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The Evolution of the Universe Based on Principal Bundle Geometry

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Abstract

This paper proposes a novel geometric framework for unifying dark matter, dark energy, and cosmic evolution through a modified gauge-theoretic approach. By introducing a scalar function $\gamma(x)=\tanh(kx)$, where $x=\frac{a}{a^0}$ (with a as the local "absolute acceleration" and $a^0 \approx 10^{-8}$ cm/s² a reference scale), we construct an extended principal bundle $\tilde{P}=P\times\mathbb{R}$ and redefine the connection form as $\tilde{\omega}=\omega_G+d\gamma$, ensuring curvature invariance while modulating gravitational sources. In the Newtonian limit, the field equation simplifies to $\gamma\nabla^2\phi=-4\pi\rho$, revealing γ's role in scaling gravitational effects. Distinctively, dark matter and dark energy emerge as geometric consequences of spacetime curvature rather than exotic components: $\gamma\to 1$ restores standard gravity at small scales, $0<\gamma<1$ enhances gravity (mimicking dark matter) at galactic scales, and $\gamma\to 0^+$ drives cosmic acceleration via $\Lambda(\gamma)\to\infty$. Crucially, allowing x to take negative values introduces $\gamma\to 0^-$ with $\Lambda(\gamma)\to-\infty$, enabling cyclic universe scenarios through curvature oscillations. The derived force law $F=\gamma ma$ aligns with MOND phenomenology, bridging modified dynamics and geometric modulation. Furthermore, the model constructs a closed "Taiji-like" cosmic evolution loop in the $\gamma-x$ plane, linking asymptotic states ($\gamma=\pm 1$ at $x=\pm\infty$) via hyperbolic perturbations, symbolizing cyclic rebirth. By unifying cosmic dynamics under a geometric gauge framework, this work offers testable predictions for parameter calibration (k, a_0, Λ_0) and establishes a philosophically resonant synthesis of mathematical physics and cosmological evolution.

Keywords

Generalized Gauge Transformation, Dark Matter, Dark Energy, Einstein

Equation, Universal Evolution

1. Introduction

General Relativity as the classical theory of gravitational interaction has been rigorously validated across scales from the solar system to cosmology [1]. However, the flattening of galaxy rotation curves [2] and cosmic accelerated expansion [3] reveal limitations of the classical framework at specific scales. While dark matter and dark energy hypotheses are widely adopted to explain these anomalies [4], their elusive microscopic nature has spurred research into modified gravity theories. Although MOND (Modified Newtonian Dynamics) [5] successfully reproduces galactic-scale observations, its lack of a covariant framework remains a constraint.

Recent breakthroughs in geometric modified gravity include: the 2022 nonsingular bouncing universe model based on f(R) gravity supporting cosmic cycles [6], and the 2021 unified inflation-contraction framework within scalar-tensor theory [7], both highlighting the central role of spacetime geometry in cosmic evolution. This work proposes an innovative gauge transformation approach by introducing a scaling function $\gamma(x)=\tanh(kx)$ (where $x=a/a_0$, with a as the local absolute acceleration and $a_0 \approx 10^{-8}$ cm/s² as the critical acceleration). We construct an extended principal bundle $\tilde{P}=P\times\mathbb{R}$ and redefine the connection form as $\tilde{\omega}=\omega_G+d\gamma$, achieving curvature invariance ($\tilde{R}=R$) while enabling nonlinear modulation of gravitational sources [8]. The proposed action is:

$$S=\int_M\left[\frac{1}{16\pi}\left(\gamma(x)R-4\pi A\left(\frac{1-\gamma(x)}{\gamma(x)}\right)\right)+\mathcal{L}_m\right]\sqrt{-g}\,\mathrm{d}^4x \tag{1}$$

Derived field equations read:

$$\gamma G_{\mu\nu}+2\pi A\left(\frac{1-\gamma}{\gamma}\right)g_{\mu\nu}=8\pi T_{\mu\nu} \tag{2}$$

In the Newtonian limit, this reduces to $\gamma\nabla^2\phi=-4\pi\rho$, demonstrating dynamic modulation of the gravitational source ρ by $\gamma(x)$. The theory's breakthrough lies in unifying dark matter and dark energy as geometric effects: standard gravity is recovered at small scales ($\gamma\to1$); enhanced gravity mimicking dark matter emerges at galactic scales ($0<\gamma<1$); and divergent $\Lambda(\gamma)\to\infty$ drives cosmic acceleration at cosmological scales ($\gamma\to0^+$) [9]. When allowing negative x values, $\gamma\to0^-$ induces $\Lambda(\gamma)\to-\infty$, triggering contraction phases and dynamic chaos zones, geometrically encoding cyclic universe scenarios [10]. The modified dynamical law $F=\gamma ma$ derived in the Newtonian limit aligns precisely with MOND phenomenology [11].

