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# Base-X Conjecture and Collatz Conjecture Proof

**Sheng Zhao**

Vancouver, Canada
Email: zhao@shaw.ca

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## Abstract

A new Base-X Conjecture was introduced in this paper, and Collatz Conjecture is just one case of Base-X Conjecture - Base-3 (Ternary). Based on Base-X number system property and Collatz Conjecture iteration, it has been proved that for any positive integer *D*, there are *n* and *m* which exist for $D_n + Y_n = 2^m$. $D_n + Y_n$ is just the result built up by collecting divided by 2 of Collatz Conjecture iteration. Divided by $2^m$ will make the Collatz Conjecture get a result of 1 for any positive integer. Also, the Collatz Tree showed that for any odd positive number, there is only one route existing in the Collatz Tree down to 1 on Collatz Conjecture iteration.

## Keywords

Collatz Conjecture, Base-X Conjecture, Collatz Spiral, Collatz Ring, Collatz Tree

## 1. Introduction

Collatz Conjecture is one of the most famous unsolved problems in mathematics. It is simply stated, easily understood. Even many efforts to solve the problem have been made [1] [2], the Collatz conjecture itself remains open. Professor Paul Erdos said about the Collatz conjecture: "Mathematics may not be ready for such problems" [3]. Professor Jeffrey Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present day mathematics" [4]. Even more research on Collatz Conjecture [5]-[16], it is still not able to prove that Collatz Conjecture will turn to 1 for any positive integer. But what Collatz Conjecture exactly is, and why is $3n + 1$? It is suggested in this paper that Collatz Conjecture is related to such a conjecture called Base-X Conjecture, in a Base-X number system, for any positive integer, if divisible by the maximum number of the base $X_1$ (even-x number), divide it by $X_1$ (even-step), if not (odd-x number), make it be divisible by $X_1$ (odd-step), repeat this iteration, the result 1 could be reached. A number is called ***pure even-x number*** if can be expressed as $X_1^k$. Number 1 is a special number in Base-X number system, for number 1 itself is an odd-x number, but $X_1^0$ is a ***pure even-x number*** by definition and still equals to 1.

## 2. Base-X Number System

A Base-X Number System uses $X$ different digits (from 0 to $X_1$) to represent numbers, with each digit's positional value increasing by a power of $X$ as moving left in the number. The digits can be paired in $X_1$'s compliment that sum of each pair equals $X_1$. Pair [0, $X_1$] are even-x digit, the rest are odd-x digit. see Figure 1: Base-X.

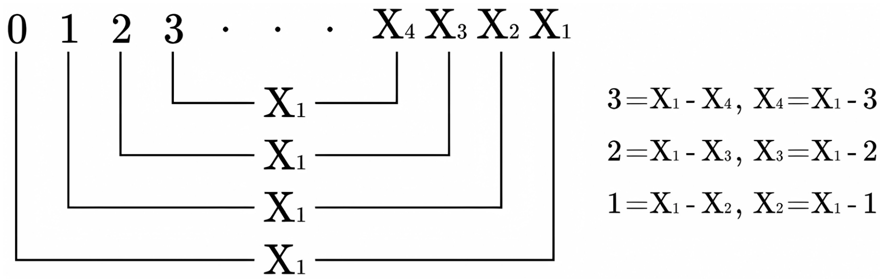


Figure 1. Base-X.

If $X$ is an odd number, the middle one is just paired itself. See Figure 2: Base-7 (Septenary).

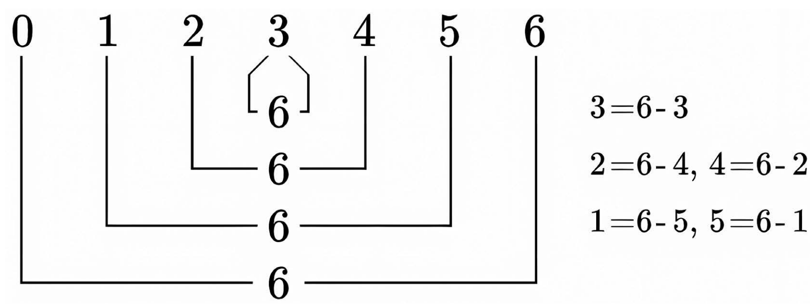


Figure 2. Base-7 (Septenary).

### Simple Properties of Base-X Number System

(1) Any positive number $D$ in Base-X Number System can be expressed as the product of a ***pure even-x number*** and an odd-x number:

$$D = X_1^k\left(Q * X_1 + r\right) \qquad r = 1, 2, 3, \cdots, X_4, X_3, X_2 \tag{1}$$

$Q$ is either odd-x or even-x number.

For any odd-x number $D$, $k = 0$, $D = Q * X_1 + r$.

If $Q = 0$ and $k = 0$, $D$ is a single digit odd-x number, $D = r$.

If $Q = 0$ and $r = 1$, $D$ is a ***pure even-x number***, $D = X_1^k$.

(2) multiple $D$ by $X$ did not affect $k$ and $r$ in property (1)

$$\begin{aligned} D*X &= X_1^k\left(Q*X_1+r\right)X \\ &= X_1^k\left(Q*X_1+r\right)\left(X_1+1\right) \\ &= X_1^k\left[\left(Q*X_1+r\right)X_1+Q*X_1+r\right] \\ &= X_1^k\left[\left(Q*X_1+r+Q\right)X_1+r\right] \\ &= X_1^k\left(Q_1*X_1+r\right) \end{aligned} \tag{2}$$

(3) from property (2) can be derived that multiple $D$ by $X^n$ did not affect $k$ and $r$

$$D*X^n = X_1^k\left(Q_n*X_1+r\right) \tag{3}$$

if $D$ is a single digit $r$, the remainder of $r*X^n$ divided by $X_1$ is $r$ itself.

(4) any odd-x $D$ can be converted to even-x number by:

$$\begin{aligned} & D*X^n+r' \qquad r' \text{ is } X_1\text{'s compliment of } r \\ & = Q_n*X_1+r+r' \\ & = Q_n*X_1+X_1 \\ & = \left(Q_n+1\right)X_1 \end{aligned} \tag{4}$$

(5) for any integer $D$, adding up all the digits of $D$ until to get a single digit $r$, if the result $r = X_1$, $D$ is even-x number, otherwise $D$ is odd-x number, $r$ is the remainder of $D$ divided by $X_1$.

