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A Note on Average Codegrees of All Proper Subgroups of Finite Groups

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Abstract

Let G be a finite group and let $\mathrm{T}(G)$ be the sum of codegrees of complex irreducible characters of a group G. Say $f(G)=\frac{\mathrm{T}(G)}{|G|}$. We prove that a finite group G such that for all proper subgroup H, $1.1<f(H)<1.3$, is solvable.

Keywords

Simple Group, Character Codegree Sum, Proper Subgroup

1. Introduction

All groups are finite under consideration. Let G be a group and let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. Some authors studied the influence of character degrees on group structure, for instance, see [1]-[4]. Qian [5] gave the definition of the codegree of a finite group as follows: for an irreducible character $\chi\in\mathrm{Irr}(G)$, the codegree of a character χ of G is defined as $\mathrm{cod}\,\chi=\frac{|G:\ker\chi|}{\chi(1)}$. Now we have $\mathrm{cod}\,\chi=1$ when $\chi=1_G$, the principal character of G. Let $\mathrm{cd}_{\mathrm{cod}}(G)$ denote the set of irreducible character codegrees of G, *i.e.* $\mathrm{cd}_{\mathrm{cod}}(G)=\{\mathrm{cod}\,\chi:\chi\in\mathrm{Irr}(G)\}$. Some good results are gotten between codegree (degree) and group structure; see [6]-[12] for instance.

Wang, Qian, Lv and Chen studied the relation between average codegree and group structure [12]. Define $\mathrm{T}(G)=\sum_{\chi\in\mathrm{Irr}(G)}\mathrm{cod}\,\chi$ and $f(G)=\frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\mathrm{cod}\,\chi$. Let $k(G)$ be the number of conjugacy classes of G. Then $k(G)=|\mathrm{Irr}(G)|$.

Inspired by [12]-[14], we change the condition from "the codegree sum of a finite group" to "the codegree sums of all proper subgroups of a finite group". In order to talk in short, the following notion is defined.

Definition 1.1. *Then call* G *a* CS*-group if* $1.3 > f(G) > 1.1$.

Let $\mathrm{PSL}_2(q)$ be the projective special linear group of degree 2 over a finite field of order q and A_n be the alternating group on n symbols. Now we give the following result.

Theorem 1.2. *If* G *is a nonabelian simple group with* $f(G) > 1.1$, *then* G *is isomorphic to* A_5 *or* $\mathrm{PSL}_2(7)$.

Now by Theorem 1.2, we have the following corollary.

Corollary 1.3. *If all subgroups of* G *are* CS*-groups, then* G *is solvable.*

If we only consider the proper subgroups of a group G, then we introduce the definition as follows.

Definition 1.4. *Let* $\mathbf{prop}(G)$ *be the set of proper nonabelian subgroups of* G. *Then call* G *an* SCS*-group if for every* $H \in \mathbf{prop}(G)$, H *is a* CS*-group.*

Note that a non-trivial abelian group G has $f(G) \geq 1.5$ as $f(C_3) = \frac{1+3+3}{3} \approx 2.33$ where C_n is a cyclic group of order n. Thus in the hypothesis of the Definition, we only consider the non-abelian subgroups of a group.

As an application of Theorem 1.2, we also can prove the following result.

Theorem 1.5. *Let* G *be an* SCS*-group. Then* G *is solvable.*

Remark 1.6. *We know from* ([15], p. 2) *that* A_4, D_{10} *and* S_3 *are the maximal subgroups of* A_5 *and by* [16], $f(A_4) = \frac{11}{12}$, $f(D_{10}) = \frac{13}{10}$ *and* $f(S_3) = \frac{5}{6}$. *Thus the condition of Theorem* 1.5 "*for all* $H \in \mathbf{prop}(G)$, H *is a* CS*-group*" *is the best possible.*

Remark 1.7. *We do not know whether in Theorem* 1.5, *the condition* "$f(G) > 1.1$" *can be changed into* "$f(G) > 1$".

