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Supereulerian Indices of Some Classes of Graphs

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Abstract

Researching Supereulerian index of a graph G is NP-hard. In this paper, we consider Supereulerian indices of some classes of graphs, Supereulerian index means the minimum integer k of iterated line graph $L^k(G)$ of a graph G such that $L^k(G)$ is Supereulerian. We show that Supereulerian indices of those graphs obtained by replacing every vertex of Petersen graph with n-cycle or a complete graph of order n, or adding n pendant edges to each vertex of Petersen graph are both 1. Concurrently, we show that Supereulerian indices of partial Generalized Petersen graphs are also 1.

Keywords

Petersen Graph, Generalized Petersen Graph, Supereulerian Index, Iterated Line Graph

1. Introduction

In this paper, we consider finite undirected simple graphs and follow the notation and terminology of [1].

The *line graph* $L(G)$ of a graph G is a graph whose vertices correspond to the edges of G, where two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint. For $k \geq 1$, the k-*time iterated line graph* $L^k(G)$ is defined recursively as follows: $L^0(G)=G$; $L^1(G)=L(G)$; $L(L^k(G))=L(L^{k-1}(G))$ for $k \geq 2$, with the condition that $E(L^{k-1}(G)) \neq \varnothing$ (*i.e.*, the edge set of $L^{k-1}(G)$ is non-empty). This definition ensures that the line graph operation can be applied iteratively as long as the previous iteration yields a non-empty graph. In a graph G, the *contraction* of edge e (denoted by G/e) with endpoints u and v is the operation that merges u and v into a single new vertex. The incident edges of this new vertex are all edges originally incident to either u or v, except for e itself (and any resulting loops are deleted). For $X \subseteq E(G)$, the *contraction* G/X is obtained from G by contracing each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$.

A *trail* of a graph G, denoted by T, is a sequence $T: v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$, whose terms are alternately vertices and edges of G such that v_{i-1} and v_i are the ends of $e_i (1 \le i \le l)$ and its edge terms are distinct. A *spanning trail* of a graph G is a trail containing all vertices of G. A *spanning closed trial* of a graph G is a trail containing all vertices of G, and $v_0 = v_l$. Supereulerian graphs are such graphs which have a spanning closed trial. The *Supereulerian index* means the minimum integer k of iterated line graph $L^k(G)$ of a G such that $L^k(G)$ is Supereulerian, denoted by $s(G)$. The cycle and complete graph with n vertices are denoted C_n and K_n, respectively.

The *Petersen Graph* is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

The concept of *Generalized Petersen graphs* was originally introduced by Watkins in 1969 [2]. Since their introduction, these graphs have become a significant subject of research in graph theory. Frucht [3] later investigated a canonical representation specifically for trivalent (3-regular) Hamiltonian Generalized Petersen graphs. Building on this work, Alspach [4] provided a complete classification of Hamiltonian Generalized Petersen graphs, establishing in particular the following key theorem:

The vertex set of the Generalized Petersen graphs $GP(n,k)$, $1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$ is defined as $V(GP(n,k)) = \{x_i, y_i \mid 1 \le i \le n\}$, and the edge set $E(GP(n,k)) = \{x_i x_{i+1}, x_i y_i, y_i y_{i+k} \mid 1 \le i \le n\}$, where the indices are taken as modulo n. Especially, Petersen graph is $GP(5,2)$.

Theorem 1. [4] The Generalized Petersen graph $GP(n,k)$ is Hamiltonian if and only if it is neither

(I) $GP(n,2) \cong GP(n,n-2) \cong GP\left(n, \frac{(n-1)}{2}\right) \cong GP\left(n, \frac{(n+1)}{2}\right)$,

$n \equiv 5 \pmod 6$, nor

(II) $GP\left(n, \frac{n}{2}\right)$, $n \equiv 0 \pmod 4$ and $n \ge 8$.

The Hamiltonian index was first introduced by Chartrand in 1968 [5], where he established its existence by proving that every connected graph (except paths) admits such an index. Chartrand *et al.* in [6] proved a result of Hamiltonian index, as follows:

Theorem 2. [6] Let G be a connected graph and the minimum degree of vertices in G is at least 3, then $h(G) \le 2$.

Later, in 1983, Clark and Wormald [7] extended this notion by introducing the Hamiltonian-like index, providing a broader framework for studying Hamiltonian properties. Further developments came in 1990 when Catlin *et al.* [8] investigated

Hamiltonian cycles and supereulerian graphs within iterated line graphs. Building on these foundations, Han, Lai, *et al.* [9] established a key relationship between a graph's Hamiltonian index and its Supereulerian index.

Theorem 3. [9] Let G be a connected graph but isn't a path, then $s(G) \le h(G) \le s(G)+1$.

In 2005, Xiong and Yan in [10] proved the supereulerian index of the tree T.

