

The Seidel Eigenvalues Polynomial and Spectrum of the Petersen Graph

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Abstract

For a simple undirected graph G , let $A(G)$ be the $(0, 1)$ adjacency matrix of G . The Seidel matrix of G , is defined as $S(G) = J - I - 2A(G)$, where J is the all-one matrix and I is the identity matrix. The Seidel eigenvalues polynomial of the graph G is $S_G(\lambda) = \det(\lambda I - S(G))$. If all the Seidel eigenvalues of the graph G are integers, then G is called a Seidel integer graph. In this paper, we apply methods from algebraic and matrix theory to obtain the Seidel eigenvalue polynomials of the Petersen graph. Furthermore, we determine whether the Petersen graph is a Seidel integral graph.

Keywords

Petersen Graph, Seidel Eigenvalues Polynomial, Seidel Spectral, Seidel Integral Graph

1. Introduction

Throughout this paper, we consider finite undirected simple graphs and follow the notation and terminology of [1].

Let G be a simple graph with n vertices. We use $A(G)$ and $S(G) = J - I - 2A(G)$ to denote the adjacency matrix and the Seidel matrix of G , respectively. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct eigenvalues of $S_G(\lambda)$ with the corresponding multiplicities k_1, k_2, \dots, k_n . We denote the Seidel spectrum of G as $\text{Spec}(G) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$. A graph is called an integral graph when all of its eigenvalues are integers. The eigenvalues of $A(G)$ (resp., $S(G)$) will be referred to as eigenvalues (resp., Seidel eigenvalues) of G .

The research of graph theory originated in 1957 and was mentioned by L. Collatz and U. Sinogowitz in reference [2]. The spectrum of a graph characterizes

some structural properties of the graph, which plays an important role in the research of other problems. In particular, it is closely related to the energy of the graph and the indices of the graph. The concept of integral graphs was first introduced by F. Harary and A. J. Schwenk [3] in 1973. They showed that if a graph is integral, then its complement and line graphs are also integral. Furthermore, if G_1 and G_2 are two integral graphs, their union, join, Cartesian product, and strong product graphs are also integral. Additionally, they characterized certain cases of integral trees.

Since then, much attention has been paid to this topic, but they mainly focus on undirected graphs and integral trees.

Theorem 1.1 (Lv Shengmei [4]) The Seidel eigenvalues polynomial and its spectrum for the complete graph are given, and it is proven that the complete graph is a Seidel integral graph.

Theorem 1.2 (Lv Shengmei [5]) All trees with a diameter of 2 are Seidel integral trees, and the sufficient and necessary conditions for a tree with a diameter of 3 to be a Seidel integral tree have been provided.

In order to study the problem of equiangular line systems in Euclidean space, Van Lint and Seidel introduced the concept of the Seidel matrix of a graph G in [6]. This problem has also been widely studied.

The adjacency matrix of G is $A = A(G) = (a_{ij})_{n \times n}$, which is an n -order square matrix. Its definition is as follows:

$$a_{ij} = \begin{cases} 1 & v_i \sim v_j \\ 0 & v_i \not\sim v_j \end{cases}, \text{ where } a_{ii} = 0, \quad a_{ij} = a_{ji}$$

Therefore, $A(G)$ is a real symmetric matrix.

Let $A(G)$ be the $(0, 1)$ adjacency matrix of G . The Seidel matrix of G , is defined as $S(G) = J - I - 2A(G)$. The Seidel eigenvalues polynomial of G is $S_G(\lambda) = \det(\lambda I - S(G))$, which can be written as $S_G(\lambda) = s_0 \lambda^n + s_1 \lambda^{n-1} + \dots + s_n \lambda^0$, where J is the all-one matrix and I is the identity matrix.

In 2008, Zeng in [7] proved the main eigenvalue of the Seidel matrix can be obtained from the principal eigenvalue and the corresponding eigenvector of its adjacency matrix.

Theorem 1.3 (Zeng Changxiong [7]) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the main eigenvalues of $A(G)$, a_1, a_2, \dots, a_m be the corresponding unit orthogonal vectors.

In 2014, Wang *et al.* presented in [8] the necessary and sufficient conditions for a complete multipartite graph to be a Seidel integral graph, and also constructed some Seidel integral graphs. Berman *et al.* in [9] gave the sufficient conditions for a complete multipartite graph determined by the Seidel spectrum under the Seidel transformation. Greaves in [10] gave the necessary and sufficient conditions for a graph with exactly three distinct Seidel eigenvalues to be a regular graph. In [11], Haemers studied the Seidel energy of graphs and proposed a conjecture about the lower bound of the Seidel energy: for all graphs G with n vertices, the complete graph has the minimum Seidel energy.

Another interesting problem in the spectral theory of graphs is whether the spectrum of a graph can be predicted based on its structure. This problem can be solved by using various operations on graphs. Up to now, graph spectrum theory has yielded many results, yet most of them are related to the Adjacency eigenvalue polynomial and its spectrum, the Laplacian eigenvalue polynomial and its spectrum, and the Signless Laplacian eigenvalue polynomial and its spectrum [12]. There are relatively few results on the research of the Seidel eigenvalue polynomial and its spectrum, and the main references found in [4]-[15].

