1-k^2}{4}, \\ &\text{then } F_{28.1}(\xi)=\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)}; \text{or } F_{28.2}(\xi)=\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi). \end{aligned} \tag{76}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{28.1}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{cn}(x-ct)}{1\pm\operatorname{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{28.1}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{cn}(x-ct)}{1\pm\operatorname{sn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \quad (77)$$

or

$$\begin{cases} u_{28.2}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(\operatorname{nc}(x-ct)\pm\operatorname{sc}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{28.2}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(\operatorname{nc}(x-ct)\pm\operatorname{sc}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \quad (78)$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 29:

$$\text{If } c_0 = -\frac{(1-k^2)^2}{4}, c_2 = \frac{k^2+1}{2}, c_4 = -\frac{1}{4}, \quad \text{then } F_{29}(\xi) = k\operatorname{cn}(\xi)\pm\operatorname{dn}(\xi). \quad (79)$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{29}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(k\operatorname{cn}(x-ct)\pm\operatorname{dn}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{29}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\left(k\operatorname{cn}(x-ct)\pm\operatorname{dn}(x-ct)\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \quad (80)$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 30:

$$\text{If } c_0 = \frac{1}{4}, c_2 = \frac{k^2+1}{2}, c_4 = \frac{(1-k^2)^2}{4}, \quad \text{then } F_{30}(\xi) = \frac{\operatorname{sn}(\xi)}{\operatorname{dn}(\xi)\pm\operatorname{cn}(\xi)}. \quad (81)$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{30}(x,t) = \pm\sqrt{\dfrac{\delta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{dn}(x-ct)\pm\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{30}(x,t) = \pm\sqrt{\dfrac{\beta c_4 - c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{\operatorname{dn}(x-ct)\pm\operatorname{cn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \quad (82)$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 31:

$$\text{If } c_0=\frac{1}{4}, c_2=\frac{k^2-2}{2}, c_4=\frac{k^4}{4},$$
$$\text{then } F_{31.1}(\xi)=\frac{\operatorname{cn}(\xi)}{\sqrt{1-k^2}\pm\operatorname{dn}(\xi)}; \tag{83}$$
$$\text{or } F_{31.2}(\xi)=\frac{\operatorname{sn}(\xi)}{1\pm\operatorname{dn}(\xi)}.$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{31.1}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{cn}(x-ct)}{\sqrt{1-k^2}\pm\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{31.1}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{cn}(x-ct)}{\sqrt{1-k^2}\pm\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{84}$$

or

$$\begin{cases} u_{31.2}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{1\pm\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{31.2}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\operatorname{sn}(x-ct)}{1\pm\operatorname{dn}(x-ct)}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{85}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 32:

$$\text{If } c_0=0, c_2>0, c_4<0,$$
$$\text{then } F_{32}(\xi)=\sqrt{-\frac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}\xi\right). \tag{86}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{32}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{32}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{87}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 33:

$$\text{If } c_0=0, c_2>0, c_4>0, \text{then } F_{33}(\xi)=\sqrt{\frac{c_2}{c_4}}\operatorname{csch}\left(\sqrt{c_2}\xi\right). \tag{88}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{33}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{\dfrac{c_2}{c_4}}\operatorname{csch}\left(\sqrt{c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{33}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{\dfrac{c_2}{c_4}}\operatorname{csch}\left(\sqrt{c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{89}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 34:

$$\text{If } c_0=\frac{c_2^2}{4c_4}, c_2<0, c_4>0, \text{then } F_{34}(\xi)=\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right). \tag{90}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{34}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{2c_4}}\tanh\left(\sqrt{-\dfrac{c_2}{2}}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{34}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{2c_4}}\tanh\left(\sqrt{-\dfrac{c_2}{2}}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{91}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 35:

$$\begin{aligned} &\text{If } c_0=0, c_2<0, c_4>0, \text{then } F_{35.1}(\xi)=\sqrt{-\frac{c_2}{c_4}}\sec\left(\sqrt{-c_2}\xi\right); \\ &\text{or } F_{35.2}(\xi)=\sqrt{-\frac{c_2}{c_4}}\csc\left(\sqrt{-c_2}\xi\right). \end{aligned} \tag{92}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{35.1}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\sec\left(\sqrt{-c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{35.1}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\sec\left(\sqrt{-c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right); \end{cases} \tag{93}$$

or

$$\begin{cases} u_{35.2}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\csc\left(\sqrt{-c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{35.2}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{-\dfrac{c_2}{c_4}}\csc\left(\sqrt{-c_2}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{94}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

Case 36:

$$\text{If } c_0=\frac{c_2^2}{4c_4}, c_2>0, c_4>0, \text{then } F_{36}(\xi)=\sqrt{\frac{c_2}{2c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right). \tag{95}$$

Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{36}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{\dfrac{c_2}{2c_4}}\tan\left(\sqrt{\dfrac{c_2}{2}}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{36}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\sqrt{\dfrac{c_2}{2c_4}}\tan\left(\sqrt{\dfrac{c_2}{2}}x-ct\right)\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \end{cases} \tag{96}$$

where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ and c are all real constants.

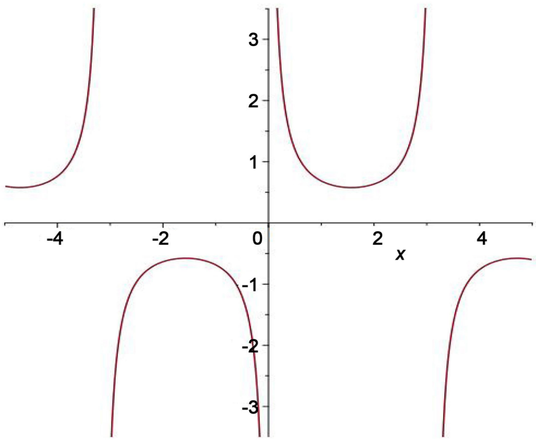
Case 37:

$$\text{If } c_0=0, c_2=1, c_4=\frac{1}{2}, \text{then } F_{37}(\xi)=\pm\frac{\sqrt{2-2\tanh^2(D_4-\xi)}}{\tanh(D_4-\xi)}. \tag{97}$$

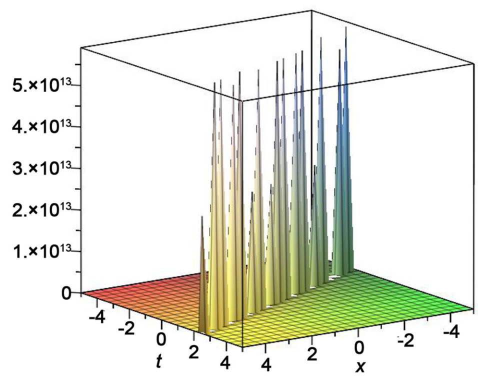
Thus, the solutions for CGPE (6) admit,

$$\begin{cases} u_{37}(x,t)=\pm\sqrt{\dfrac{\delta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{2-2\tanh^2(D_4-(x-ct))}}{\tanh(D_4-(x-ct))}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right), \\ v_{37}(x,t)=\pm\sqrt{\dfrac{\beta c_4-c_4\gamma}{\beta\delta-\gamma^2}}\dfrac{\sqrt{2-2\tanh^2(D_4-(x-ct))}}{\tanh(D_4-(x-ct))}\exp\left(ic\left(x-\dfrac{c^2-c_2}{2c}t\right)\right). \end{cases} \tag{98}$$

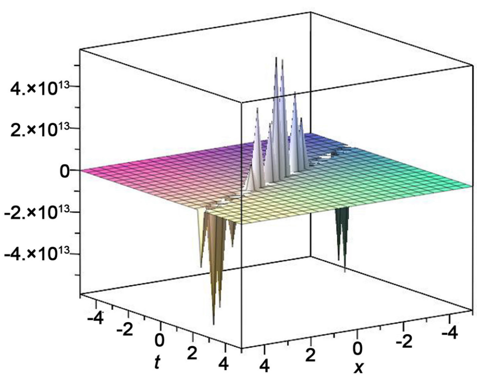
where $0<k<1$ denotes the modulus of the Jacobian elliptic function, β,δ,γ,D_4 and c are all real constants.