Furthermore, through the principal bundle framework, this work unifies dark matter, dark energy, and curvature oscillations as gauge field effects, systematically connecting to our prior generalized gauge transformation theories [10]-[14].

The core innovation lies in constructing a cosmic cycle diagram on the $\gamma - x$ plane: the central S -shaped evolution curve $\gamma(x) = \tanh(kx)$ interconnects virtual and real outer loops, forming a Taiji-like topological closure at $x = \pm\infty$ ($\gamma = \pm 1$), symbolizing the philosophical concept of cosmic rebirth.

The paper is structured in six sections: Section II details the geometric construction of $\tilde{P}$; Section III derives field Equation (2) via variational principles; Section IV establishes correspondence with MOND under Newtonian approximations; Section V unveils cosmic cyclic evolution mechanisms through **Figure 1**; Section VI summarizes findings, discusses the geometric significance of generating γ and outlines observational calibration pathways for parameters (k, a_0, Λ_0).

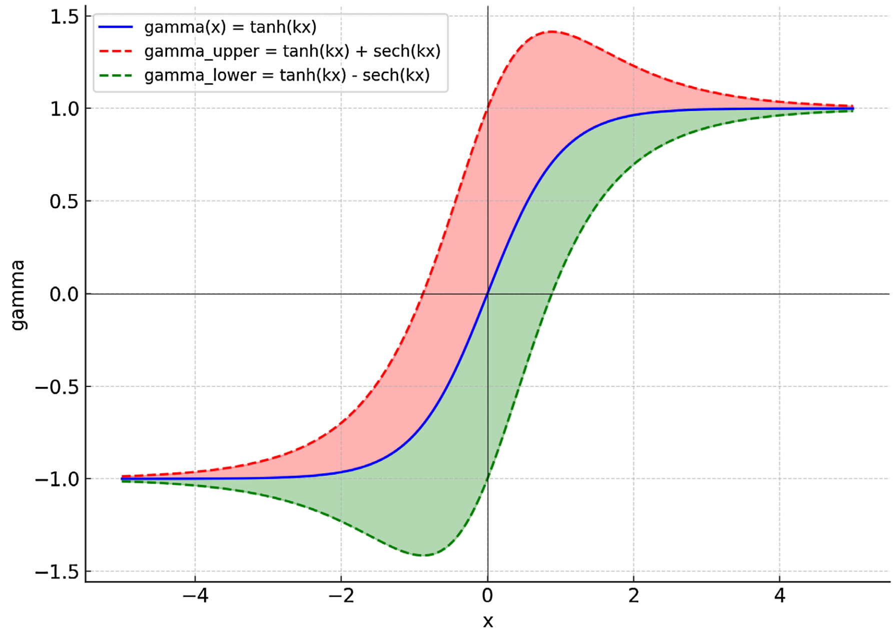

Figure 1. Shows that on the $\gamma - x$ plane, the central solid line $\gamma(x) = tanh(kx)$ represents the cyclic evolution of the two branches of the universe: when $0 < \gamma < 1$, $G_{eff} = G/\gamma > G$, which can explain the rotation curve without the need for dark matter particles; for dark energy, $\gamma \to 0$, gravity disappears, Λ dominates the expansion; $\gamma < 0$, repulsive force increases, accelerating expansion (or gravity begins to increase and the universe contracts). Upper outer circle $\gamma_{upper} = \tanh(kx) + \text{sech}(kx)$ (red dotted line) and lower outer circle $\gamma_{lower} = \tanh(kx) - \text{sech}(kx)$ (green dotted line) are virtually closed at $\gamma = \pm 1, x \to \pm\infty$, forming a Tai Chi-like diagram.