Any positive integer D can be expressed as:

$$D=\sum_{i=0}^{n} d_i X^i \quad d=1,2,3,\cdots,X_3,X_2,X_1$$

from property (3), each item $d_iX^i$ divided by $X_1$, the remainder should be $d_i$, so the remainder $R$ of $D$ divided by $X_1$ should be:

$$R=\sum_{i=0}^{n} d_i \quad d=1,2,3,\cdots,X_3,X_2,X_1$$

$R$ is a new number with less digits than $D$, repeating above processing, a single digit could be obtained.

Take Hexadecimal for example ($X_1 = F$):

for hex number $2E35F9BD67$

$$2+E+3+5+F+9+B+D+6+7=55$$

$$5+5=A$$

so it is an odd-16 number, and the remainder of divided by $F$ is $A$.

$$2E35F9BD67=314AA3FD3*F+A$$

for hex number $7E35F9BD67$

$$7+E+3+5+F+9+B+D+6+7=5A$$

$$5+A=F$$

so it is an even-16 number.

$$7E35F9BD67=869FF9529*F$$

## 3. Base-X Conjecture

For the simply stated Base-X Conjecture in Section 1, property (4) was used to make an odd-x number ($k = 0$) to an even-x number.

### 3.1. $n = 0$

For any odd-x positive integer $D$

$$D + r' = Q * X_1 + r + r' = (Q + 1) X_1$$

divided by $X_1$ result is $Q + 1$. The Base-X Conjecture iteration is converging due to $D > Q + 1$, so the Conjecture is true.

### 3.2. $n \geq 2$

The Base-X Conjecture iteration is mostly diverging, so the Conjecture could be false. This will not be discussed in this paper.

### 3.3. $n = 1$

$$\begin{aligned} & D * X + r' \\ &= (Q * X_1 + r) X_1 + r + r' \\ &= (Q * X_1 + r) X_1 + X_1 \\ &= (Q * X_1 + r + 1) X_1 \end{aligned}$$

(a) for a single digit number $Q = 0$

$$r * X + r' = r(X_1 + 1) + r' = r * X_1 + (r + r') = r * X_1 + X_1 = (r + 1) * X_1$$

the result $r$ is increased by 1 after one odd-step and one even-step. If the result ($r$ + 1) = $X_1$, take another even-step that will get 1, otherwise repeat the iteration until the result ($r$ + 1) = $X_1$. Base-X conjecture is true for a single digit number.

(b) if ($r$ + 1) = $X_1$ and ($Q$ + 1) = $X_1^m$, Base-X conjecture is true.

$D = (X_1^{2n} - 1) / X$, ($n = 1, 2, 3, \cdots$) meets this condition.

### 3.4. Base-X Conjecture Definition

Based on above analysis, use property (4) of Base-X number system with $n = 1$ to convert odd-x number to even-$x$.

- If the number is even-x, divide it by $X_1$.
- If the number is odd-x, left shift 1 digit (multiply by $X$) and add $X_1$'s compliment of the remainder divided by $X_1$.

In modular arithmetic notation, define the function $f$ as follows:

$$f(n) = \begin{cases} n/X_1 & \text{if } n \equiv 0 (\operatorname{mod} X_1) \\ X * n + r' & \text{if } n \equiv r (\operatorname{mod} X_1) \end{cases}$$

for Collatz Conjecture, it should be one of Base-X conjecture - Base-3 (Ternary). For Base-3 (Ternary), $X_1$ = 2, and there is only one remainder 1 if odd-3 (odd) number divided by 2, 2's compliment of 1 is still 1, here comes $3n$ + 1. The Base-X conjecture for Base-3 (Ternary) is as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 (\operatorname{mod} 2) \\ 3 * n + 1 & \text{if } n \equiv 1 (\operatorname{mod} 2) \end{cases}$$

It is exactly the same as Collatz Conjecture.

### 3.5. Base-X Conjecture Verification

Till now, the Base-X Conjecture is neither proved nor disproved, but a preliminary verification can be done by computer. There are a total of 28 bases from Base-3 (Ternary) to Base-30 (Tricenary) up to 10 digits have been verified, the verification results are shown in Table 1.

Table 1. The verification results from Base-3 (Ternary) to Base-30 (Tricenary).

| Base | | Digit Number | | | | | | | | | | Loop#1 Steps | | | Loop#2 Steps | | | Loop#3 Steps | | | True False |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | Name | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Odd | Even | Total | Odd | Even | Total | Odd | Even | Total | |
| 3 | Ternary (Tertial) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | | | | | | | TRUE? |
| 4 | Quaternary (Quartal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 3 | 5 | 7 | 8 | 16 | | | | FALSE |
| 5 | Quinary (Quintal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 4 | 7 | 12 | 14 | 26 | | | | FALSE |
| 6 | Senary (Sextal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 5 | 9 | | | | | | | TRUE? |
| 7 | Septenary (Septimal) | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 6 | 11 | 11 | 12 | 23 | 23 | 25 | 48 | FALSE |
| 8 | Octonary (Octal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 7 | 13 | | | | | | | TRUE? |
| 9 | Nonary (Nonal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 8 | 15 | | | | | | | TRUE? |
| 10 | Denary (Decimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 9 | 17 | 20 | 21 | 41 | | | | FALSE |
| 11 | Undenary (Undecimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 10 | 19 | 47 | 49 | 96 | | | | FALSE |
| 12 | Duodenary (Duodecimal) | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 11 | 21 | 55 | 57 | 112 | | | | FALSE |
| 13 | Terdenary (Tredecimal) | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 | 12 | 23 | 93 | 96 | 189 | 31 | 32 | 63 | FALSE |
| 14 | Quattuordenary (Quattuordecimal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 12 | 13 | 25 | | | | | | | TRUE? |
| 15 | Quindenary (Quindecimal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 13 | 14 | 27 | | | | | | | TRUE? |
| 16 | Senidenary (Hexadecimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 14 | 15 | 29 | 40 | 41 | 81 | | | | FALSE |
| 17 | septendenary (Septendecimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 15 | 16 | 31 | 45 | 46 | 91 | | | | FALSE |
| 18 | Octodenary (Octodecimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 16 | 17 | 33 | 48 | 49 | 97 | | | | FALSE |
| 19 | Novendenary (Novendecimal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 17 | 18 | 35 | | | | | | | TRUE? |
| 20 | Vigenary (Vigesimal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 18 | 19 | 37 | | | | | | | TRUE? |
| 21 | Viginti-unary (Viginti-unal) | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 19 | 20 | 39 | 59 | 60 | 119 | 60 | 61 | 121 | FALSE |
| 22 | Viginti-binary (Viginti-dual) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | 20 | 21 | 41 | | | | | | | TRUE? |
| 23 | Viginti-ternary (Viginti-tertial) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | - | 21 | 22 | 43 | | | | | | | TRUE? |
| 24 | Viginti-quaternary (Viginti-quartal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | - | - | - | 22 | 23 | 45 | 71 | 72 | 143 | | | | FALSE |
| 25 | Viginti-quinary (Viginti-quintal) | 1 | 2 | 0 | 0 | 0 | 0 | 0 | - | - | - | 23 | 24 | 47 | 77 | 78 | 155 | 155 | 157 | 312 | FALSE |
| 26 | Viginti-senary (Viginti-sextal) | 1 | 0 | 1 | 0 | 0 | 0 | 0 | - | - | - | 24 | 25 | 49 | 82 | 83 | 165 | | | | FALSE |
| 27 | Viginti-septenary (Viginti-septimal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | - | 25 | 26 | 51 | | | | | | | TRUE? |
| 28 | Viginti-octonary (Viginti-octal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | - | - | - | 26 | 27 | 53 | 90 | 91 | 181 | | | | FALSE |
| 29 | Viginti-nonary (Viginti-nonal) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | - | - | - | 27 | 28 | 55 | | | | | | | TRUE? |
| 30 | Tricenary (Trigesimal) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | - | - | - | 28 | 29 | 57 | 96 | 97 | 193 | | | | FALSE |