Here we introduce the structure of this paper. In Section 2, we compute the $\mathrm{T}(\mathrm{PSL}_2(q))$. In Section 3, some results about CS-groups are given and then prove the solvability of an SCS-group. All other symbols are standard, see [15] and [17] for instance.

2. Results for Section 3

Lemma 2.1. *Let* $H < G$.

(1) Let H be subnormal in G and let ψ be an irreducible constituent of χ_M. Then $\mathrm{cod}(\psi) \mid \mathrm{cod}(\chi)$.

(2) For $\chi \in \mathrm{Irr}(G) \setminus \{1_G\}$, $\mathrm{cod}\,\chi \geq \chi(1) + \frac{1}{\chi(1)} > \chi(1)$.

Proof. (1) It follows from Lemma 5.11 of [17].

(2) $\mathrm{cod}(\chi) = \frac{|G:\ker\chi|}{\chi(1)} \geq \frac{1+\chi(1)^2}{\chi(1)} > \frac{\chi(1)^2}{\chi(1)} = \chi(1)$.

Lemma 2.2. *Let* q *be a prime power and let* $G = \mathrm{PSL}_2(q)$. *Then*

$|G| = \frac{1}{(2, q-1)} q(q^2-1)$. *Furthermore,*

$$
\mathrm{T}(G)=\begin{cases} q\left(q^2+1\right), & q\equiv 0\left(\operatorname{mod} 2\right),\\ \dfrac{q^3+7q^2+8q+2}{4}, & q\equiv -1\left(\operatorname{mod} 4\right),\\ \dfrac{q^3+7q^2-6q+2}{4}, & q\equiv 1\left(\operatorname{mod} 4\right).\end{cases}
$$

Proof. We know that for a nonabelian simple group G, G has faithful irreducible characters except for 1_G, the principal character. By ([18], pp. 402-403), we have that, if $q\equiv 0\left(\operatorname{mod} 2\right)$, then

$$
\begin{aligned}
\mathrm{T}(G)&=1\cdot 1+\frac{1}{2}q\cdot\frac{q\left(q^2-1\right)}{q-1}+1\cdot\frac{q\left(q^2-1\right)}{q}+\left(\frac{1}{2}q-1\right)\cdot\frac{q\left(q^2-1\right)}{q+1}\\
&=q^3+q\\
&=q\left(q^2+1\right).
\end{aligned}
$$

$$
|G|=q\left(q^2-1\right);
$$

if $q=-1\left(\operatorname{mod} 4\right)$, then

$$
\begin{aligned}
\mathrm{T}(G)&=1+1\cdot\frac{q\left(q^2-1\right)/2}{q}+2\cdot\frac{q\left(q^2-1\right)/2}{(q-1)/2}+\frac{q-3}{4}\cdot\frac{q\left(q^2-1\right)/2}{q+1}\\
&\quad+\frac{q-3}{4}\cdot\frac{q\left(q^2-1\right)/2}{q-1}\\
&=1+\frac{q^2-1}{2}+2\left(q^2+q\right)+\frac{1}{8}(q-3)\left(q^2-q\right)+\frac{1}{8}(q-3)\left(q^2+q\right)\\
&=\frac{q^3+7q^2+8q+2}{4}.
\end{aligned}
$$

$$
|G|=\frac{1}{2}q\left(q^2-1\right);
$$

if $q=1\left(\operatorname{mod} 4\right)$, then

$$
\begin{aligned}
\mathrm{T}(G)&=1+1\cdot\frac{q\left(q^2-1\right)/2}{q}+2\cdot\frac{q\left(q^2-1\right)/2}{(q+1)/2}+\frac{q-5}{4}\cdot\frac{q\left(q^2-1\right)/2}{q+1}\\
&\quad+\frac{q-1}{4}\cdot\frac{q\left(q^2-1\right)/2}{q-1}\\
&=\frac{q^3+7q^2-6q+2}{4}.
\end{aligned}
$$

$$
|G|=\frac{1}{2}q\left(q^2-1\right).
$$

This completes the proof.

3. Results

In this section, we first show the solvability of a CS-group, and then give that an SCS-group is solvable, and finally some other common-use results for character degree sums are given.