2. Principal Bundle and Structural Group

We introduce the principal bundle is defined as:

$$\tilde{P} = P \times \mathbb{R} \tag{3}$$

Where P is the principal bundle of the original gauge group $G = GL(m)$, that is, the frame bundle FM [10]-[14], and the fiber is expanded to the additive group $\mathbb{R}$ (rather than the multiplicative group). Here the structure group is:

$$\tilde{G} = G \times \mathbb{R} \tag{4}$$

where the element $\gamma(x) \in \mathbb{R}$ can be positive, negative or zero, indicating the "displacement" on the fiber. Then the local cross section is:

$$\tilde{s}(x) = (s(x), \gamma(x)), \quad s(x) \in P \tag{5}$$

Here, $\gamma(x)$ is the $\mathbb{R}$ fiber coordinate. The connection form is expanded to:

$$\tilde{\omega} = \omega_G + \omega_\gamma \tag{6}$$

where $\omega_G \in \Omega^1(P, \mathrm{g})$ is a connection of G, and $\Omega^1(P, \mathrm{g})$ is a set of 1-forms with Lie algebra g as range on the principal bundle P. Since $\mathbb{R}$ is an Abelian additive group, its Lie algebra is $\mathbb{R}$, with generator D, and the connection form is a translational differentiation:

$$\omega_\gamma = d\gamma \tag{7}$$

So, the gauge transformations are driven by elements $(h(x), \tilde{\gamma}(x))$ of the extended structure group $\tilde{G} = G \times \mathbb{R}$, where:

- $h(x) \in G$ is a group element of G;
- $\tilde{\gamma}(x) \in \mathbb{R}$ is a group element of $\mathbb{R}$.

These gauge transformations act as group operations on the principal bundle $\tilde{P}$, altering the choice of local sections. Here, $h(x)$ and $\tilde{\gamma}(x)$ are independent gauge parameters corresponding to the degrees of freedom of G and $\mathbb{R}$, respectively. The field $\gamma(x)$ originates as the fiber coordinate in the original section $\tilde{s}(x) = (s(x), \gamma(x))$, representing the "position" along the $\mathbb{R}$-fiber. Meanwhile, $\tilde{\gamma}(x)$ acts as the incremental parameter in gauge transformations, encoding translations along the $\mathbb{R}$-fiber.

Crucially, without introducing $\tilde{\gamma}(x)$, the gauge transformations of the $\mathbb{R}$-fiber cannot be described. This is because $\gamma(x)$ is fixed as the intrinsic value of the original section $\tilde{s}(x)$, whereas gauge transformations require modifying the fiber coordinate $\gamma(x)$ to generate a new section $\tilde{s}'(x)$. The absence of $\tilde{\gamma}(x)$, would "freeze" the gauge freedom of the $\mathbb{R}$-fiber, preventing adjustments to $\gamma(x)$.

However, regardless of gauge transformations, the connection $\tilde{\omega}$ and curvature $\tilde{\Omega}$ on the principal bundle remain invariant. Their projections onto the base manifold manifest as components satisfy the generalized gauge equations; this is a foundational principle underlying the grand unification of physical interactions.

So the original section is $\tilde{s}(x) = (s(x), \gamma(x))$, is a local gauge choice of the bottom manifold M to $\tilde{P}$. After the gauge transformation $U = (h(x), \tilde{\gamma}(x))$ acts:

$$\tilde{s}'(x) = \tilde{s}(x) \cdot (h(x), \tilde{\gamma}(x)) = (s(x)h(x), \gamma(x) + \tilde{\gamma}(x)) \tag{8}$$

where $s(x) \to s(x)h(x)$ is the action of G; $\gamma(x) \to \gamma(x) + \tilde{\gamma}(x)$ is the additive action of $\mathbb{R}$.

Hence, the original connection is equation (6), that is, $\tilde{\omega} = \omega_G + \omega_\gamma$, where ω_γ is based on $\tilde{s}(x)$. After the gauge transformation:

$$\omega_G' = Ad_{h^{-1}}\omega_G + h^{-1}dh \tag{9}$$

$$\omega_\gamma' = d\left(\gamma + \tilde{\gamma}\right) = d\gamma + d\tilde{\gamma} \tag{10}$$

where $\tilde{\gamma}(x)$ is a gauge parameter, its differential $d\tilde{\gamma}$ reflects the transformation effect in the $\mathbb{R}$ direction. So for $\left(h(x),\tilde{\gamma}(x)\right) \in G \times \mathbb{R}$, we have:

$$\tilde{s}'(x) = \left(s(x)h(x), \gamma(x) + \tilde{\gamma}(x)\right) \tag{11}$$

$$\omega_G' = Ad_{h^{-1}}\omega_G + h^{-1}dh \tag{12}$$

$$\omega_\gamma' = d\left(\gamma + \tilde{\gamma}\right) = d\gamma + d\tilde{\gamma} \tag{13}$$

The above explanation of the gauge construction is given in Appendix A.