Note: - not verified.

The table showed loops found for different digits number, and the steps for the loops found. There are a total of 28 bases verified, 16 bases are false due to more than 1 loop found. 12 bases could be true including Collatz Conjecture (Base-3). There is no diverging found, and no more loops found for more than 3 digits to maximum checked digits. Now we will try to prove the Base-3 (Collatz Conjecture) for it is the simple one, only dealing with a single digit 1.

## 4. Collatz Conjecture Proof

If a positive integer $D$ after $n$ odd-step and $m$ even-step turn to 1, this can be expressed as:

$$D_n + Y_n = 2^m \tag{5}$$

where $D_n = 3^n * D$. For any positive integer $D$, if such $Y_n$ existed based on Collatz Conjecture iteration, Collatz Conjecture should be true. Now what is needed to do is to figure out how $Y$ is built up along with Collatz Conjecture iteration. From now on, odd and even are used instead of odd-3 and even-3 for Base-3 (Ternary). Decimal number will be used for convenience if not stated otherwise.

### 4.1. Y Built up

It is already stated that "Collatz Conjecture, it is just one case of Base-X conjecture for Base-3 (Ternary)" in 3.4, so simple properties of Base-X number system could be used to figure it out. For any positive integer $D$ in Base-3 can be expressed as following based on property (3):

$$D = 2^{k(0)}(Q*2+1) \qquad k(0) = 0,1,2,3,\cdots \tag{6}$$

Now we can apply Collatz Conjecture iteration to the odd part and collect the divided by 2 and put it to the even part. Any odd number to another odd number, one odd-step and one or more even-step (simply called one step together) needed. For example:

1) Number 3:

$$3*3+1=10 \to 5,$$

one odd step and one even step, collect one divided by 2 → $2^{k(0)+1}$.

2) Number 9:

$$3*9+1=28 \to 14 \to 7,$$

one odd step and two even steps, collect two divided by 2 → $2^{k(0)+2}$.

3) Number 13:

$$3*13+1=40 \to 20 \to 10 \to 5,$$

one odd step and three even steps, collect three divided by 2 → $2^{k(0)+3}$.

4) Number 37:

$$3*37+1=112 \to 56 \to 28 \to 14 \to 7,$$

one odd step and four even steps, collect four divided by 2 → $2^{k(0)+4}$.

Generally for Equation [6], just apply $\boldsymbol{3n+1}$ ***and divided by*** **2** to the odd part

$(Q * 2 + 1)$ and pass divided by 2 to $2^k$ portion we got:

$$\begin{aligned} &2^{k(0)}\left[3(Q*2+1)+1\right] \\ &=3*\left[2^{k(0)}(Q*2+1)\right]+2^{k(0)} \\ &=3*D+2^{k(0)} && \text{collect } 0 \\ &=D_1+Y_1 && D_1=3^1*D, Y_1=2^{k(0)}, D_1>Y_1 \end{aligned} \tag{7}$$

in another way,

$$\begin{aligned} &2^{k(0)}\left[3(Q*2+1)+1\right] \\ &=2^{k(0)}\left[3*Q*2+4\right] \\ &=2^{k(0)+1}\left[3*Q+2\right] \\ &=2^{k(0)+1}*2^x*(Q_1*2+1) \\ &=2^{k(0)+(1+x)}*(Q_1*2+1) && \text{collect } 1+x, x=0,1,2,3,\cdots \\ &=2^{k(1)}*(Q_1*2+1) && k(1)=k(0)+(1+x) \end{aligned} \tag{8}$$

after one $3n + 1$ and one or more divided by 2 we got a new equation from Equation [6]:

$$D_1+Y_1=2^{k(1)}*(Q_1*2+1) \quad k(1)>k(0) \tag{9}$$

Compare to Equation [6]:

$$D \to D_1+Y_1,\ 2^{k(0)} \to 2^{k(1)}=2^{k(0)+(1+x)},\ Q \to Q_1$$

Apply another $\mathbf{3n + 1}$ ***and divided by*** **2** to the odd part $(Q_1 * 2 + 1)$ of Equation [9] we got:

$$\begin{aligned} &2^{k(1)}\left[3(Q_1*2+1)+1\right] \\ &=3*\left[2^{k(1)}(Q_1*2+1)\right]+2^{k(1)} \\ &=3*D_1+3*Y_1+2^{k(1)} \\ &=D_2+Y_2 && D_2=3^2*D, Y_2=3*Y_1+2^{k(1)} \end{aligned}$$

in other way,

$$\begin{aligned} &2^{k(1)}\left[3(Q_1*2+1)+1\right] \\ &=2^{k(1)}\left[3*Q_1*2+4\right] \\ &=2^{k(1)+1}\left[3*Q_1+2\right] \\ &=2^{k(1)+(1+x)}(Q_2*2+1) && \text{collect } 1+x, x=0,1,2,3,\cdots \\ &=2^{k(2)}(Q_2*2+1) && k(2)>k(1) \end{aligned}$$

same as Equation [9] we got

$$D_2+Y_2=2^{k(2)}(Q_2*2+1)$$

repeat the $\mathbf{3n + 1}$ ***and divided by*** **2** on the odd part to step $n$ we could get:

$$D_n+Y_n=2^{k(n)}(Q_n*2+1) \tag{10}$$

The above process could be interpreted as it approaches a certain ***pure even-x***