3.1. CS-Groups

In this subsection, we will prove Theorem 1.2 and Corollary 1.3. By Lemma 2.2, $f(A_5)=\frac{68}{60}=\frac{17}{15}$ and $f(\mathrm{PSL}_2(7))=\frac{186}{168}=\frac{31}{28}$. We will first determine the structures of CS-groups.

Theorem 3.1. *Let G be a nonabelian simple group with $f(G)>1.1$. Then G is isomorphic to A_5 or $\mathrm{PSL}_2(7)$.*

Proof. As G is a nonabelian simple group, we have that for all $\chi\in\mathrm{Irr}(G)\setminus\{1_G\}$, $\ker\chi=1$. By the definition of $f(G)$ and Lemma 2.1, we have that

$$\begin{aligned} f(G) &= \frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{|G:\ker\chi|}{\chi(1)} \\ &= \sum_{\chi\in\mathrm{Irr}(G)}\frac{1}{|\ker\chi|\chi(1)}=\sum_{\chi\in\mathrm{Irr}(G)}\frac{1}{\chi(1)} \\ &= 1+\sum_{\chi\in\mathrm{Irr}(G)\setminus\{1_G\}}\frac{1}{\chi(1)}\geq 1+\frac{(k-1)\sqrt{k-1}}{\sqrt{|G|-1}}. \end{aligned}$$

where $k=k(G)$ and the last inequality is gotten by using the well-known inequality $\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\leq\sqrt{\frac{\sum_{i=1}^{n}x_i^2}{n}}$. Note that $k(A_5)=5$, so

$$f(G)\geq 1+\frac{(5-1)\sqrt{5-1}}{\sqrt{|G|-1}}=1+\frac{8}{\sqrt{|G|-1}}.$$

By hypothesis, we can assume that $1+\frac{8}{\sqrt{|G|-1}}>1.1$. It follows that $60\leq|G|\leq 6401$ as a nonabelian simple group G is of order ≥ 60. Now by ([15], p. 239), the possibilities for G are $A_5\cong\mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7)$, $A_6\cong\mathrm{PSL}_2(9)$, $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(17)$, A_7, $\mathrm{PSL}_2(19)$, $\mathrm{PSL}_2(16)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$ and $\mathrm{PSL}_2(23)$.

First we consider $\mathrm{PSL}_2(q)$ with $q\geq 5$. By Lemma 2.2, we have that

$$f(\mathrm{PSL}_2(q))=\begin{cases}\frac{q^2+1}{q^2-1}\geq 1.1, & \text{if } q\equiv 0 \pmod 2,\\ \frac{q^3+7q^2+8q+2}{2q(q^2-1)}\geq 1.1, & \text{if } q\equiv -1 \pmod 4,\\ \frac{q^3+7q^2-6q+2}{2q(q^2-1)}\geq 1.1, & \text{if } q\equiv 1 \pmod 4,\end{cases}$$

i.e.

$$0.1q^2-2.1\leq 0,\ \text{if } q\equiv 0 \pmod 2,$$

$$1.2q^3-7q^2-10.2q-2\leq 0,\ \text{if } q\equiv -1 \pmod 4,$$

$$1.2q^3-7q^2+1.5q-2\leq 0,\ \text{if } q\equiv 1 \pmod 4.$$

It follows from $q \geq 5$ that q is equal to 5 or 7. Now by ([15], pp. 2-3), we obtain that $f(A_5) = \frac{68}{60} \approx 1.13 > 1.1$ and $f(\mathrm{PSL}_2(7)) = \frac{186}{168} \approx 1.107 > 1.1$.

Now we consider these groups A_7, $\mathrm{PSL}_3(3)$ and $\mathrm{PSU}_3(3)$. By ([15], p. 10, 13-14), we obtain that

$$f(A_7) = \frac{1645}{2520} \approx 0.652 \ngtr 1.1,$$

$$f(\mathrm{PSL}_3(3)) = \frac{3305}{5616} \approx 0.588 \ngtr 1.1,$$

$$f(\mathrm{PSU}_3(3)) = \frac{5931}{6048} \approx 0.98 \ngtr 1.1.$$

Therefore $G \cong \mathrm{PSL}_2(q)$ with $q = 5$ or 7.