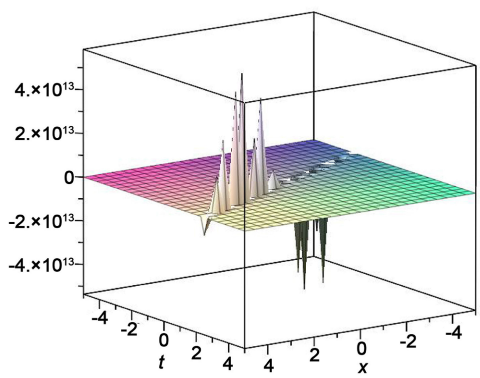
(a) The figure of the amplitude of u_2 in 2D



(b) The figure of the amplitude of u_2 in 3D



(c) The real part of u_2 in 3D



(d) The imaginary part of u_2 in 3D

Figure 1. (a) The figure of the amplitude of the solution u_2 shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_2 shown in the three-dimensional space. (c) The real part of the solution u_2 shown in the three-dimensional space. (d) The imaginary part of the solution u_2 shown in the three-dimensional space.

4. Numerical Simulations Experiments of the Solutions

In this section, we will investigate some of the exact solutions for CGPE (6) and interpret some of the solutions in the perspective of their physical meaning.

According to (15), we see that the figures of $u(x,t)$ and $v(x,t)$ have similar structures, so we only provide the figures of $u(x,t)$.

Example 1. In this example, for the solution $u_2(x,t)$, that is (19), we assume the following parameters

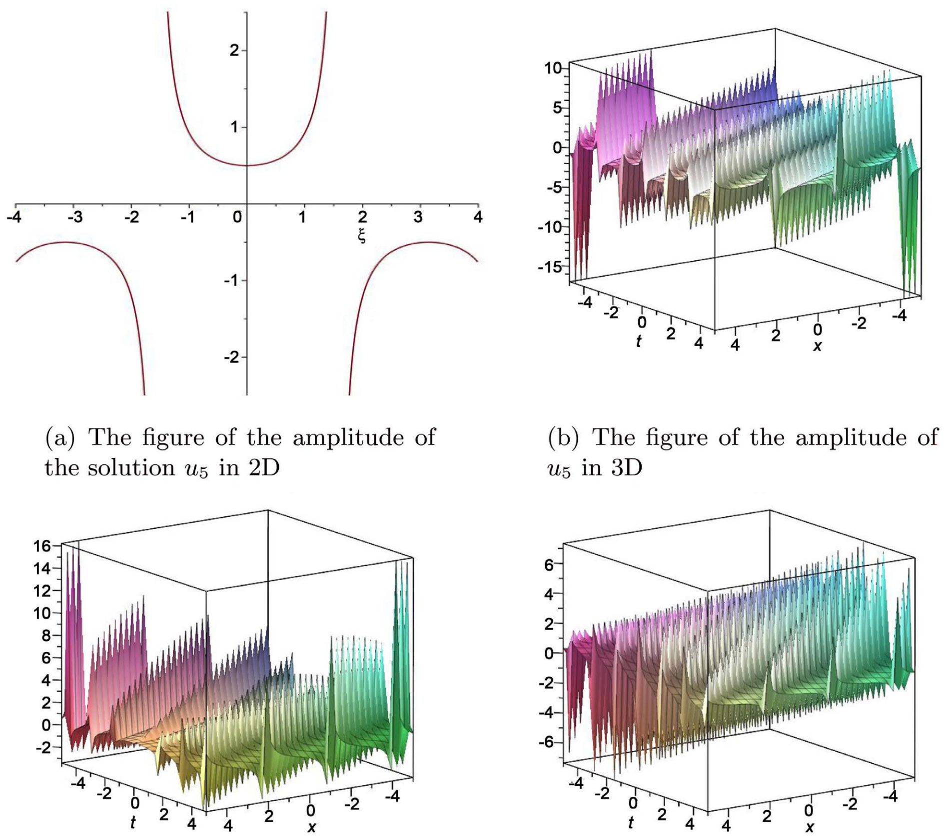
$$\beta=\delta=2,\gamma=1,k=\frac{1}{2},c=2,$$

so the figures of $u_2(x,t)$ for (6) are like to Figure 1.

Example 2. In this example, for the solution $u_5(x,t)$, that is (25), we assume the following parameters

$$\beta=1,\delta=5,\gamma=1,k=\frac{\sqrt{3}}{2},c=2,$$

so the figures of $u_5(x,t)$ for (6) are like to Figure 2.



(a) The figure of the amplitude of the solution u_5 in 2D

(b) The figure of the amplitude of u_5 in 3D

(c) The real part of u_5 in 3D

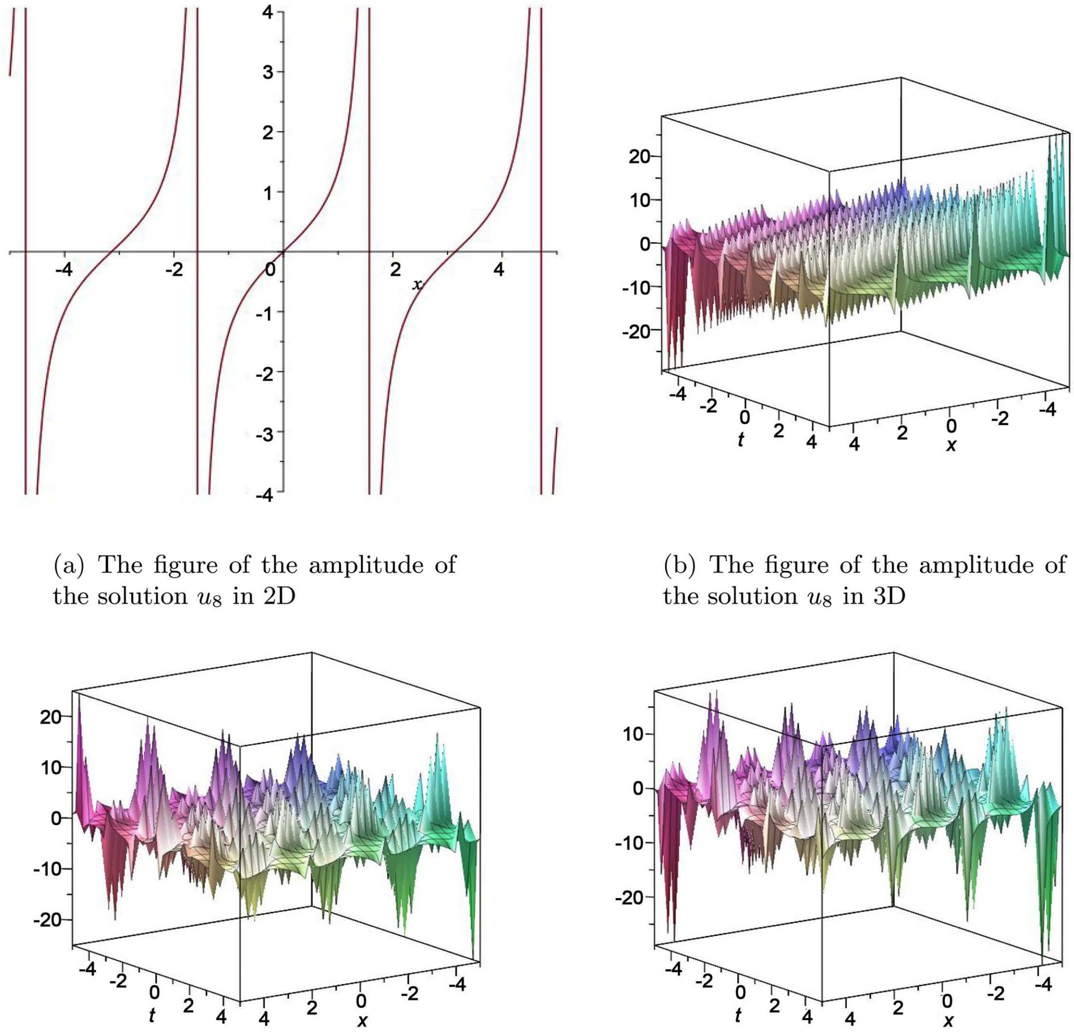
(d) The imaginary part of u_5 in 3D

Figure 2. (a) The figure of the amplitude of the solution u_5 shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_5 shown in the three-dimensional space. (c) The real part of the solution u_5 shown in the three-dimensional space. (d) The imaginary part of the solution u_5 shown in the three-dimensional space.

Example 3. In this example, for the solution $u_8(x,t)$, that is (31), we assume the following parameters

$$\beta = 1, \delta = 5, \gamma = 1, k = \frac{1}{2}, c = 2,$$

so the figures of $u_8(x,t)$ for (6) are like to **Figure 3**.