***number*** $2^m$ with continuing iteration, then we can get the following equation by combining Equation [5] and [10]:

$$2^{m-1} < D_n + Y_n = 2^{k(n)}\left(Q_n * 2 + 1\right) \le 2^m \tag{11}$$

let $Y_0 = 0$, then

$$Y_1 = 2^{k(0)} = 3 * Y_0 + 2^{k(0)}$$

it can be figured out that $Y$ is built up as:

$$Y_0 = 0, Y_1 = 3 * Y_0 + 2^{k(0)}, Y_2 = 3 * Y_1 + 2^{k(1)}, \cdots, Y_n = 3 * Y_{n-1} + 2^{k(n-1)}$$

$$k(0) < k(1) < \cdots < k(n-1) < k(n)$$

The minimum step of $k$ is 1, and for any odd number $k(0) = 0$, the minimum $k$ sequence should be:

$$k = 0, 1, 2, 3, \cdots$$

and the minimum $Y$ sequence is as:

$$Y = 0, 1, 5, 19, 65, 211, \cdots, 3^n - 2^n \quad [17].$$

### 4.2. Find $Y_n$

As described above, $D$ and $Y$ are both multiplied by 3, $D$ is not changed, but $Y$ is added $2^k$ each time, so that $Y$ is getting bigger and bigger, at the beginning $Y_1 < D_1$, along with the iteration continuing, $Y$ will exceed $D$ and turn to $Y > D$ from $Y \le D$ at a certain step. **Figure 3** shows the change in Base-3 (Ternary) - $D$ is simply shifting left and $Y$ is growing up along with left shifting.

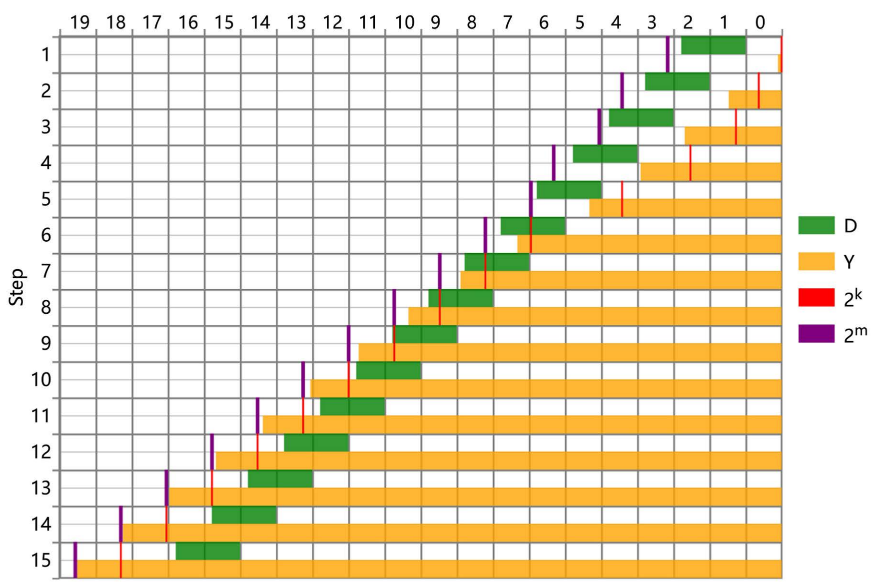


**Figure 3.** $D$-$Y$ change in Base-3 (Ternary).

Assuming $Y_{n-1} \le D_{n-1}$ at step $n-1$, and $Y_n > D_n$ at step $n$, from Equation (11) we got:

$$2^{m-1} < D_n + Y_n = D_n + 3 * Y_{n-1} + 2^{k(n-1)} \le 2^m \tag{12}$$

$$2^{m-1} < D_n + 3 * Y_{n-1} + \Delta_n = 2^m \tag{13}$$

See Figure 4 [A, B]. To ensure from $Y_{n-1} \le D_{n-1}$ to $Y_n > D_n$, $2^{k(n-1)}$ (difference between $D_n$ and $3^* Y_{n-1}$) must take the possible maximum value. From Equation (12), the maximum value of $2^{k(n-1)}$ could be $2^{m-1}$. But if $2^{k(n-1)} = 2^{m-1}$, $D_n + 3^* Y_{n-1}$ should be $\le 2^{m-1}$, and result in $3^* Y_{n-1} = 0$ and $D_n = 2^{m-1}$. For $Y_{n-1}$ cannot be 0, so that $2^{k(n-1)}$ must be less than $2^{m-1}$, and the maximum possible value should be $2^{m-2}$. So we got:

$$2^{m-2} \le D_n < 2^{m-1}$$

$$2^{m-2} \le 3 * Y_{n-1} < 2^{m-1}$$

$$2^{m-1} < D_n + 3 * Y_{n-1} \le 2^m$$

See Figure 4 [C, D]


Figure 4. Find $Y_n$.

From above analysis, we can get:

$$\Delta_n = 2^{k(n-1)} = 2^{m-2}$$

so that

$$3*Y_{n-1}+\Delta_n=3*Y_{n-1}+2^{k(n-1)}=Y_n \quad \text{(see Figure 4 [F, G, H])}$$

$Y_n$ satisfies Equation [5], Collatz Conjecture should be TRUE for any positive integer.

Please note that, the $Y_n$ based on condition " $Y_{n-1}\le D_{n-1}$ to $Y_n > D_n$ " is not the first one satisfies Equation [5], it is just used to prove such $Y_n$ exists. The test results shown that the first $Y$ satisfies Equation [5] should be 2 steps earlier. See Figure 5 (for number 7). At step 7, $Y$ changed from less than $D$ to greater than $D$, and the first $Y$ satisfies Equation [5] happened at step 5.