Here we rewrite Corollary 1.3 here.

Corollary 3.2. *If all subgroups of* G *are* CS*-groups, then* G *is solvable.*

Proof. Assume the result is not true with minimal order $|G|$, then G is nonsolvable but its proper subgroups are solvable. Thus, we can assume that G is a nonabelian simple group. Now hypothesis shows that G is a CS-group and by Lemma 3.1, G is isomorphic to A_5 or $\mathrm{PSL}_2(7)$. Thus, two cases are done within the following.

Case 1: A_5.

We know from ([15], p. 2) that A_4 is a maximal subgroup of A_5. Now by [16], $f(A_4) = \frac{11}{12} \ngtr 1.1$, a contradiction.

Case 2: $\mathrm{PSL}_2(7)$.

Then by ([15], p. 3), $\mathrm{PSL}_2(7)$ has S_4 as a maximal subgroup. Now $f(S_4) = \frac{22}{24} \ngtr 1.1$ arrives at a contradiction.

The two contradictions show that G is solvable.

We also can get the following results.

Theorem 3.3. *A finite group with* $f(G) > \frac{17}{15}$ *is solvable.*

Proof. If the result is wrong, then G is nonsolvable with minimal order $|G|$. Thus we can assume that G is a nonabelian simple group. By Theorem 3.1, we see that for a nonabelian simple group G, G has order at most 6401 with $f(G) > \frac{17}{15} > 1.1$. We can check these groups by [15] and get that $f(G) \leq \frac{17}{15}$, a contradiction. Thus G is solvable.

Corollary 3.4. *Let* G *be a nonabelian simple group. Then* $f(G) \leq \frac{17}{15}$.

Proof. If the theorem is not true, then $f(G) > \frac{17}{15}$, a contradiction to Theorem 3.3.

3.2. SCS-Groups

In this subsection, we give the proof of Theorem 1.5. We first need the following result which is due to Thompson [19].

Lemma 3.5 (Corollary 1 of [19]). *Every minimal simple group is isomorphic to one of the following minimal simple groups*:

(1) $\mathrm{PSL}_2(2^p)$ *for* p *a prime*;

(2) $\mathrm{PSL}_2(3^p)$ *for* p *an odd prime*;

(3) $\mathrm{PSL}_2(p)$, *for* p *any prime exceeding* 3 *such that* $p^2+1\equiv 0(\mathrm{mod}\,5)$;

(4) $Sz(2^p)$ *for* p *an odd prime*;

(5) $\mathrm{PSL}_3(3)$.

In order to argue in brief, we introduce the notation $[x]$ which denotes the maximal integer part of a rational number x, for example $[-\pi]=-4$ and $[\pi]=3$. Let

$$\ker_m(G)=\{[1_G(1),|G|,1],[\chi_1(1),|\ker\chi_1|,m_1],\cdots,[\chi_s(1),|\ker\chi_s|,m_s]\}$$

where m_i denotes the number of $\chi_i\in\mathrm{Irr}(G)$ with the same $|\ker\chi_i|$.

To control the structure of a group, we also need the following result.

Lemma 3.6. *Let* D_{2n} *be a dihedral group of order* $2n$.

(1) *Let* n *be odd. Then*

$$\ker_m(D_{2n})=\left\{[1,2n,1],[1,n,1],\left[2,1,\frac{n-1}{2}\right]\right\}.$$

(2) *Let* n *be even. Then*

$$\ker_m(D_{2n})=\left\{[1,2n,1],[1,n,1],\left[2,\left(3+(-1)^j\right)/2,\frac{n}{2}-1\right]\right\}.$$

(3) $\mathrm{T}(D_{2n})=1+2+2\cdot 2+\sum_{j=1}^{\left[\frac{n}{2}\right]-1}\frac{2n}{3+(-1)^j}$.