Let $B(G)$ denote the set of branches of G, and let $B_1(G)=\{b \in B(G) \mid b \text{ has at least one end vertex in } V_1(G)\}$. Define $C_B(G)=\{b \in B(G): \text{any edge of } b \text{ is a cut edge of } G\}$ and $C_{B_1}(G)=B_1(G)$. Define $k(G)=0$ if G is 2-connected; $k(G)=1$ if G is not 2-connected and $C_B(G)=\varnothing$; otherwise,
$k(G)=\max\{\max\{|E(b)|: b \in C_B(G) \setminus C_{B_1}(G)\}, \max\{|E(b)|: b \in C_{B_1}(G)\}\}$.

Theorem 4. [10] Let T be a tree. Then $s(T)=k(T)$.

In 2010, Xiong and Li established that the supereulerian index of a claw-free graph remains stable under contractions and closures, their results are as follows:

Theorem 5. [11] Let G be a graph and H be a collapsible subgraph of G. Then $s(G)=s(G/H)$.

Theorem 6. [11] Let G be a connected claw-free graph with at least three edges other than a path. Then $s(G)=s(cl(G))$.

Although Hamiltonian graphs are necessarily supereulerian, the converse fails in general. Therefore, it is meaningful to study the Supereulerian index of a graph G.

2. Our Main Results

Motivated by these researches, we consider Supereulerian indices of some graphs which obtained from Petersen graph and Generalized Petersen graphs, respectively.

Before presenting our main findings, several supplementary essential concepts and theorems are introduced: If $P=u_1u_2\ldots u_k$ be a path in a graph G. For subgraphs $S,T \subseteq G$: P is called an (S,T)-*path* if $u_1 \in V(T)$ and $u_k \in V(S)$. The *distance* between S and T, denoted $d_G(S,T)$, it is the minimum length among all (S,T)-*paths*. For any vertex $v \in v(G)$: $d_G(v)$ denotes the degree of a vertex v. $V_i(G)=\{v \in V(G): d_G(v)=i\}$ defines the i-degree vertex set (for $i \ge 0$). A *branch* in G is a nontrivial path satisfying: All internal vertices have degree 2, both end vertices have degree other than 2. If a branch has length 1, then it has no internal vertex. For any subgraph $H \subseteq G$, let $B_H(G)=\{b \in B(G) \mid \text{all edges of } b \text{ belong to } E(H)\}$.

Theorem 7. [12] Let G be a connected graph with at least 3 edges. Then $L^k(G)$ is supereulerian if and only if $S_k(G) \ne \varnothing$. Let $S_k(G)$ denote the set of all subgraphs $H \subseteq G$ satisfying the following properties:

(I) $\forall x \in V(H), d_H(x) \equiv 0 \pmod 2$;

(II) $V_0(H) \subseteq \bigcup_{i \ge 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) Every branch $b \in B(G)$ with $E(b) \cap E(H) = \varnothing$ satisfies $|E(b)| \le k+1$;

(IV) $|E(b)| \le k$ for any branch $b \in B_1(G)$;

(V) $d_G(H_1, H - H_1) \le k$ for every subgraph H_1 of H and $d_G(H_1, H - H_1) = 0$ means that H is connected;

Based on above results, we obtain the following results:

Theorem 8. Let G be a Petersen graph, then $s(G) = 1$.

Theorem 9. Let G be the graphs obtained from the Petersen graph by vertex replacement with cycles $C_n (n \ge 3)$. We have $s(G) = 1$.

Theorem 10. Let G be the graphs obtained from the Petersen graph by vertex replacement with complete graph $K_n (n \ge 4)$. We have $s(G) = 1$.

Theorem 11. Let G be the graphs obtained from the Petersen graph by adding n pendant edges to every vertex. We have $s(G) = 1$.

Theorem 12. Let G be the Generalized Petersen graph satisfying $GP(n,2) \cong GP(n,n-2) \cong GP\left(n, \frac{n-1}{2}\right) \cong GP\left(n, \frac{n+1}{2}\right)$, $n \equiv 5(\mathrm{mod}\, 6)$, we have $s(G) = 1$.

Theorem 13. Let G be the Generalized Petersen graph satisfying $GP\left(n, \frac{n}{2}\right)$, $n \equiv 0(\mathrm{mod}\, 4)$ and $n \ge 8$, we have $s(G) = 1$.

3. Proof of Main Results

Proof of Theorem 8. We prove this result by Theorem 7, we only prove $S_1(G) \ne \varnothing$. Suppose that $H \subseteq S_1(G)$, let H be a subgraph of G formed as $H = \bigcup_{i=1}^{10} H_i(G)$, where each $H_i(G)$ is the vertices in Petersen graph. Then H satisfies conditions (I)-(V) of $S_1(G)$, with the following properties:

(I) $\forall x \in V(H)$, $d_H(x) \equiv 0(\mathrm{mod}\, 2)$;

(II) $V_0(H) \subseteq \bigcup_{i \ge 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (III) is satisfied. Since $E(H) = 0$, it is obviously for branch $b \in B(G)$ with $E(b) \cap E(H) = \varnothing$, $|E(b)| = 1 \le 1 + 1 = k + 1$, that is $k = 1$.