(a) The figure of the amplitude of the solution u_8 in 2D

(b) The figure of the amplitude of the solution u_8 in 3D

(c) The real part of u_8 in 3D

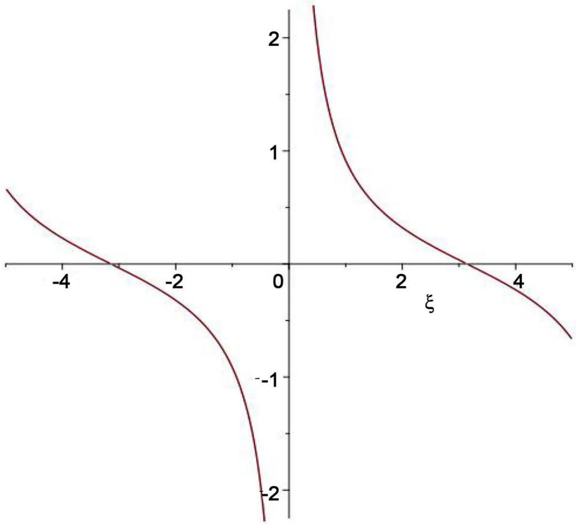
(d) The imaginary part of u_8 in 3D

Figure 3. (a) The figure of the amplitude of the solution u_8 shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_8 shown in the three-dimensional space. (c) The real part of the solution u_8 shown in the three-dimensional space. (d) The imaginary part of the solution u_8 shown in the three-dimensional space.

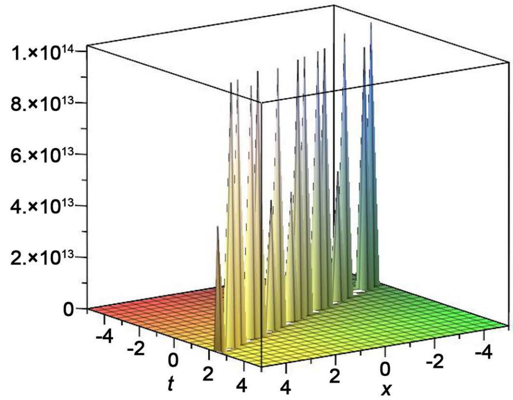
Example 4. In this example, for the solution $u_{11}(x,t)$, that is (37), we assume the following parameters

$$\beta = 1, \delta = 5, \gamma = 1, k = \frac{1}{2}, c = 2,$$

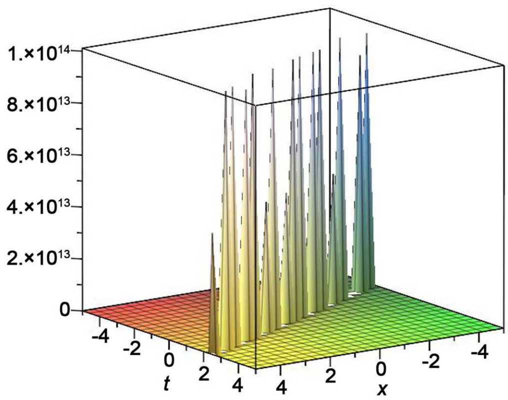
so the figures of $u_{11}(x,t)$ for (6) are like to **Figure 4**.



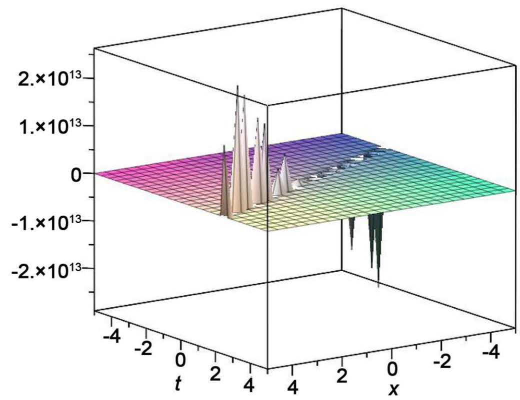
(a) The figure of the amplitude of the solution u_{11} in 2D



(b) The figure of the amplitude of the solution u_{11} in 3D



(c) The real part of u_{11} in 3D



(d) The imaginary part of u_{11} in 3D

Figure 4. (a) The figure of the amplitude of the solution u_{11} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{11} shown in the three-dimensional space. (c) The real part of the solution u_{11} shown in the three-dimensional space. (d) The imaginary part of the solution u_{11} shown in the three-dimensional space.

Example 5. In this example, for the solution $u_{14.1}(x,t)$, that is (43), we assume the following parameters

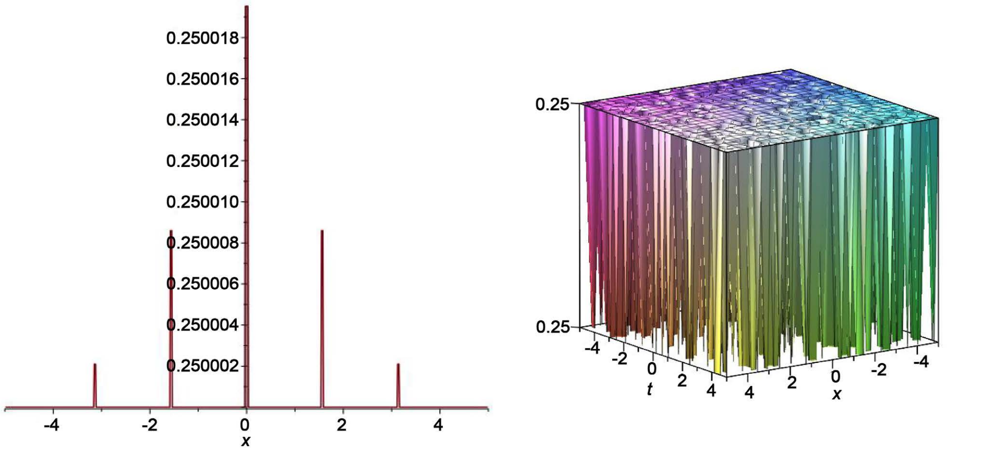
$$\beta = 1, \delta = 5, \gamma = 1, k = \frac{1}{2}, c = 3,$$

so the figures of $u_{14.1}(x,t)$ for (6) are like to **Figure 5**.

Example 6. In this example, for the solution $u_{17}(x,t)$, that is (50), we assume the following parameters

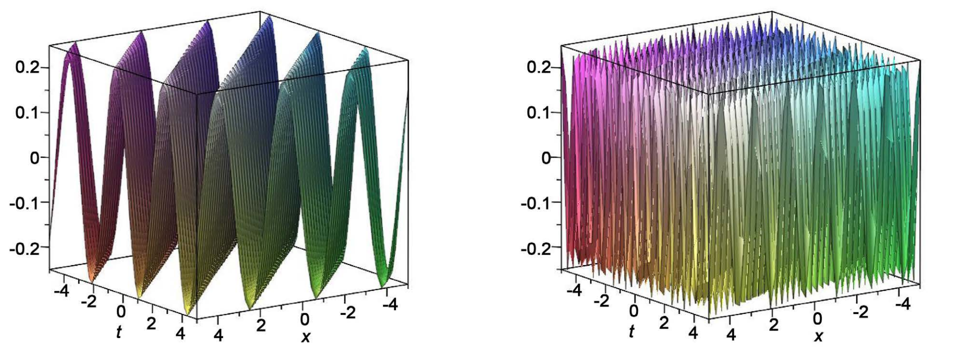
$$\beta = 1, \delta = 5, \gamma = 1, k = \frac{1}{2}, c = \frac{1}{2}, D_1 = 2$$

so the figures of $u_{17}(x,t)$ for (6) are like to **Figure 6**.



(a) The figure of the amplitude of the solution $u_{14.1}$ in 2D

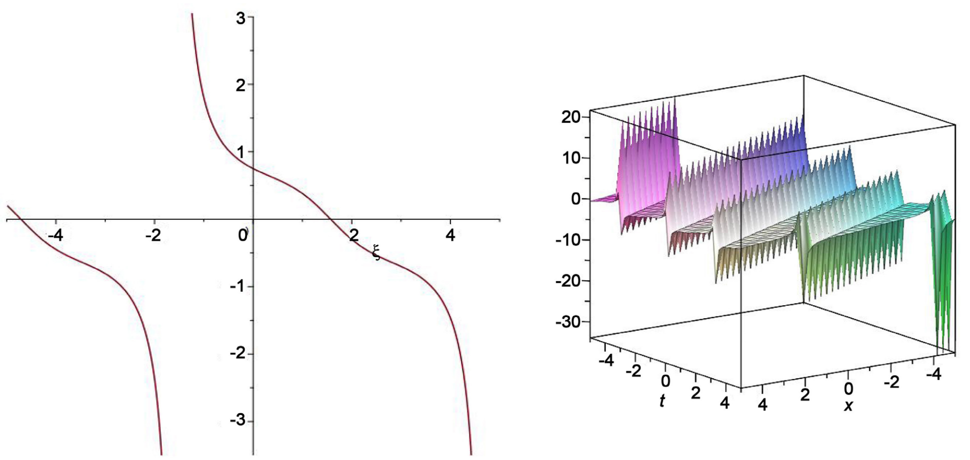
(b) The figure of the amplitude of the solution $u_{14.1}$ in 3D