Table 1. Character table of D_{2n} with n odd ([20], p. 182).

g_i	1	$a^r\ (1\le r\le (n-1)/2)$	b
$\lvert C_{D_{2n}}(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	−1
ψ_j $(1\le j\le (n-1)/2)$	2	$\varepsilon^{jr}+\varepsilon^{-jr}$	0

In Table 1, $\varepsilon=\mathrm{e}^{2\pi\mathrm{i}/n}$.

Proof. (1) By Table 1, we have that $\chi_1(g)=\chi_1(1)$ for $g\in G$, so $\ker\chi_1=G$; $\chi_2(a^r)=\chi_2(1)$, so $|\ker\chi_2|=1+\frac{2n}{n}\cdot\frac{n-1}{2}=n$; if $\psi_j(g)=\psi_i(1)$, then $g=1$, so $\ker\psi_j=1$. Thus, we have

$$\mathrm{T}(D_{2n})=1+\frac{(2n/n)}{\chi_2(1)}+\sum_{1\le j\le(n-1)/2}\frac{(2n/1)}{\psi_j(1)}=3+n\cdot\frac{n-1}{2}.$$

(2) Obviously $\ker\chi_1 = G$, so $\operatorname{cod}\chi_1 = 1$.

We see $|\ker\chi_2| = 1+1+\frac{2n}{n}\cdot(m-1) = 2m = n$ and $\operatorname{cod}\chi_2 = \frac{(2n/n)}{1} = 2$.

Now $|\ker\psi_j|$ equals to 1 + 1 when j is even or 1 when j is odd, so

$$|\ker\psi_j| = 1+\frac{2n}{2n}\cdot\left(1+(-1)^j\right)/2.$$

Thus $\operatorname{cod}\psi_j = \frac{2n}{3+(-1)^j}$.

To compute the kernels of the remaining irreducible characters of a group, by Table 2, we need to consider when $(-1)^k = 1$. As the computation methods of $\ker\chi_3$ and $\ker\chi_4$ are similar, we only consider to compute $\ker\chi_3$. We divide the computation into two cases in light of m.

Table 2. Character table of D_{2n}, $n = 2m$ ([20], p. 183).

g_i	1	a^m	$a^r\ (1\le r\le m-1)$	b	ab
$\lvert C_{D_{2n}}(g_i)\rvert$	$2n$	$2n$	n	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	$(-1)^m$	$(-1)^r$	1	−1
χ_4	1	$(-1)^m$	$(-1)^r$	−1	1
ψ_j $(1\le j\le m-1)$	2	$2(-1)^j$	$\varepsilon^{jr}+\varepsilon^{-jr}$	0	0

In Table 2, $\varepsilon = e^{2\pi i/n}$.

Case 1: m is even.

Then in the interval $[1, m-1]$, there are $\left[\frac{m-1}{2}\right]$ even numbers r such that $(-1)^r = 1$. Thus

$$|\ker\chi_3| = 1+\frac{2n}{2n}+\frac{2n}{n}\cdot\left[\frac{\frac{n}{2}-1}{2}\right]+\frac{2n}{4} = 2+2\left(\frac{n-4}{4}\right)+\frac{n}{2} = n.$$

so $\operatorname{cod}\chi_3 = 2 = \operatorname{cod}\chi_4$.

Therefore $\mathrm{T}(D_{2n}) = 1+2+2\cdot 2+\sum_{j=1}^{\frac{n}{2}-1}\frac{2n}{3+(-1)^j}$.

Case 2: m is odd.

Then there are $\left[\frac{m-1}{2}\right]$ even integers r in the interval $[1, m-1]$ such that $(-1)^r = 1$, so

$$
\begin{aligned}
|\ker \chi_3| &= 1+\frac{2n}{n}\cdot\left[\frac{m-1}{2}\right]+\frac{2n}{4} \\
&= 1+2\cdot\left(\frac{m-1}{2}\right)+\frac{n}{2}=n.
\end{aligned}
$$

It follows that $\operatorname{cod}\chi_3 = 2 = \operatorname{cod}\chi_4$.

Now we have $\mathrm{T}(D_{2n}) = 1+2+2\cdot 2+\sum_{j=1}^{\frac{n}{2}-1}\frac{2n}{3+(-1)^j}$.