The Petersen graph (Figure 1) is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G . The energy of a graph is related to the eigenvalues of the graph, that is, it is related to the graph spectrum. Haemers defined in [11] the Seidel energy as the sum of the absolute values of the eigenvalues of the Seidel matrix. Besides, he also proved the upper and lower bounds of the Seidel energy and proposed a conjecture that among all graphs with n vertices, the complete graph K_n has the minimum Seidel energy. This conjecture was proved to be true for $n \leq 10$ by Robin Swinkels.

Based on the previous results, we want to consider the Seidel eigenvalues of the Petersen graph class. However, when each vertex of the Petersen graph is replaced by a cycle, the structure of the graph becomes complex and the Seidel matrix grows extremely large, which requires the aid of algorithms to solve. Thus, in this paper, we mainly characterized the Seidel polynomial and spectrum of the Petersen graph.

Theorem 1.4 Let G be the Petersen graph, then the Seidel eigenvalues polynomial of the graph G is

$$S_G(\lambda) = \lambda^{10} - 45\lambda^8 + 818\lambda^6 - 7218\lambda^4 + 324\lambda^3 + 32805\lambda^2 + 5.1 \times 10^{-12}\lambda - 59049.$$

Theorem 1.5 Let G be the Petersen graph, then G is a Seidel integral graph.

2. Preliminary Lemmas

In this section, we propose several essential preliminary lemmas, which are very useful for the proof of our main result.

Definition 2.1 (Huang Youdu, Di Chengen, Zhu Shixin [10]). In an n order square matrix $A = (a_{ij})$, arbitrarily select k rows and k columns with the same row and column indices ($1 \leq k \leq n$). The sum of the k order determinants composed of k^2 elements, located at the intersections of these k rows and k columns in their original positions, is called the k order trace of the square matrix A , denoted as $tr^{[k]}(A)$, that is,

$$tr^{[k]}(A) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \quad (k = 1, 2, \dots, n) \quad (1)$$

where, $a_{i_t i_s}$ ($1 \leq i_t, i_s \leq n$, $t, s = 1, 2, \dots, k$; $i_1 < i_2 < \dots < i_k$), it is the element located at the intersection of the i_t row and the i_s column in A .

The k order trace of an n order square matrix, $tr^{[k]}(A)$ is the sum of C_n^k determinants of order k .

According to Equation (1), we can obtain that:

$$\begin{aligned} tr^{[1]}(A) &= \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}; \\ tr^{[2]}(A) &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-1} & a_{n,n} \end{vmatrix}; \\ tr^{[3]}(A) &= \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & a_{i_1 i_3} \\ a_{i_2 i_1} & a_{i_2 i_2} & a_{i_2 i_3} \\ a_{i_3 i_1} & a_{i_3 i_2} & a_{i_3 i_3} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} & a_{15} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & a_{1,n-1} & a_{1,n} \\ a_{n-1,1} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,n-1} & a_{nn} \end{vmatrix} \\ &\quad + \dots + \begin{vmatrix} a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} \\ a_{n,n-1} & a_{n,n} & a_{n,n+1} \\ a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} \end{vmatrix} \end{aligned}$$

In the same way, we can get:

$$tr^{[n]}(A) = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = |A|$$

Lemma 2.2 (Huang Youdu, Di Chengen, Zhu Shixin [16]). The eigenvalues polynomial $f(\lambda) = |\lambda I - A|$ of the n order square matrix A , can be written in the following form:

$$f(\lambda) = \sum_{k=1}^n (-1)^k tr^{[k]}(A) \lambda^{n-k}$$

$$\text{where, } tr^{[0]}(A) = 1, \quad tr^{[k]}(A) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \quad (k = 1, 2, \dots, n).$$

For the convenience of subsequent calculations, we use $m(tr^{[k]}(A))$ to represent the number of $tr^{[k]}(A)$.

3. Proof of Main Results

In what follows, we verify Theorem 1.4.

Proof of Theorem 1.4. Let G be the Petersen graph (Figure 1), then $A(G)$ be the $(0, 1)$ adjacency matrix of the Petersen graph.

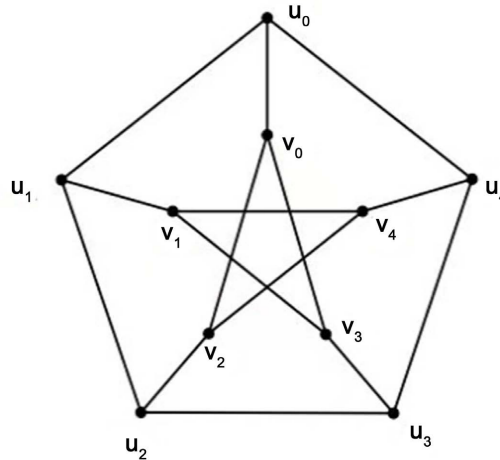


Figure 1. Petersen graph.