(c) The real part of $u_{14.1}$ in 3D

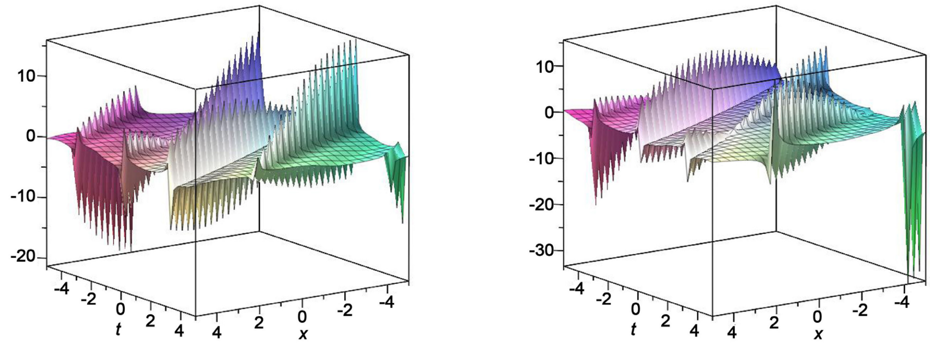
(d) The imaginary part of $u_{14.1}$ in 3D

Figure 5. (a) The figure of the amplitude of the solution $u_{14.1}$ shown in the two-dimensional space. (b) The figure of the amplitude of the solution $u_{14.1}$ shown in the three-dimensional space. (c) The real part of the solution $u_{14.1}$ shown in the three-dimensional space. (d) The imaginary part of the solution $u_{14.1}$ shown in the three-dimensional space.



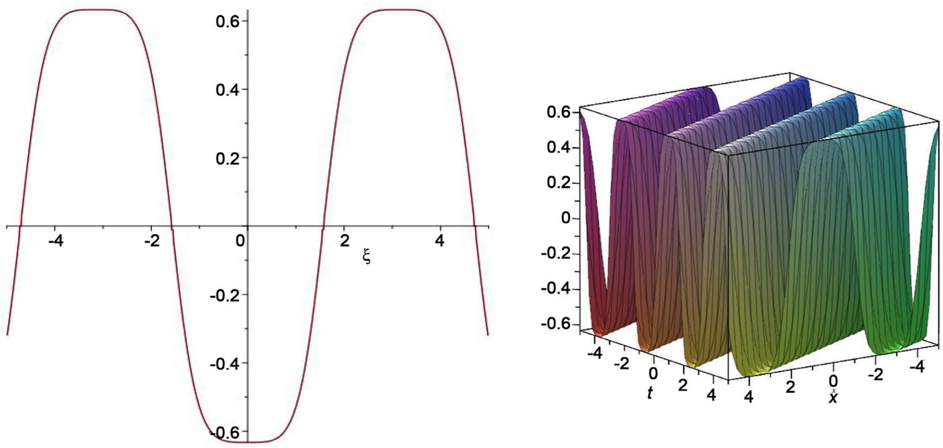
(a) The figure of the amplitude of the solution u_{17} in 2D

(b) The figure of the amplitude of the solution u_{17} in 3D

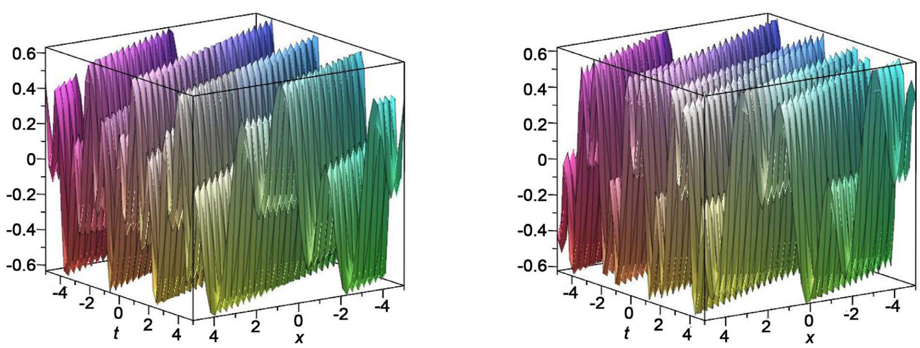


(c) The real part of u_{17} in 3D (d) The imaginary part of u_{17} in 3D

Figure 6. (a) The figure of the amplitude of the solution u_{17} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{17} shown in the three-dimensional space. (c) The real part of the solution u_{17} shown in the three-dimensional space. (d) The imaginary part of the solution u_{17} shown in the three-dimensional space.



(a) The figure of the amplitude of the solution u_{19} in 2D (b) The figure of the amplitude of the solution u_{19} in 3D



(c) The real part of u_{19} in 3D (d) The imaginary part of u_{19} in 3D

Figure 7. (a) The figure of the amplitude of the solution u_{19} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{19} shown in the three-dimensional space. (c) The real part of the solution u_{19} shown in the three-dimensional space. (d) The imaginary part of the solution u_{19} shown in the three-dimensional space.

Example 7. In this example, for the solution $u_{19}(x,t)$, that is (54), we assume the following parameters

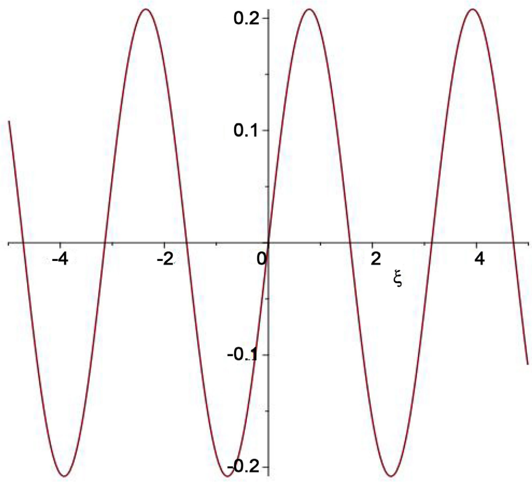
$$\beta = \delta = 2, \gamma = 3, k = \frac{1}{2}, c = 2,$$

so the figures of $u_{19}(x,t)$ for (6) are like to **Figure 7**.

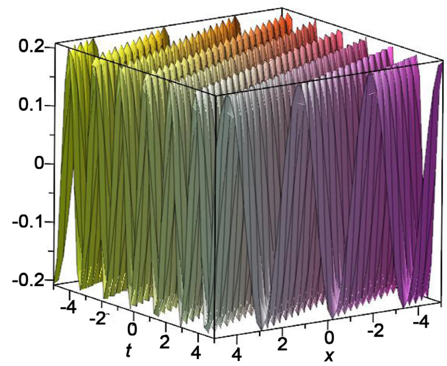
Example 8. In this example, for the solution $u_{21}(x,t)$, that is (58), we assume the following parameters

$$\beta = \delta = 2, \gamma = 3, k = \frac{\sqrt{3}}{2}, c = 2,$$

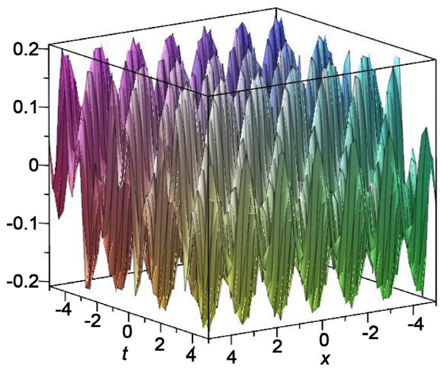
so the figures of $u_{21}(x,t)$ for (6) are like to **Figure 8**.