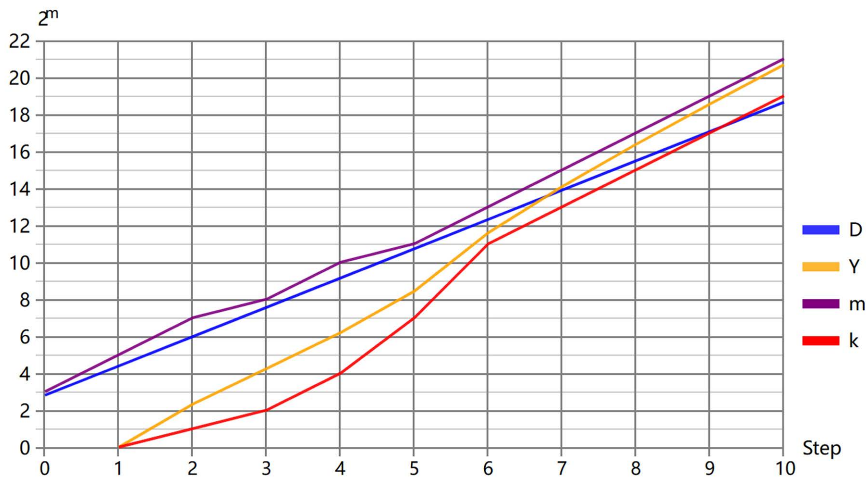


Figure 5. Changes of $D$, $Y$, $m$ and $k$.

From step 5, $m$ became a straight line which means the odd part of Equation [6] keeps the same number. The minimum difference between $m$ and $k$ should be 4 (from number 5, $3 \times 5 + 1 = 16 = 2^4$, will be explained later) at the first time that $Y$ satisfies Equation [5] (step 5 in Figure 5). From step 6, $k$ became a straight line, and $m - k = 2$, which means the odd part of Equation [6] goes into the 1 → 4 → 2 → 1 loop as Collatz Conjecture iteration.

## 5. Collatz Conjecture Analysis

From now on, odd and even are used instead of odd-3 and even-3 for Base-3 (Ternary). Any even number can be converted to odd number divided by 2 according Collatz Conjecture, the analysis will only focus on odd numbers. At first, group the odd numbers into 3 groups according to the last digit $d$ in Base-3 (ternary), same as the result mod by 3 in decimal.

On Collatz Conjecture iteration, for any odd number $D = Q * 2 + 1$.

- Group 0: $d = 0$, there should no any odd number come to this group on Collatz Conjecture iteration, go to Group 2 if $Q$ is odd, go to Group 1 if $Q$ is even.
- Group 1: $d = 1$, go to Group 2 if $Q$ is odd, stay in Group 1 if $Q$ is even.
- Group 2: $d = 2$, go to Group 1 if $Q$ is even, stay in Group 2 if $Q$ is odd.

For convenience, decimal is used for the following analysis instead of Base-3 (ternary).

### 5.1. Collatz Conjecture Odd Number Groups

All odd number are grouped by $4n + 1$ sequence for Collatz Conjecture. All numbers in the same sequence will go to the same odd number on Collatz Conjecture iteration with one odd step and one or more even steps:

$$3*(4n+1)+1=3*4n+3+1=3*4n+4=4*(3n+1)\rightarrow 3n+1$$

#### 5.1.1. Odd Number 1

Start from 1 we got:

$\boldsymbol{S}(1)$ = 1, 5, 21, 85, 341, ...

the $n$th number is: $\boldsymbol{S}(1)[n]=\left(2^{2n}-1\right)/3 \quad n=1,2,3,\cdots$

All numbers in $\boldsymbol{S}(1)$ are going to 1 on one odd-step and 2, 4, 6, 8, 10, ... even-step. (even sequence).

#### 5.1.2. Odd Number 3

$\boldsymbol{S}(3)$ = 3, 13, 53, 213, 841, ...

the $\boldsymbol{n}$th number is: $\boldsymbol{S}(3)[n]=2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

All numbers in $\boldsymbol{S}(3)$ are going to 5 (SeqN) on one odd-step and 1, 3, 5, 7, 9, ... even-step. (odd sequence).

#### 5.1.3. Odd Number 5

5 is 2nd number of $\boldsymbol{S}(1)$.

#### 5.1.4. Odd Number 7

$\boldsymbol{S}(7)$ = 7, 29, 117, 469, 1877, ... - odd sequence, SeqN = 11.

the $n$th number is: $\boldsymbol{S}(7)[n]=3*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.5. Odd Number 9

$\boldsymbol{S}(9)$ = 9, 37, 149, 597, 2389, ... - even sequence, SeqN = 7.

the $n$th number is: $\boldsymbol{S}(9)[n]=2*2^{2n}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.6. Odd Number 11

$\boldsymbol{S}(11)$ = 11, 45, 181, 725, 2901, ... - odd sequence, SeqN = 17.

the $n$th number is: $\boldsymbol{S}(11)[n]=5*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.7. Odd Number 13

13 is the 2nd number of $\boldsymbol{S}(3)$.

#### 5.1.8. Odd Number 15

$\boldsymbol{S}(15)$ = 15, 61, 245, 981, 3925, ... - odd sequence, SeqN = 23.

the $n$th number is: $\boldsymbol{S}(15)[n]=7*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.9. Odd Number 17

$\boldsymbol{S}(17)$ = 17, 69, 277, 1109, 4437, ... - even sequence, SeqN = 13.

the $n$th number is: $\boldsymbol{S}(17)[n]=4*2^{2n}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.10. Odd Number 19

*S*(19) = 19, 77, 309, 1237, 4949, ... - odd sequence, SeqN = 29.

the *n*th number is: $\boldsymbol{S}(19)[n]=9*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.11. Odd Number 21

21 is the 3rd number of *S*(1).

#### 5.1.12. Odd Number 23

*S*(23) = 23, 93, 373, 1493, 5973, ... - odd sequence, SeqN = 35.

the *n*th number is: $\boldsymbol{S}(23)[n]=11*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.13. Odd Number 25

*S*(25) = 25, 101, 405, 1621, 6485, ... - even sequence, SeqN = 19.

the *n*th number is: $\boldsymbol{S}(25)[n]=6*2^{2n}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.14. Odd Number 27

*S*(27) = 27, 109, 437, 1749, 6997, ... - odd sequence, SeqN = 41.

the *n*th number is: $\boldsymbol{S}(3)[n]=13*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.15. Odd Number 29

29 is the 2nd number of *S*(7).

#### 5.1.16. Odd Number 31

*S*(31) = 31, 125, 501, 2005, 8021, ... - odd sequence, SeqN = 47.

the *n*th number is: $\boldsymbol{S}(31)[n]=15*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.17. Odd Number 33

*S*(33) = 33, 133, 533, 2133, 8533, ... - even sequence, SeqN = 25.

the *n*th number is: $\boldsymbol{S}(33)[n]=8*2^{2n}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.18. Odd Number 35

*S*(35) = 35, 141, 565, 2261, 9045, ... - odd sequence, SeqN = 53.

the *n*th number is: $\boldsymbol{S}(35)[n]=17*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$

#### 5.1.19. Odd Number 37

37 is the 2nd number of *S*(9).