Therefore, the curvature form can be obtained from the Cartan second structural equation:

$$\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = d\omega_G + \omega_G \wedge \omega_G + d\left(d\gamma\right) \tag{14}$$

Here, $d\left(d\gamma\right) = 0$, and it can be proved that the cross term $\omega_G \wedge d\gamma = 0$, because $\left[T_a, D\right] = 0$, where T_a is the generator (basis) of g, such as $a = 1, 2, \cdots, dim(\mathrm{g})$, so it is also the generator of ω_G; D is the Lie algebra generator of $\mathbb{R}$, choose $D = 1$ (standard basis), see Appendix B for detailed proof. So, we have the result:

$$\tilde{\Omega} = \Omega_G \tag{15}$$

where we define the field strength as:

$$\hat{F}_{\mu\nu} = s^*\Omega_G \tag{16}$$

Its scaling form is:

$$\hat{F}_{\mu\nu}' = \gamma(x)\hat{F}_{\mu\nu} \tag{17}$$

where, $\gamma(x) \in \mathbb{R}$ can be negative, and a negative value indicates a reversal of the field strength direction (the attractive force turns into a repulsive force).

3. New Field Equation

Considering that gravitational interactions are dominant in large-scale spacetime, the standard form of the interaction quantity we constructed is:

$$S = \int_M \left[\frac{1}{16\pi}\left(R - 2\Lambda\right) + \mathcal{L}_m\right]\sqrt{-g}\,\mathrm{d}^4x \tag{18}$$

Then the generalized gauge transformation proposed above is performed. Here, the transformation of $\Lambda(\gamma)$ refers to the relationship between the cosmological constant and $\gamma(x)$ proposed by the author in the literature [10] [11], that is,

$$R \to \gamma(x)R,\quad \Lambda(\gamma) \to 2\pi A\left(\frac{1-\gamma}{\gamma}\right) \tag{19}$$

where R is the scalar curvature, A is an undetermined constant. In this way, we construct the action as:

$$S = \int_M \left[\frac{1}{16\pi}\left(\gamma(x)R - 4\pi A\left(\frac{1-\gamma(x)}{\gamma(x)}\right)\right) + \mathcal{L}_m\right]\sqrt{-g}\,\mathrm{d}^4x \tag{20}$$

here, $\mathcal{L}_m$ is the Lagrangian of matter. Then the action variation is:

$$\delta S = \int_M \left[\frac{1}{16\pi} \delta \left(\gamma(x) R - 4\pi A \left(\frac{1-\gamma(x)}{\gamma(x)} \right) \right) + \delta \mathcal{L}_m \right] \sqrt{-g}\, \mathrm{d}^4 x \tag{21}$$

Hence we obtain:

$$\delta \left(\gamma(x) R - 4\pi A \left(\frac{1-\gamma(x)}{\gamma(x)} \right) \right) \sqrt{-g} = \delta \gamma R \sqrt{-g} - 4\pi \delta \left(A \frac{1-\gamma}{\gamma} \sqrt{-g} \right) \tag{22}$$

1) Gravitational term:

$$\frac{1}{16\pi} \delta \gamma R \sqrt{-g} = \gamma \frac{1}{16\pi} \delta R \sqrt{-g} = \frac{1}{16\pi} \gamma G_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} \tag{23}$$

2) Cosmological term:

$$\begin{aligned} & -\frac{4\pi}{16\pi} \delta \left(A \frac{1-\gamma}{\gamma} \sqrt{-g} \right) = -\frac{1}{4} A \left(\frac{1-\gamma}{\gamma} \right) \delta \sqrt{-g} \\ & = -\frac{1}{4} A \left(\frac{1-\gamma}{\gamma} \right) \left(-\frac{1}{2} g_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} = \frac{1}{8} A \left(\frac{1-\gamma}{\gamma} \right) g_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} \end{aligned} \tag{24}$$

where, $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$

3) Matter term:

$$\delta \mathcal{L}_m \sqrt{-g} = -\frac{1}{2} T_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} \tag{25}$$