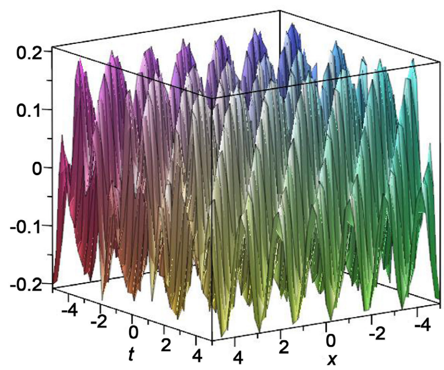
(a) The figure of the amplitude of the solution u_{21} in 2D



(b) The figure of the amplitude of the solution u_{21} in 3D



(c) The real part of u_{21} in 3D



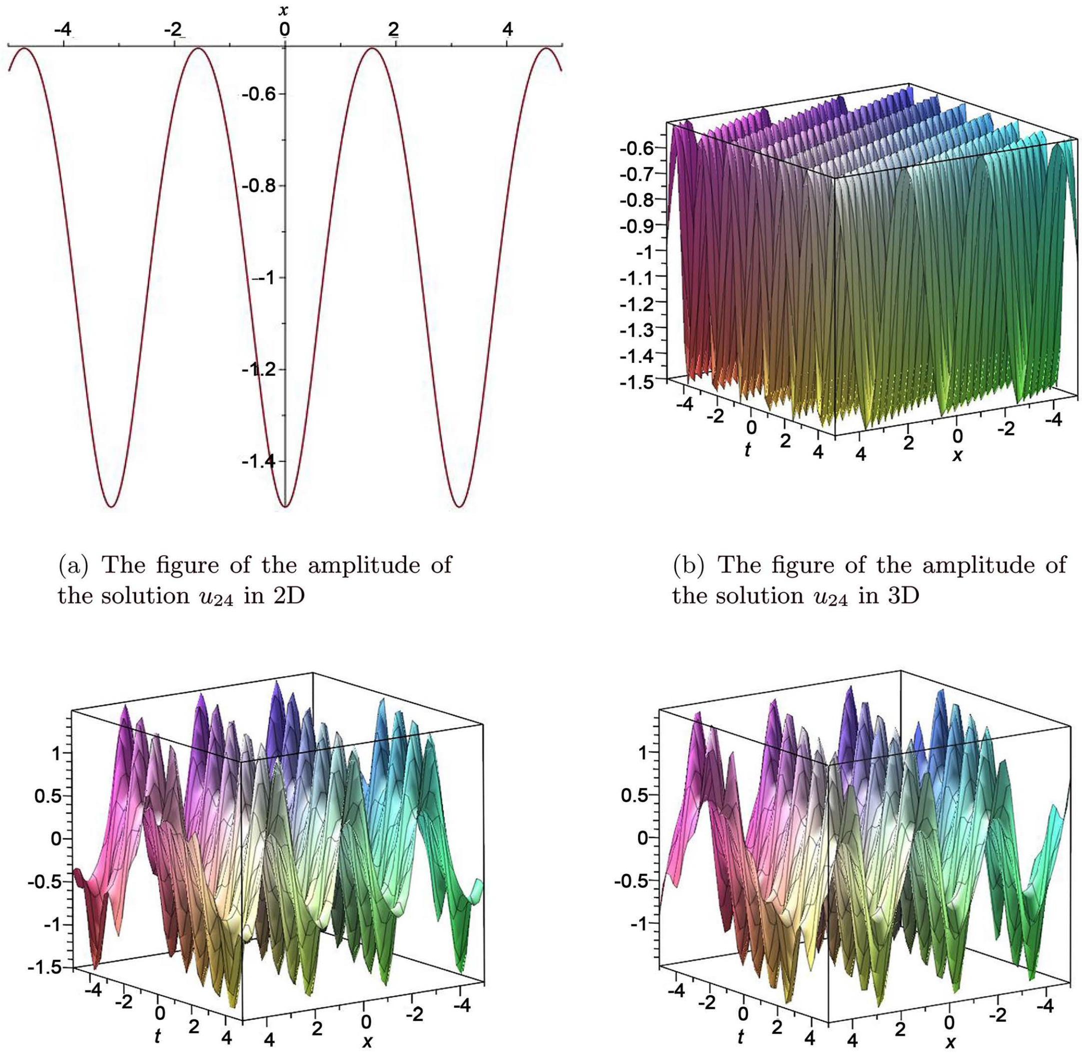
(d) The imaginary part of u_{21} in 3D

Figure 8. (a) The figure of the amplitude of the solution u_{21} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{21} shown in the three-dimensional space. (c) The real part of the solution u_{21} shown in the three-dimensional space. (d) The imaginary part of the solution u_{21} shown in the three-dimensional space.

Example 9. In this example, for the solution $u_{24}(x,t)$, that is (64), we assume the following parameters

$$\beta = 5, \delta = 1, \gamma = 2, D_2 = 3, k = \frac{1}{2}, c = 2,$$

so the figures of $u_{24}(x,t)$ for (6) are like to **Figure 9**.



(a) The figure of the amplitude of the solution u_{24} in 2D

(b) The figure of the amplitude of the solution u_{24} in 3D

(c) The real part of u_{24} in 3D

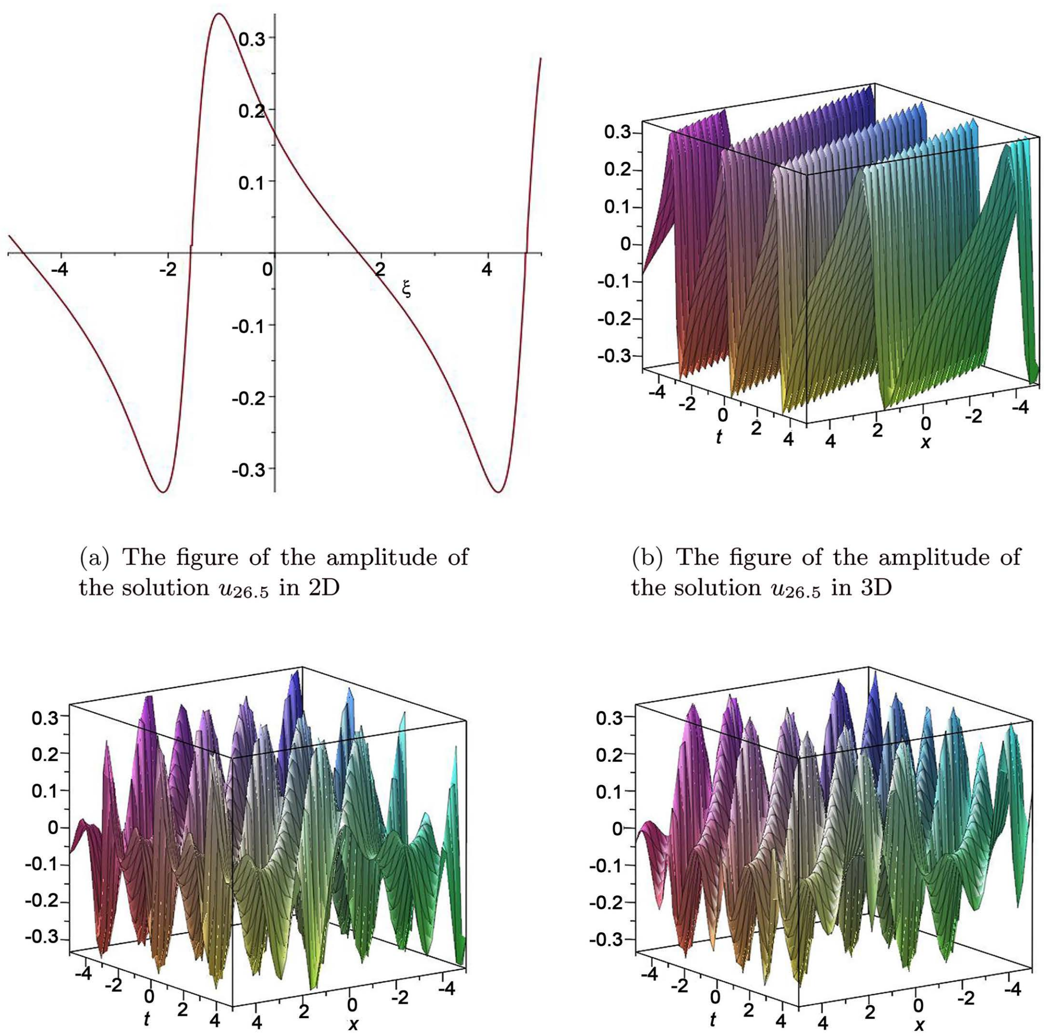
(d) The imaginary part of u_{24} in 3D

Figure 9. (a) The figure of the amplitude of the solution u_{24} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{24} shown in the three-dimensional space. (c) The real part of the solution u_{24} shown in the three-dimensional space. (d) The imaginary part of the solution u_{24} shown in the three-dimensional space.

Example 10. In this example, for the solution $u_{26.5}(x,t)$, that is (72), we assume the following parameters

$$\beta = 5, \delta = 2, \gamma = 1, K = \frac{1}{2}, c = 2,$$

so the figures of $u_{26.5}(x,t)$ for (6) are like to **Figure 10**.