We know there is no $V_1(G)$, so it is obviously that $B_1(G) \cap G = \varnothing$, so $|E(b)| = 0 \le 1$. We can get $k = 1$ which satisfies condition (IV).

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H - H_1) = 1 \le 1$, that is $k = 1$. So it follows that H complies with condition (V).

Thus, $S_1(G) \ne \varnothing$, then $L^1(G)$ is Supereulerian, *i.e.*, $s(G) \le 1$. We cannot find a spanning closed trail in the Petersen graph, we can be sure that $s(G) \ne 0$, so $s(G) = 1$. □

Proof of Theorem 9. By Theorem 7, we only need to prove $S_1(G) \ne \varnothing$. Suppose that $H \subseteq S_1(G)$, let H be a subgraph of G formed as $H = \bigcup_{i=1}^{10} H_i(G)$, where each $H_i(G)$ is a n-cycle in graph. Then H satisfies conditions (I)-(V) of

$S_1(G)$, with the following properties:

(I) $\forall x \in V(H)$, $d_H(x) \equiv 0 \pmod 2$;

(II) $V_0(H) \subseteq \bigcup_{i \geq 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (III) is satisfied. It is obviously for branch $b \in B(G)$ with $E(b) \cap E(H) = \varnothing$, $|E(b)| = 1 \leq 1+1 = k+1$, that is $k = 1$.

We know there is no $V_1(G)$, so it is obviously that $B_1(G) \cap G = \varnothing$, so $|E(b)| = 0 \leq 1$. We can get $k = 1$ satisfies condition (IV).

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H - H_1) = 1 \leq 1$, that is $k = 1$. So it follows that H complies with condition (V).

Thus, $S_1(G) \neq \varnothing$, then $L^1(G)$ is Supereulerian, *i.e.*, $s(G) \leq 1$. Since the graph G obtained from the Petersen graph by vertex replacement with cycles $C_n(n \geq 3)$, it is clear that there is no way for a spanning closed trail to exist in G, that is $s(G) \neq 0$, so $s(G) = 1$. □

Proof of Theorem 10. By Theorem 7, we only need to prove $S_1(G) \neq \varnothing$. Suppose that $H \subseteq S_1(G)$, let H be a subgraph of G formed as $H = \bigcup_{i=1}^{10} H_i(G)$, where each $H_i(G)$ is a n-cycle which induced by every complete graph K_n in graph. Then H satisfies conditions (I)-(V) of $S_1(G)$, with the following properties:

(I) $\forall x \in V(H), d_H(x) \equiv 0 \pmod 2$;

(II) $V_0(H) \subseteq \bigcup_{i \geq 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (III) is satisfied. It is obviously for branch $b \in B(G)$ with $E(b) \cap E(H) = \varnothing$, $|E(b)| = 1 \leq 1+1 = k+1$, that is $k = 1$.

We know there is no $V_1(G)$, so it is obviously that $B_1(G) \cap G = \varnothing$, so $|E(b)| = 0 \leq 1$. We can get $k = 1$ satisfies condition (IV).

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H - H_1) = 1 \leq 1$, that is $k = 1$. So it follows that H complies with condition (V).

Thus, $S_1(G) \neq \varnothing$, then $L^1(G)$ is Supereulerian, *i.e.*, $s(G) \leq 1$. By Theorem 5, we can know that $s(G) = s(G/H)$, every complete graph can be contracted to a vertex, so the Supereulerian index of this graph is equal to Petersen graphs, we can make sure that there exists no spanning closed trail in G, then $s(G) \neq 0$, so $s(G) = 1$. □

Proof of Theorem 11. We also prove this result by Theorem 7. Suppose that $H \subseteq S_1(G)$, let H be a subgraph of G formed as $H = \bigcup_{i=1}^{10} H_i(G)$, where each $H_i(G)$ is a vertex in Petersen graph. Then H satisfies conditions (I)-(V) of $S_1(G)$, with the following properties:

(I) $\forall x \in V(H), d_H(x) \equiv 0 \pmod 2$;

(II) $V_0(H) \subseteq \bigcup_{i\geq 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (III) is satisfied. Since $E(H)=0$, it is obviously for branch $b \in B(G)$ with $E(b)\cap E(H)=\varnothing$, $|E(b)|=1\leq 1+1=k+1$, that is $k=1$.