So we get the action variation as:

$$\delta S = \int_M \left[\frac{1}{16\pi} \gamma G_{\mu\nu} + \frac{1}{8} A \left(\frac{1-\gamma}{\gamma} \right) g_{\mu\nu} - \frac{1}{2} T_{\mu\nu} \right] \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x = 0 \tag{26}$$

The new field equation is derived as follows:

$$\frac{1}{16\pi} \gamma G_{\mu\nu} + \frac{1}{8} A \left(\frac{1-\gamma}{\gamma} \right) g_{\mu\nu} - \frac{1}{2} T_{\mu\nu} = 0 \tag{27}$$

$$\gamma G_{\mu\nu} + 2\pi A \left(\frac{1-\gamma}{\gamma} \right) g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{28}$$

where

$$\Lambda(\gamma) = 2\pi A \left(\frac{1-\gamma}{\gamma} \right) = \Lambda_0 \left(\frac{1-\gamma}{\gamma} \right) \tag{29}$$

We can assume $\Lambda_0 = 2\pi A$, where A is a constant and is determined experimentally.

4. Newtonian Approximation

The above new field equation is that when $\gamma \to 1$, $\Lambda(\gamma) = 2\pi A \left(\frac{1-\gamma}{\gamma} \right) \to 0$, but $\gamma G_{\mu\nu}$ still maintains $\gamma G_{\mu\nu}$, then $\gamma m a = F$ can be deduced under Newton's approximation, that is, the modified Newton's second law can be derived. The following is the derivation process:

Starting from the new field Equation (29),

$$\gamma G_{\mu\nu} + 2\pi A\left(\frac{1-\gamma}{\gamma}\right)g_{\mu\nu} = 8\pi T_{\mu\nu}$$

where, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$; $\gamma = \gamma(x)$ is a scalar function; A is a constant or a function that does not depend on $g_{\mu\nu}$; $T_{\mu\nu}$ is the energy-momentum tensor. In the Newtonian approximation, we have:

1) Weak field limit:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \approx -1 - 2\phi \tag{30}$$

where, $\eta_{\mu\nu} = diag(-1,1,1,1)$; $h_{00} = -2\phi(\phi \ll 1)$, $h_{ij} \approx \delta_{ij}$.

2) Low speed limit:

$$T_{00} = \rho,\quad T_{0i} \approx 0,\quad T_{ij} \approx 0 \tag{31}$$

3) Static Assumption:

$$\partial_t \phi = 0 \tag{32}$$

4) Space and time condition, $\gamma \to 1$ (Such as Solar system scale);

$$\Lambda(\gamma) = 2\pi A\left(\frac{1-\gamma}{\gamma}\right) \to 0$$

Thus, there is only the time-time component G_{00}:

$$R_{00} \approx -\nabla^2\phi,\quad R \approx -2\nabla^2\phi \tag{33}$$

Therefore, we can get:

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R \approx -\nabla^2\phi - \frac{1}{2}(-1)(-2\nabla^2\phi) = -2\nabla^2\phi \tag{34}$$

$$g_{00} \approx -1,\quad T_{00} = \rho \tag{35}$$

Substituting equations (34) and (35) into the field Equation (28), we obtain

$$\gamma(-2\nabla^2\phi) + 2\pi A\left(\frac{1-\gamma}{\gamma}\right)(-1) = 8\pi\rho \tag{36}$$

$$\gamma\nabla^2\phi + \pi A\left(\frac{1-\gamma}{\gamma}\right) = -4\pi\rho \tag{37}$$

In the Newtonian approximation (solar system scale approximation), $\gamma \to 1$; -$\frac{1-\gamma}{\gamma} \to 0$; $\Lambda(\gamma) \to 0$; so we have

$$\gamma\nabla^2\phi + \pi A\left(\frac{1-\gamma}{\gamma}\right) = -4\pi\rho \Rightarrow \gamma\nabla^2\phi = -4\pi\rho \tag{38}$$