(a) The figure of the amplitude of the solution $u_{26.5}$ in 2D

(b) The figure of the amplitude of the solution $u_{26.5}$ in 3D

(c) The real part of $u_{26.5}$ in 3D

(d) The imaginary part of $u_{26.5}$ in 3D

Figure 10. (a) The figure of the amplitude of the solution $u_{26.5}$ shown in the two-dimensional space. (b) The figure of the amplitude of the solution $u_{26.5}$ shown in the three-dimensional space. (c) The real part of the solution $u_{26.5}$ shown in the three-dimensional space. (d) The imaginary part of the solution $u_{26.5}$ shown in the three-dimensional space.

Example 11. In this example, for the solution $u_{28.2}(x,t)$, that is (78), we assume the following parameters

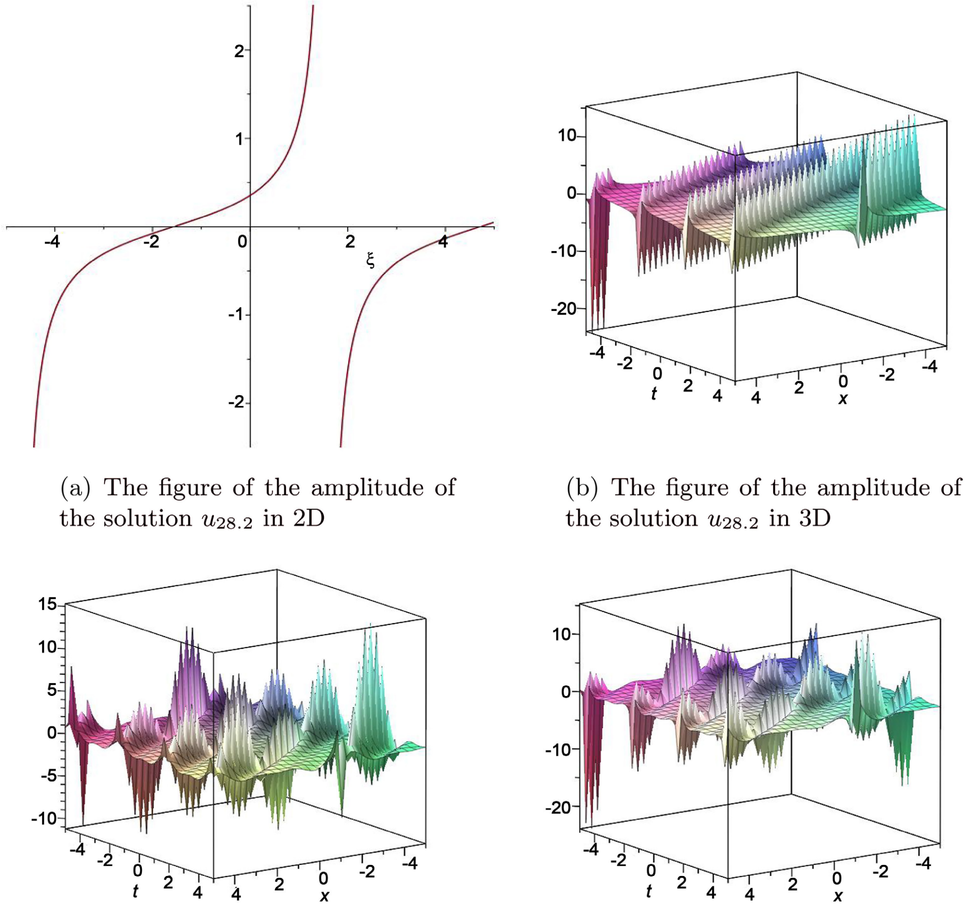
$$\beta = 1, \delta = 9, \gamma = 1, K = \frac{\sqrt{2}}{2}, c = 3,$$

so the figures of $u_{28.2}(x,t)$ for (6) are like to **Figure 11**.

Example 12. In this example, for the solution $u_{30}(x,t)$, that is (82), we assume the following parameters

$$\beta = 1, \delta = 6, \gamma = 2, k = \frac{1}{2}, c = \frac{1}{2},$$

so the figures of $u_{30}(x,t)$ for (6) are like to **Figure 12**.



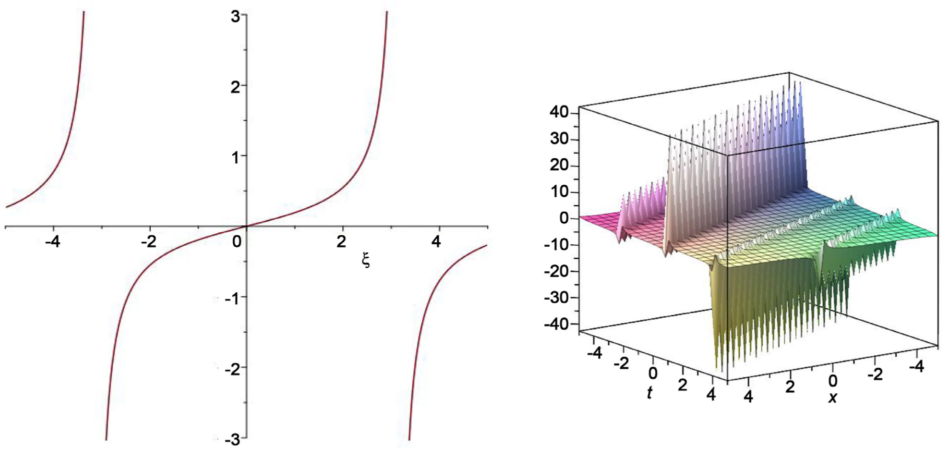
(a) The figure of the amplitude of the solution $u_{28.2}$ in 2D

(b) The figure of the amplitude of the solution $u_{28.2}$ in 3D

(c) The real part of $u_{28.2}$ in 3D

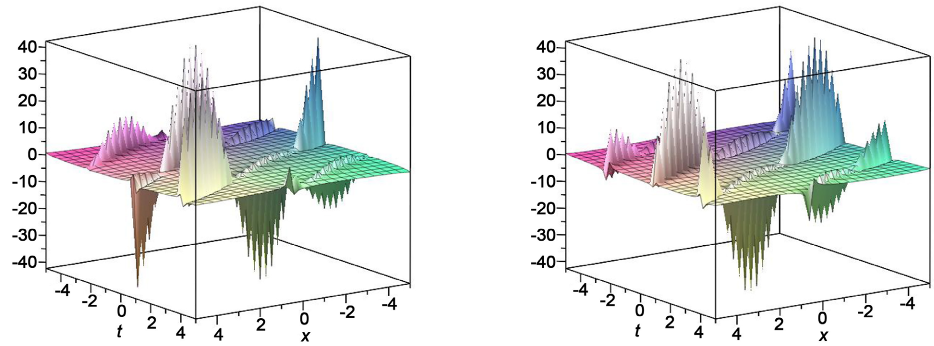
(d) The imaginary part of $u_{28.2}$ in 3D

Figure 11. (a) The figure of the amplitude of the solution $u_{28.2}$ shown in the two-dimensional space. (b) The figure of the amplitude of the solution $u_{28.2}$ shown in the three-dimensional space. (c) The real part of the solution $u_{28.2}$ shown in the three-dimensional space. (d) The imaginary part of the solution $u_{28.2}$ shown in the three-dimensional space.



(a) The figure of the amplitude of the solution u_{30} in 2D

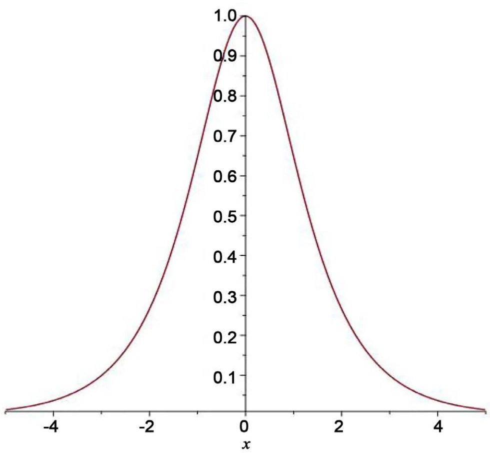
(b) The figure of the amplitude of the solution u_{30} in 3D



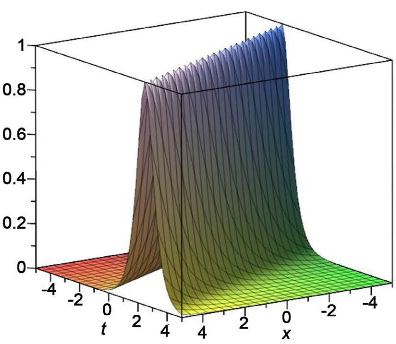
(c) The real part of u_{30} in 3D

(d) The imaginary part of u_{30} in 3D

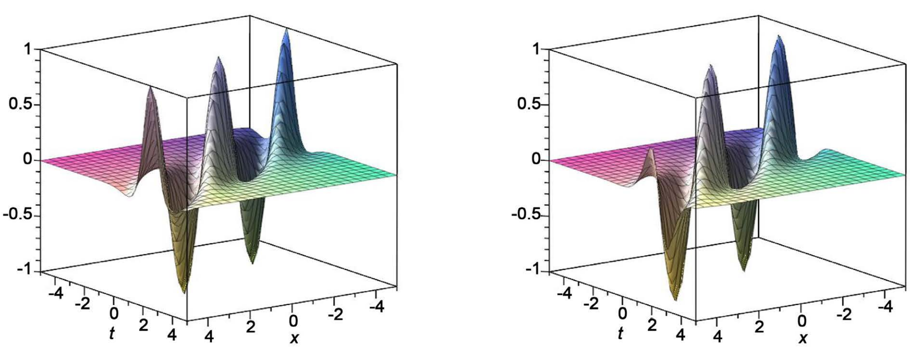
Figure 12. (a) The figure of the amplitude of the solution u_{30} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{30} shown in the three-dimensional space. (c) The real part of the solution u_{30} shown in the three-dimensional space. (d) The imaginary part of the solution u_{30} shown in the three-dimensional space.