For the condition (IV), since the graph G contains pendant edges, there must exist $V_1(G)$, it is obviously that $|E(b)|=1\leq 1=k$ for any branch $b\in B_1(G)$, where $k=1$.

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H-H_1)=1\leq 1$, that is $k=1$. So it follows that H complies with condition (V).

Thus, $S_1(G)\neq\varnothing$, then $L^1(G)$ is Supereulerian, *i.e.*, $s(G)\leq 1$. Since the graph G obtained from the Petersen graph by vertex replacement with pendant edges with n. then there exists no spanning closed trail in G, $s(G)\neq 0$, so $s(G)=1$. □

Proof of Theorem 12. Due to the equivalence relation in Theorem 8, we only need to prove the Supereulerian index for one of the cases. Now we prove the result when $k=2$, $n\equiv 5(\mathrm{mod}\,6)$, let $n=6t+5$. For convinience, we denote the vertices of the inner-cycle as $v_0, v_1, v_2, \cdots, v_{6t+4}$, the vertices of the outer-cycle as $u_0, u_1, \cdots, u_{6t+4}$ of the Generalized Petersen graph, the vertices of the inner and outer cycles are connected by $u_i v_i$, where $0\leq i\leq 6t+4$, respectively. Next, we use the same method to prove.

(I) When $t=0$, $n=5$, that is a Petersen graph, according to our result in Theorem 8, $s(G)=1$.

(II) when $t\neq 0$, $n=6t+5$, we prove this result by theorem 7. Suppose that $H\subseteq S_1(G)$, let H be a subgraph of G formed as $H=\bigcup_{i=1}^{2} H_i(G)$, $H_1(G)$ is inner-cycle of the Generalized Petersen graph which has odd order vertices, $H_2(G)$ is outer-cycle of the Generalized Petersen graph which has odd order vertices. Then H satisfies conditions (I)-(V) of $S_1(G)$, with the following properties:

(I) $\forall x\in V(H), d_H(x)\equiv 0(\mathrm{mod}\,2)$;

(II) $V_0(H) \subseteq \bigcup_{i\geq 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (iii) is satisfied. It is obviously for branch $b\in B(G)$ with $E(b)\cap E(H)=\varnothing$, $|E(b)|=1\leq 1+1=k+1$, that is $k=1$.

We know there is no $V_1(G)$, so it is obviously that $B_1(G)\cap G=\varnothing$, so $|E(b)|=0\leq 1$. We can get $k=1$ which satisfies condition (IV).

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H-H_1)=1\leq 1$, that is $k=1$. So it follows that H complies with condition (V).

Thus, $S_1(G)\neq\varnothing$, then $s(G)\leq 1$. Since these graphs G exists no spanning closed trail in G, $s(G)\neq 0$, so $s(G)=1$.

(III) When $t = 0$, $n = 6(t+1)+5$, we prove it by same way, so $s(G)=1$.

Therefore, $L(P(n,2))$ is Supereulerian, that is, $s(P(n,2))=1$. So $s(P(n,k))=1$ (where $k=2$, $\frac{n-1}{2}$, $\frac{n+1}{2}$ or $n-2$). □

Proof of Theorem 13. When $n \ge 8$ and $n \equiv 0 (\bmod 4)$, we prove this result by Theorem 7, Suppose that $H \subseteq S_1(G)$, let H be a subgraph of G formed as $H = \bigcup_{i=1}^{\frac{n}{2}+1} H_i(G)$, where $H_2(G), H_3(G), \dots, H_{\frac{n}{2}}(G)$ is a C_2 formed by the vertices v_i and v_{i+4} inside the Generalized Petersen graph, $H_1(G)$ is outer-cycle C_n of the Generalized Petersen graph which has even order vertices. Then H satisfies conditions (I)-(V) of $S_1(G)$, with the following properties:

(I) $\forall x \in V(H), d_H(x) \equiv 0 (\bmod 2)$;

(II) $V_0(H) \subseteq \bigcup_{i \ge 3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

Now, we demonstrate that (III) is satisfied. It is obviously for branch $b \in B(G)$ with $E(b) \cap E(H) = \varnothing$, $|E(b)| = 1 \le 1+1 = k+1$, that is $k=1$.

We know there is no $V_1(G)$, so it is obviously that $B_1(G) \cap G = \varnothing$, so $|E(b)| = 0 \le 1$. We can get $k=1$ satisfies condition (IV).

Regarding condition (V), we can take every $H_i(G)$ as H_1. Subsequently, we have $d_G(H_1, H - H_1) = 1 \le 1$, that is $k=1$. So it follows that H complies with condition (V).

Thus, $S_1(G) \ne \varnothing$, then $L^1(G)$ is Supereulerian, *i.e.*, $s(G) \le 1$. Since this graph G exists no spanning closed trail in G, $s(G) \ne 0$, so $s(G)=1$. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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