... ...

For any odd number $D = Q*2 + 1$,

- start a new odd sequence if $Q$ is odd number.
- start a new even sequence if $Q$ is even and $Q/2$ is odd number.
- a sequence member otherwise.

if the integer quotient of SeqN of *S*($x$) divided by 3 is $q$, for a odd sequence, the *n*th number is:

$$\boldsymbol{S}(x)[n]=q*2^{2n-1}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$$

for an even sequence, the *n*th item is:

$$\boldsymbol{S}(x)[n]=q*2^{2n}+\boldsymbol{S}(1)[n] \quad n=1,2,3,\cdots$$

### 5.2. Collatz Spiral

If line up the sequences described above, and put the odd numbers in a chart, we can get a spiral as shown in **Figure 6** Collatz Spiral. All numbers on the same radial line are in one sequence. An odd sequence will go to an odd number greater than the first sequence number. An even sequence will go to an odd number less than the first sequence number. $\mathcal{S}(1)$ is a special even sequence just loop on its first number 1.

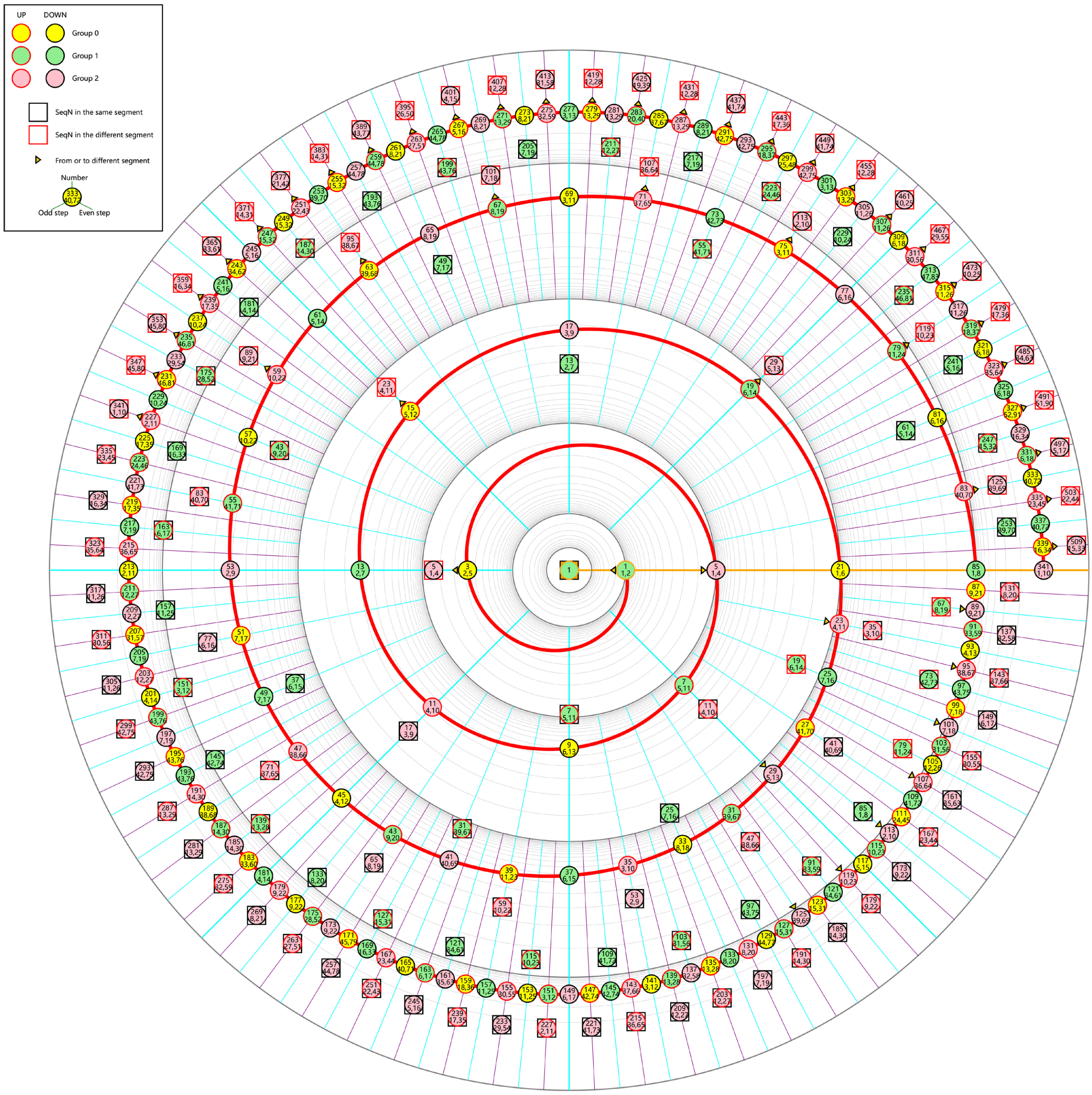


**Figure 6.** Collatz spiral.

### 5.3. Collatz Ring

From **Figure 6** we can see that each time the number across $\mathcal{S}(1)$, the new sequences will be added. If breaks at that point and makes rings, the sequences can show the sequences more clearly as shown in **Figure 7**. Expanding the ring by applying $4n + 1$ on each number, 3 new sequences added between any adjacent two numbers.