(a) The figure of the amplitude of the solution u_{32} in 2D



(b) The figure of the amplitude of the solution u_{32} in 3D



(c) The real part of u_{32} in 3D

(d) The imaginary part of u_{32} in 3D

Figure 13. (a) The figure of the amplitude of the solution u_{32} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{32} shown in the three-dimensional space. (c) The real part of the solution u_{32} shown in the three-dimensional space. (d) The imaginary part of the solution u_{32} shown in the three-dimensional space.

Example 13. In this example, for the solution $u_{32}(x,t)$, that is (87), we assume the following parameters

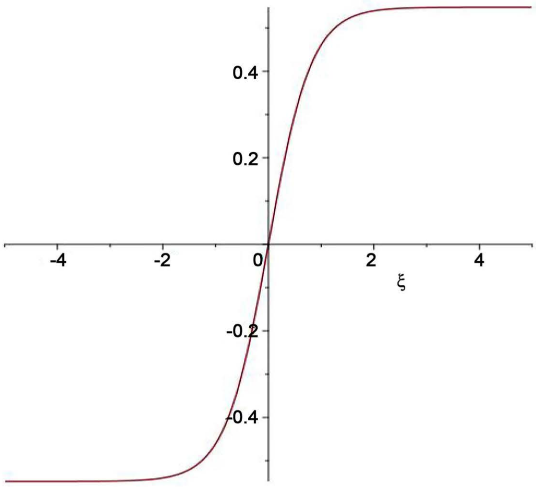
$$\beta = 5, \delta = 1, \gamma = 2, c_2 = 1, c_4 = -1, c = 2,$$

so the figures of $u_{32}(x,t)$ for (6) are like to Figure 13.

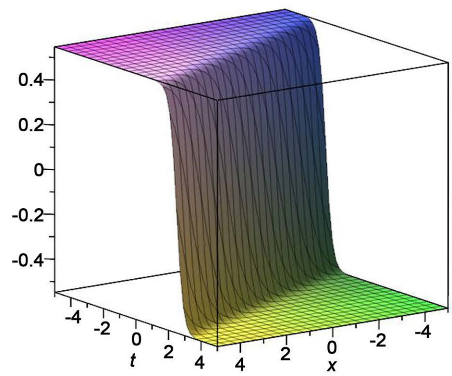
Example 14. In this example, for the solution $u_{34}(x,t)$, that is (91), we assume the following parameters

$$\beta = 3, \delta = 2, \gamma = 1, c_2 = -3, c_4 = 2, c = 2,$$

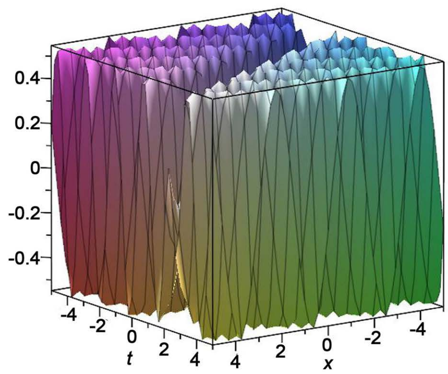
so the figures of $u_{34}(x,t)$ for (6) are like to Figure 14.



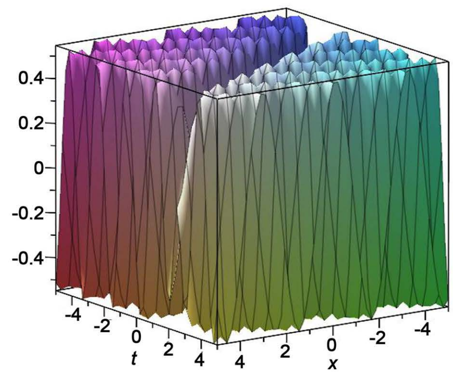
(a) The figure of the amplitude of the solution u_{34} in 2D



(b) The figure of the amplitude of the solution u_{34} in 3D



(c) The real part of u_{34} in 3D



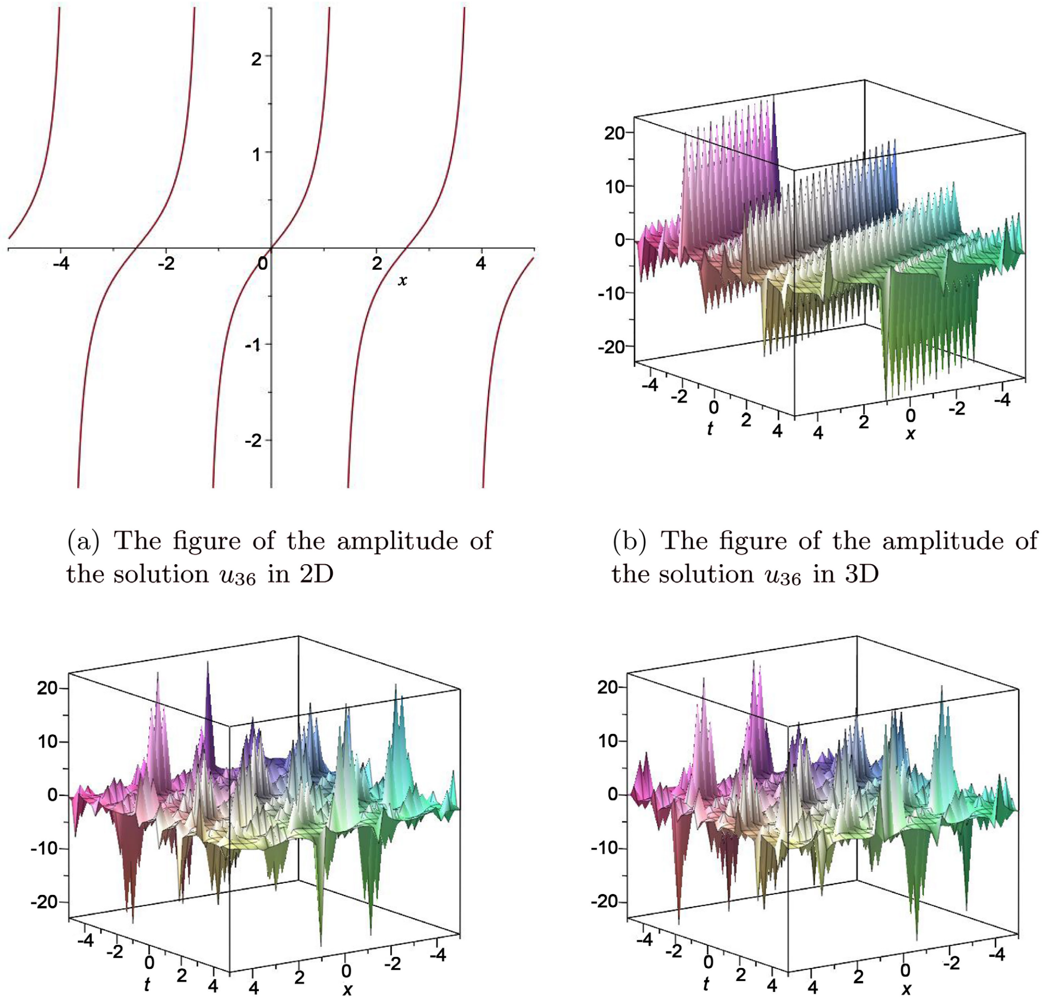
(d) The imaginary part of u_{34} in 3D

Figure 14. (a) The figure of the amplitude of the solution u_{34} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{34} shown in the three-dimensional space. (c) The real part of the solution u_{34} shown in the three-dimensional space. (d) The imaginary part of the solution u_{34} shown in the three-dimensional space.