The lemma is complete.

Let $\max G$ or $\max(G)$ denote the set of maximal proper subgroups with regard to subgroup-order divisibility. Let $F:Q$ or $|F|:|Q|$ be the Frobenius group with kernel F and complement Q respectively. In order to read easily, we write Theorem 1.5 here.

Theorem 3.7. *Let* G *be an* SCS*-group. Then* G *is solvable.*

Proof. Assume the theorem is wrong with minimal order $|G|$. Let $H \in \mathbf{prop}(G)$ be a nontrivial minimal normal subgroup in G. Then H is a CS-group, and by Theorem 1.3, H is solvable. Thus H is abelian. If there is a normal subgroup K with $H < K < G$ such that K/H is non-abelian simple, then K is solvable, a contradiction. It means that $K = G$. If $H > 1$, then G/H is an SCS-group. It follows from the minimal choice of G that G/H solvable and so is G, a contradiction. It follows that $H = 1$ and that G is a minimal simple group such that its proper subgroups are solvable. Thus G is a minimal nonabelian simple group. So in the following, three cases are considered.

Case 1: $\mathrm{PSL}_2(q)$ for certain q.

If $q \in \{5,7,9,11\}$, then by ([15], p. 2-3, 5-7) and [16], we respectively have

$$f(D_{10}) = \frac{13}{10} = 1.3 \nless 1.3, \quad D_{10} \in \max \mathrm{PSL}_2(5),$$

$$f(7:3) = 1 \ngtr 1.1, \quad 7:3 \in \max \mathrm{PSL}_2(7),$$

$$f(3^2:4) = \frac{31}{36} \ngtr 1.1, \quad 3^2:4 \in \max \mathrm{PSL}_2(9),$$

$$f(11:5) = \frac{43}{55} \ngtr 1.1, \quad 11:5 \in \max \mathrm{PSL}_2(11).$$

Thus $q \neq 5,7,9,11$ and we can assume that $q \geq 13$ when q is odd and $q \geq 8$ when q is even, so let $n = \frac{q\pm 1}{k}$, $n \geq 6$ when q is odd and $n \geq 7$ when q is even. By Lemma 3.6 and **Table 3**, we have that $D_{2(q\pm1)/k} \in \max \mathrm{PSL}_2(q)$ and either

$$1.1 < \frac{n^2-n+6}{4n} < 1.3 \tag{1}$$

or

$$1.1 < \frac{7+\sum_{j=1}^{\frac{n}{2}-1}\frac{2n}{3+(-1)^j}}{2n} < 1.3. \tag{2}$$

For the inequality (1), we have $n = 4 \ngeq 6$, a contradiction. But the inquality (2) has no solution in $\mathbf{N}$ since $n \geq 6$.

Table 3. $\mathrm{PSL}_2(q)$, $q \geq 5$ ([21], Chap II, Theorem 8.27).

	$\max \mathrm{PSL}_2(q)$	Condition
$\mathcal{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathcal{C}_2$	$D_{2(q-1)/k}$	$q \notin \{5,7,9,11\}$
$\mathcal{C}_3$	$D_{2(q+1)/k}$	$q \notin \{7,9\}$
$\mathcal{C}_5$	$\mathrm{PSL}_2(q_0).(k,b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathcal{C}_6$	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3,5,13,27,37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}, F_q = F_p\left[\sqrt{5}\right]$

Case 2: $Sz(2^p)$ for p an odd prime.

Let $q = 2^p$. Then by ([22], p. 385), $D_{2(q-1)} \in \max Sz(q)$, and so by Lemma 3.6, we have $1.1 < \frac{(q-1)^2 - (q-1) + 6}{4(q-1)} < 1.3$, so $q = 5$ is non-even, a contradiction.

Case 3: $\mathrm{PSL}_3(3)$.

By ([15], p. 13), $13:3 \in \max \mathrm{PSL}_3(3)$ and by [16], $f(13:3) = \frac{59}{39} \approx 1.5 \nleq 1.3$, a contradiction.

From the above three cases, G is solvable, the wanted result.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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