That is, from the Newtonian approximation of the field equation $\gamma\nabla^2 G_{\mu\nu} = -8\pi\rho$, we can obtain the modified Poisson Equation (38):

$$\nabla^2\phi = -\frac{4\pi\rho}{\gamma}$$

Defined by gravitational field strength

$$g = -\nabla \phi \tag{39}$$

Directly integrating the modified Poisson equation (assuming the boundary condition $\phi \to 0$ at infinity), we obtain

$$\phi(x) = \frac{1}{\gamma(x)} \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' \tag{40}$$

However, in order to explicitly handle the gradient of γ, we need to strictly expand the gradient operator acting on ϕ, that is,

$$g = -\nabla \phi = -\nabla \left(\frac{1}{\gamma} \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' \right) \tag{41}$$

Using Leibniz's law:

$$g = -\frac{1}{\gamma} \nabla \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' - \frac{1}{\gamma^2} (\nabla \gamma) \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' \tag{42}$$

Define the standard Newtonian potential:

$$\nabla^2 \phi_{Newton} = -4\pi\rho \tag{43}$$

$$\phi_{Newton}(x) = \int \frac{\rho(x')}{|x - x'|} \mathrm{d}^3 x' \tag{44}$$

Then the gravitational field strength is:

$$g = -\nabla \phi = -\frac{1}{\gamma} \nabla \phi_{Newton}(x) + \frac{\phi_{Newton}(x)}{\gamma^2} \nabla \gamma \tag{45}$$

So the Newton force is

$$\boldsymbol{F} = m\boldsymbol{g} = -\frac{m}{\gamma} \nabla \phi_{Newton} + \frac{m\phi_{Newton}}{\gamma^2} \nabla \gamma \tag{46}$$

Decompose the Newton force above into two parts:

$$\boldsymbol{F} = m\boldsymbol{g} = \underbrace{-\frac{m}{\gamma} \nabla \phi_{Newton}}_{\text{"Main item"}} + \underbrace{\frac{m\phi_{Newton}}{\gamma^2} \nabla \gamma}_{\text{"Additional items"}} \tag{47}$$

Then we rescale the main term, that is, the main term can be written as:

$$-\frac{m}{\gamma} \nabla \phi_{Newton} = m\boldsymbol{a}_\gamma \tag{48}$$

The corrected acceleration is defined here:

$$\boldsymbol{a}_\gamma = -\frac{\nabla \phi_{Newton}}{\gamma} = \frac{\boldsymbol{a}_N}{\gamma} \tag{49}$$

The additional term comes directly from the gradient of γ:

$$\frac{m\phi_{Newton}}{\gamma^2} \nabla \gamma \tag{50}$$

Therefore, combining the main term and the additional term, the force modulated by γ is:

$$\boldsymbol{F}_\gamma = m\boldsymbol{a}_\gamma + \frac{m\phi_{Newton}}{\gamma^2}\nabla\gamma \tag{51}$$

Here, in the main term $\boldsymbol{F}_\gamma \approx \frac{\boldsymbol{F}_N}{\gamma}$, so when Newton approximation is used, $\gamma \approx 1$ (such as $\gamma = \tanh(kx)$), we may have

$$\boldsymbol{F}_N \approx \gamma m\boldsymbol{a}_\gamma \tag{52}$$

This is consistent with the MOND theory. In summary, the additional term $\frac{m\phi_{Newton}}{2\gamma}\nabla\gamma$ above comes from the spatial inhomogeneity of γ; after observing [12]-[16] if $\gamma = \tanh(kx)$, we have:

$$\nabla\gamma = \frac{\mathrm{d}\gamma}{\mathrm{d}x}\hat{x} = k\cdot\mathrm{sech}^2(kx)\hat{x} = k\left(1-\tanh^2(kx)\right)\hat{x} = k\left(1-\gamma^2\right)\hat{x} \tag{53}$$

Hence we can get

$$\boldsymbol{F}_\gamma = m\boldsymbol{a}_\gamma + m\phi_{Newton}k\left(\frac{1-\gamma^2}{\gamma^2}\right)\hat{\mathbf{x}} \tag{54}$$

However, this formula is derived based on the Newton approximation, so we have

$$\Lambda(\gamma)\to 0,\ \gamma\approx 1 \to m\phi_{Newton}k\left(\frac{1-\gamma^2}{\gamma^2}\right)\hat{\mathbf{x}}\approx \mathbf{0} \tag{55}$$