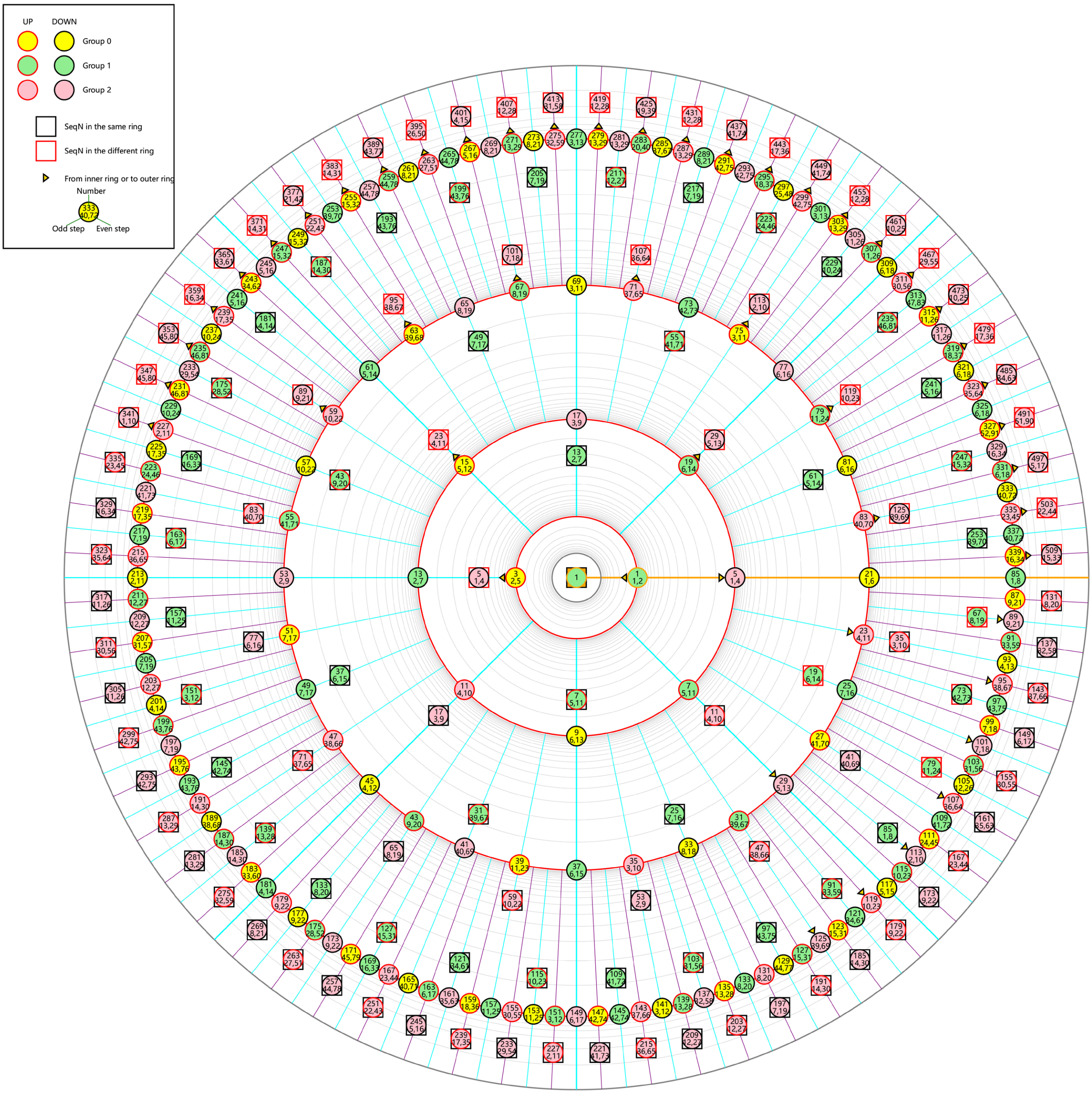


**Figure 7.** Collatz Ring.

## 5.4. Collatz Tree

From Collatz Ring, it can be found that "the corresponding relationship of an odd number and a $4n + 1$ sequence can be only one-to-one correspondence".