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the Seidel matrix of the Petersen graph is $S(G)$.

$$S(G) = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

The Seidel eigenvalues polynomial corresponding to it is denoted as $S_G(\lambda) = s_0\lambda^n + s_1\lambda^{n-1} + \dots + s_n\lambda^0$.

It can be obtained from Definition 2.1:

$$S_0 = \text{tr}^{[0]}(S(G)) = 1;$$

Since, $S_1 = (-1)^1 \text{tr}^{[1]}(S(G))$, $m(\text{tr}^{[1]}(S(G))) = C_{10}^1 = 10$. Thus,

$$S_1 = a_{11} + a_{22} + a_{33} + a_{44} + a_{55} + \cdots + a_{1010} = 0;$$

Since, $S_2 = (-1)^2 \text{tr}^{[2]}(S(G))$, $m(\text{tr}^{[2]}(S(G))) = C_{10}^2 = 45$. Thus,

$$S_2 = 15 \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + 30 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -45;$$

Since, $S_3 = (-1)^3 \text{tr}^{[3]}(S(G))$, $m(\text{tr}^{[3]}(S(G))) = C_{10}^3 = 120$. Thus,

$$\begin{aligned} S_3 = & -8 \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} + 14 \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} + 30 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 23 \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ & + 23 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} + 14 \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} + 8 \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} \\ = & 0; \end{aligned}$$

Similarly, we can obtain:

$$S_4 = (-1)^4 \text{tr}^{[4]}(S(G)), m(\text{tr}^{[4]}(S(G))) = C_{10}^4 = 210, \text{ that is, } S_4 = 818;$$

$$S_5 = (-1)^5 \text{tr}^{[5]}(S(G)), m(\text{tr}^{[5]}(S(G))) = C_{10}^5 = 252, \text{ that is, } S_5 = 0;$$

$$S_6 = (-1)^6 \text{tr}^{[6]}(S(G)), m(\text{tr}^{[6]}(S(G))) = C_{10}^6 = 210, \text{ that is, } S_6 = -7218;$$

$$S_7 = (-1)^7 \text{tr}^{[7]}(S(G)), m(\text{tr}^{[7]}(S(G))) = C_{10}^7 = 120, \text{ that is, } S_7 = 324;$$

$$S_8 = (-1)^8 \text{tr}^{[8]}(S(G)), m(\text{tr}^{[8]}(S(G))) = C_{10}^8 = 45, \text{ that is, } S_8 = 32805;$$

$$S_9 = (-1)^9 \text{tr}^{[9]}(S(G)), m(\text{tr}^{[9]}(S(G))) = C_{10}^9 = 10, \text{ that is, } S_9 = 5.1 \times 10^{-12};$$

$$S_{10} = (-1)^{10} \text{tr}^{[10]}(S(G)), m(\text{tr}^{[10]}(S(G))) = C_{10}^{10} = 1, \text{ that is,}$$

$$S_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} = 59049$$

Therefore, the Seidel eigenvalues polynomial of the Petersen graph G is

$$S_G(\lambda) = \lambda^{10} - 45\lambda^8 + 818\lambda^6 - 7218\lambda^4 + 324\lambda^3 + 32805\lambda^2 + 5.1 \times 10^{-12}\lambda - 59049.$$

That is, Theorem 1.4 is proved.

In what follows, we verify Theorem 1.5.

Proof of Theorem 1.5. Let G be the Petersen graph (Figure 1),

$$\begin{aligned} S_G(\lambda) &= \lambda^{10} - 45\lambda^8 + 818\lambda^6 - 7218\lambda^4 + 324\lambda^3 + 32805\lambda^2 + 5.1 \times 10^{-12} - 59049 \\ &= (\lambda + 3)^5 (\lambda - 3)^5 \end{aligned}$$

Let $S_G(\lambda) = 0$, then we get: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -3$,
 $\lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 3$.

Obviously, the Seidel eigenvalues of the Petersen graph are integers. That is to say, the Petersen graph is a Seidel integral graph.

4. Remarks

This paper primarily investigates whether the Petersen graph is a Seidel integral graph. It concludes that the Petersen graph is indeed a Seidel integral graph and provides the Seidel eigenvalues polynomial for the Petersen graph.

It is also noted that when each vertex of the Petersen graph is replaced with a cycle, the structure of the graph becomes complex and the Seidel matrix grows extremely large. This complexity makes it challenging to obtain its eigenvalues polynomial. Consequently, the Seidel eigenvalues polynomial of the Petersen graph in a more general case, as well as results related to Seidel integral graphs, have not been derived in this paper.

After that, we hope to use algorithms to solve the problem of Seidel eigenvalues of the Petersen graph class, and obtain the Seidel energy of this class of graphs. Additionally, we will explore the role that the graph energy plays in the quantitative structure, properties and activities.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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