Finally, we get:

$$\boldsymbol{F}_\gamma = m\boldsymbol{a}_\gamma \Rightarrow \boldsymbol{F}_N = \gamma\boldsymbol{F}_\gamma = \gamma m\boldsymbol{a}_\gamma \tag{56}$$

where:

$\boldsymbol{a}_\gamma = -\frac{\nabla\phi_{Newton}}{\gamma}$ is the acceleration modulated by γ, while the Newtonian acceleration is $\boldsymbol{a}_N = -\nabla\phi_{Newton} = \gamma\boldsymbol{a}_\gamma$; $\frac{m\phi_{Newton}}{\gamma^2}\nabla\gamma$ is the direct effect of the γ gradient. Under the above Newtonian approximation, for galaxy structures beyond the solar system (for example, the Milky Way), since γ is closer to 1, so $\frac{1-\gamma^2}{\gamma^2}\to 0$, so $\boldsymbol{F}_N = \gamma\boldsymbol{F}_\gamma = \gamma m\boldsymbol{a}_\gamma$ is consistent with MOND theory [15] [16]. Here, γ has a dual role: it modulates the gravitational source (through the field equation) and modifies the inertial mass (through the dynamical equation), which is the source of the dark matter and dark energy hypothesis.

5. Physical Effects and Cosmic Evolution

Through the above analysis of the physical behavior of $\gamma(x)$, we define:

$$\gamma(x) = \tanh(kx),\quad x = |a|/a_0 \tag{57}$$

Here a is the absolute acceleration of particles in the region (a can be regarded as the local acceleration field of the universe (such as galaxies or expansion effects)), $a_0 \approx 2\times10^{-8}\ \text{cm/s}^2$, is the Milky Way acceleration constant. k is the adjustment positive and negative and proportional coefficient.

So we can find:

For small scales (x large, $|a| \gg a_0$): $\gamma \to 1$, $\Lambda \to 0$, restore standard gravity (such as the scale of the solar system);

Galaxy scale (x medium, $0<|a|<a_0$: $0<\gamma<1$, effective gravity $G_{eff}=G/\gamma$ enhanced, simulated dark matter effect;

The cosmic scale ($x\to 0$, $|a|\to 0$): $\gamma \to 0^+$, $\Lambda \to \infty$ dominates, driving accelerated expansion; but ($x\to 0$, $|a|\to 0$): $\gamma \to 0^-$, $\Lambda \to -\infty$ dominates, driving accelerated contraction! It shows that at the edge of the evolution of the universe, expansion or contraction is extremely unstable and is a chaotic area!

Negative value region ($x<0$, k is adjusted to a negative value): $\gamma<0$, $\hat{F}'_{\mu\nu}<0$, the gravity is reversed to repulsion.

Or directly define $x=\frac{a}{a_0}$ (acceleration can be positive or negative); $a>0$, $x>0$, $\gamma\to 0^+$; $a<0$, $x<0$, $\gamma \to 0^-$. Physical meaning: the sign of a may be related to the acceleration direction of the local area of the universe. $\gamma\to 0^+$, $\Lambda\to\infty$, the expansion is consistent with the current universe; $\gamma \to 0^-$, $\Lambda \to -\infty$, contraction may occur in the future or in local areas. $\gamma \approx 0$ is the critical point, Λ diverges, and positive and negative fluctuations may cause random switching between expansion and contraction; it conforms to "edge chaos": matter is sparse on a large scale in the universe, and the fluctuation is significant.

Expansion and contraction: $\Lambda>0$, dark energy drives expansion; $\Lambda<0$, similar to negative energy density, possible local collapse. Edge chaos: $\gamma\approx 0$ may be the "phase transition point" of cosmic evolution; fluctuations may come from quantum effects or dynamic uncertainty of γ. Comparison with observations: In the current universe, $\gamma \approx 0.01$ (large-scale estimate), $\Lambda>0$ small positive value; has not reached the $\gamma=0$ chaotic region. Future edge: $\gamma\to 0$ may trigger chaotic fluctuations.