$S(1)$ -> 1

$S(3)$ -> 5

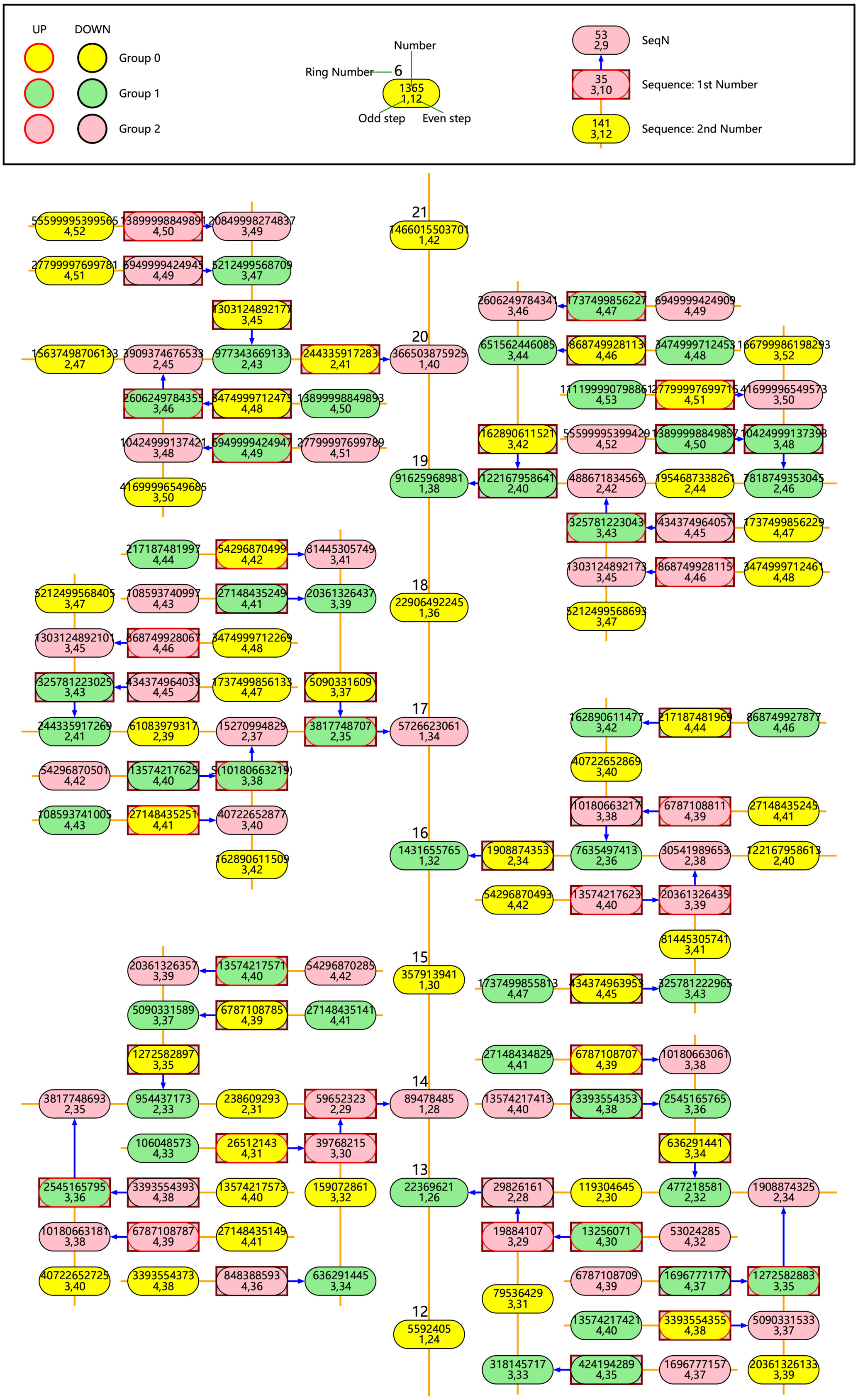


**Figure 8.** Collatz tree (A).

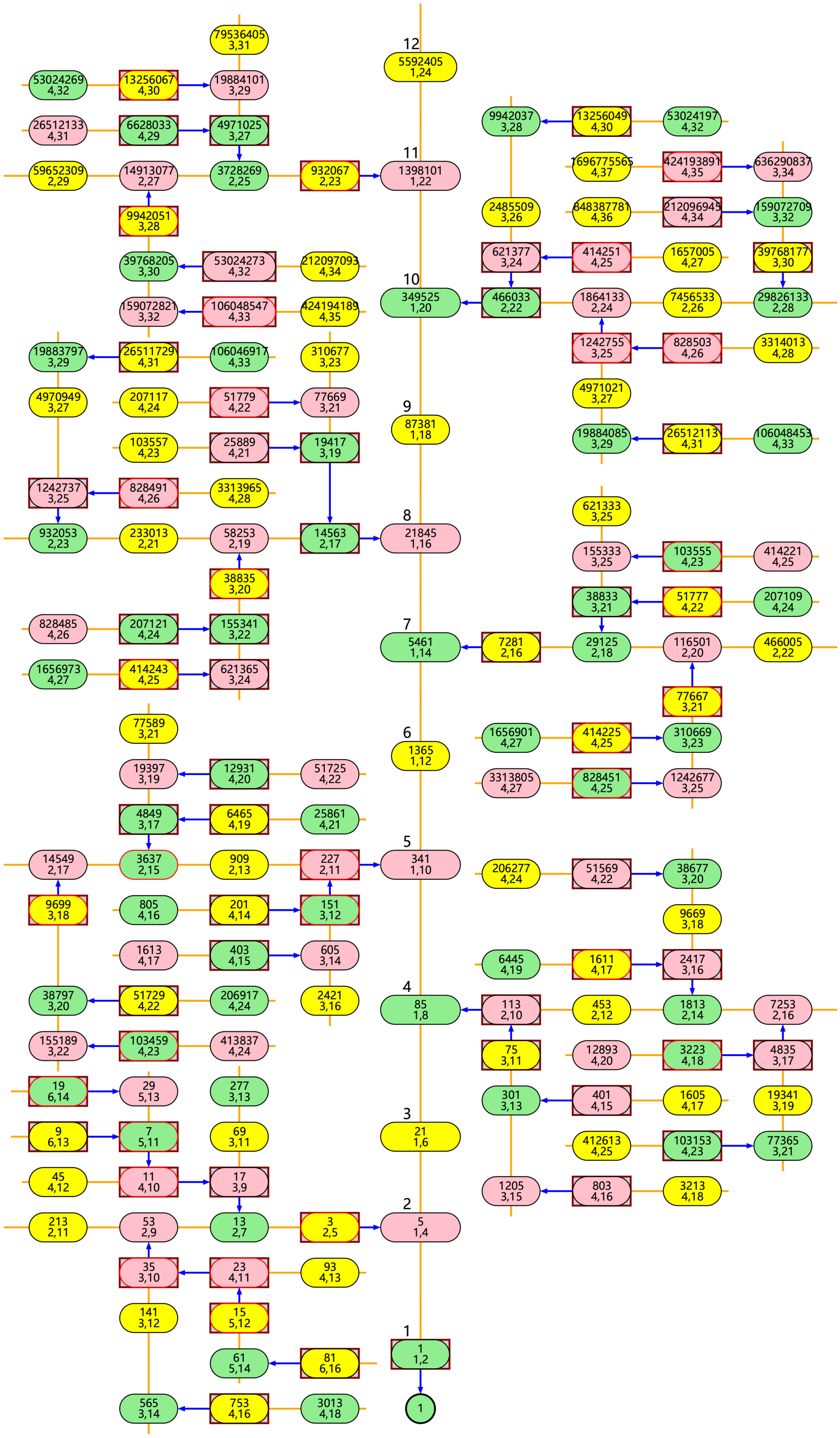


**Figure 9.** Collatz tree (B).

$S(7)$ -> 11
$S(9)$ -> 7
$S(11)$ -> 17
$S(15)$ -> 23
$S(17)$ -> 13
$S(19)$ -> 29
$S(23)$ -> 35
$S(25)$ -> 13
$S(27)$ -> 41
$S(31)$ -> 47
$S(33)$ -> 25
$S(35)$ -> 53
… …

So that only odd numbers in $S(1)$ can go to 1 on Collatz Conjecture iteration, and the minimum odd number turn to 1 is 5 (the 2nd number). To take $S(1)$ as the trunk, there should be only one branch ($4n + 1$ sequence) connected to each number in the $S(1)$. In the same way, there should be only one sub branch connected each number in the branches. For any branch, there should no loop exist. For any odd number on the tree, there should be only one route down to the trunk $S(1)$ on Collatz Conjecture iteration, then go to 1. All of the odd numbers should be on the tree which means for any odd number, there is a route and only one route to the trunk $S(1)$ then to 1 on Collatz Conjecture iteration. Figure 8 and Figure 9 show a Collatz tree with 21 numbers in the $S(1)$ trunk. There is no branch connected to Group 0 number.

## 6. Conclusion

For Collatz Conjecture is just one case of new introduced Base-X Conjecture - Base-3 (Ternary), and based on Base-X number system property and Collatz Conjecture iteration, it has been proved that for any positive integer $D$, there are $n$ and $m$ existing for $D^n + Y^n = 2^m$. $D^n + Y^n$ is just the result built up by collecting divided by 2 of Collatz Conjecture iteration. Divided by $2^m$ will make the Collatz Conjecture get a result of 1 for any positive integer. Collatz tree further confirmed that for any odd number, there is a route and only one route down to 1 on Collatz Conjecture iteration. So it could be said that "***Collatz Conjecture should be true for any positive integer***".

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## Conflicts of Interest

The author declares that there are no conflicts of interest.

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