Example 15. In this example, for the solution $u_{36}(x,t)$, that is (96), we assume the following parameters

$$\beta = 3, \delta = 2, \gamma = 1, c_2 = 3, c_4 = 2, c = 2,$$

so the figures of $u_{36}(x,t)$ for (6) are like to **Figure 15**.



(a) The figure of the amplitude of the solution u_{36} in 2D

(b) The figure of the amplitude of the solution u_{36} in 3D

(c) The real part of u_{36} in 3D

(d) The imaginary part of u_{36} in 3D

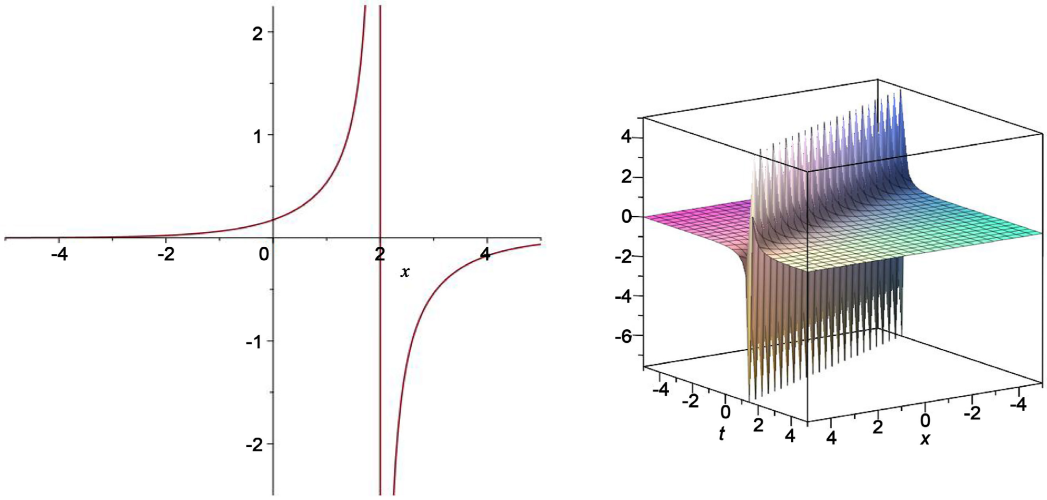
Figure 15. (a) The figure of the amplitude of the solution u_{36} shown in the two-dimensional space. (b)The figure of the amplitude of the solution u_{36} shown in the three-dimensional space. (c) The real part of the solution u_{36} shown in the three-dimensional space. (d) The imaginary part of the solution u_{36} shown in the three-dimensional space.

Example 16. In this example, for the solution $u_{37}(x,t)$, that is (98), we assume the following parameters

$$\beta = 2, \delta = 3, \gamma = 1, c = 2, D_4 = 2,$$

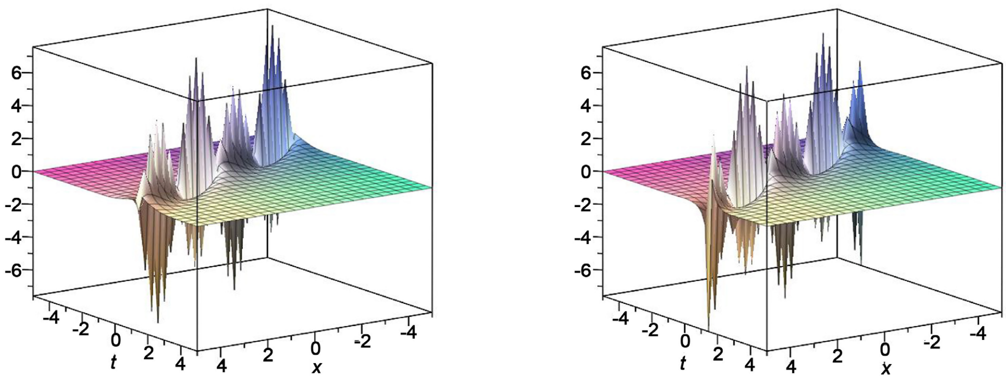
so the figures of $u_{37}(x,t)$ for (6) are like to **Figure 16**.

The CGPE without an external potential mainly describes the dynamic behavior of multiple interacting boson systems, and the dynamics of the system are determined entirely by the interactions between the particles. Naturally, by prescribing different β, δ and γ, one can obtain different a_2, and b_2. Correspondingly, more novel dynamic behaviors of derived exact solutions will be revealed.



(a) The figure of the amplitude of the solution u_{37} in 2D

(b) The figure of the amplitude of the solution u_{37} in 3D



(c) The real part of u_{37} in 3D

(d) The imaginary part of u_{37} in 3D

Figure 16. (a) The figure of the amplitude of the solution u_{37} shown in the two-dimensional space. (b) The figure of the amplitude of the solution u_{37} shown in the three-dimensional space. (c) The real part of the solution u_{37} shown in the three-dimensional space. (d) The imaginary part of the solution u_{37} shown in the three-dimensional space.

Remark: It does not seem mathematically tractable to determine the figures of the other types solutions for CGPE (6), however, there is only tedious algebraic calculation process, thus, we omit the examples and the figures about them.

5. Conclusions

In this work, we study the exact solutions for CGPE (6) with no external potential. Here, the traveling wave solution transformation and modified polynomial method are used to deduce the exact solutions for CGPE. Through the solutions of the auxiliary equation, the exact solutions for CGPE are obtained, and abundant solutions with different physical structures are derived, all of which include one or more $\text{cn}, \text{sn}, \text{dn}, \text{sech}, \tanh, \sec$, etc. functions. It includes complex elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions. These solutions can be used to reveal how the interactions between boson components affect the system and how the system evolves under different conditions. And numerical simulation experiments show the dynamic properties of the exact solutions under certain conditions.

We, here, proposed the efficient modified polynomial expansion method by the auxiliary differential Equation (11) and the modified traveling wave solution transformation, by which we obtain more new exact solutions for CGPE (6). On comparing with the polynomial expansion method and the traveling wave solution transformation in handling a huge number of nonlinear dispersive and dissipative equations, the proposed scheme is more effective, powerful and reliable to be used in identical nonlinear dispersive models. Moreover, the modified polynomial expansion method and the modified traveling wave solution transformation can be used to solve any coupled high-order complex partial differential equations. Using this modified method, we get a set of nonlinear algebraic equations that can be solved by the Maple software. Also, the Maple software was applied over for both the graphical impersonation and the emulation. Finally, we can say that the method is a very strong scheme to find more new exact solutions for CGPE.

When exploring the exact solutions of the CGPE, although many studies have employed traditional methods such as the Hirota bilinear method and Darboux transformation to obtain specific types of solutions, these studies are often confined to a limited number of solution forms, such as soliton and breather solutions. In contrast, this study introduces an improved polynomial expansion method and an improved traveling wave solution transformation method. These methods not only significantly enrich the solution set of the CGPE but also achieve remarkable breakthroughs in the novelty of the solutions. Specifically, this study successfully obtains a variety of different types of exact solutions, the number of which far exceeds the results of previous studies. These solutions include traditional elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions. The discovery of these novel solutions not only demonstrates the vastness of the solution space of the CGPE but also provides additional perspectives for understanding the physical phenomena described by the equation. Moreover, numerical simulations reveal that these solutions have unique physical structures, such as specific periodicities. The discovery of these novel solutions not only deepens our understanding of the physical properties of the CGPE but also provides new theoretical support for experimental research and applications in related fields. These achievements not only highlight the complexity and diversity of the solution space of the CGPEs but also offer new directions and ideas for future research.

In addition, CGPE plays an important role in applied mathematics, applied physics, quantum physics, engineering, Bose-Einstein condensation, nonlinear optics, biophysics, finance, and oceanography. However, we only use the modified polynomial expansion method, the coupled modified traveling wave solution transformation, and the auxiliary differential equation obtaining more new exact solutions in this paper, and according to some special parameter values, we give the figures of some solutions. In future research, we will use other methods to study the structure and properties of solutions for CGPE, and the application of the solutions in practice.

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Ethics Approval

Not unethical.

Consent for Publication

Consent for publication.

Availability of Data and Material

The data and material are transparent and available.

Code Availability

The code comes from software or customization, and all code is transparent and available.

Authors' Contributions

The main idea of this paper was proposed by the author Junliang Lu, the authors Can Xu and Jianping Li wrote the main manuscript text and the figures. At last, all authors reviewed the manuscript.

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Conflicts of Interest

The authors declare they have no financial interests.

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