In fact, from Equation (57) $\gamma(x)=\tanh(kx)$, we can find that the universe is divided into two branches and curvature oscillations:

1) Branch 1 (positive branch, current universe), that is, $0\le\gamma\le1$:

In $\gamma=1$: positive curvature, Newtonian gravity applies; $0<\gamma<1$: gravity weakens, explaining the rotation curve (dark matter effect), accelerated expansion of the universe (dark energy effect); $\gamma=0$: zero curvature, gravity disappears, chaotic region, expansion and contraction of the universe depends on $\gamma\to 0^\pm$. So, if it is found that the universe γ is gradually getting smaller, $\gamma\to 0^+$, $\Lambda\to\infty$, it will lead to the solution of the "cosmological constant problem of physicists" (regarding the quantum vacuum energy discrepancy) and the "cosmological constant problem of astronomers" (concerning the observed accelerated expansion), basically confirming that the first branch of the evolution of this model is established.

2) Branch 2 (negative branch, negative curvature universe), that is, $\gamma < 0$:

$\hat{F}'_{\mu\nu} = \gamma \hat{F}_{\mu\nu} < 0$, the gauge field is reversed, generating repulsive force; G_{ab} becomes more negative, the curvature tends to −1, and the accelerated expansion intensifies.

The evolution of the universe has the characteristics of oscillation, which is manifested as $\gamma(x) = \tanh(kx)$ changes with x (or time), γ changes from 1 to 0 to −1, and the curvature cycles between (1, 0, −1). However, the two points $\gamma = 1, -1$ are also a kind of asymptote, and the terminal of the asymptote corresponds to a certain limit point, that is, $x \to \pm\infty$, $a \to 0$. The universe is actually similar to the starting singularity of expansion (Big Bang). These two singularities may also be highly chaotic and may transform into each other, so they may be connected by dotted lines. In the end, which singularity the universe expands violently from (Big Bang) may be random. Of course, the connection of the dotted lines here is more of a philosophical guess. Therefore, we give the following description of the universe evolution diagram.

Based on the above analysis, we can construct a cosmic evolution diagram:

1) Center dividing line, using the S-curve $\gamma(x) = \tanh(kx)$ as the dividing line; Symmetry: $\gamma = 0$ when $x = 0$; Asymptotics: $x \to +\infty$, $\gamma \to 1$, $x \to -\infty$, $\gamma \to -1$.

2) Outer circle construction, that is, defining the upper and lower outer circle curves:

$$\gamma_{upper} = \tanh(kx) + \operatorname{sech}(kx),\ \gamma_{lower} = \tanh(kx) - \operatorname{sech}(kx) \tag{58}$$

where, the hyperbolic secant function $\operatorname{sech}(kx) = \dfrac{2}{e^{kx} + e^{-kx}}$ takes a maximum value of 1 at $x = 0$, and tends to 0 as x increases; $a \le 1$ (such as $a = 0.5$) limits γ to [−1, 1]; $x = 0: \gamma_{upper} = a$, $\gamma_{lower} = -a$, $x \to \pm\infty$: γ_{upper}, $\gamma_{lower} \to \pm 1$, coinciding with the center line.

In this way, the upper and lower $\gamma_{upper}, \gamma_{lower}$ form a dynamic closure. It ensures that the decay of $\operatorname{sech}(kx)$ is close to the center line when the outer circle $x \to \pm\infty$, forming a visual closure. It can also be said that the evolution diagram simulates an evolutionary Tai Chi circle, see **Figure 1**.

Figure 1 illustrates: it is the image of the $\gamma - x$ plane; center line: $\gamma(x) = \tanh(kx)$ (solid line); upper outer circle: $\gamma_{upper} = \tanh(kx) + \operatorname{sech}(kx)$ (red dotted line); lower outer circle: $\gamma_{lower} = \tanh(kx) - \operatorname{sech}(kx)$ (green dotted line).

The physical effects manifested are: dark matter: when $0 < \gamma < 1$, $G_{eff} = G/\gamma > G$, which can explain the rotation curve without the need for dark matter particles; dark energy: $\gamma \to 0$, gravity disappears, Λ dominates the expansion; $\gamma < 0$, repulsive force increases, accelerating expansion (gravity begins to increase and the universe contracts).

6. Results and Discussion

This study establishes a geometrically unified framework where the dynamics of $\gamma(x)$ is driven by the principal bundle curvature R and the cosmological constant Λ. By constructing a modified field equation without external potentials, we achieve consistency with MOND theory in the Newtonian limit, thereby unifying dark matter, dark energy, and cosmic multiverse evolution within the extended principal bundle $\tilde{P} = P \times \mathbb{R}$. The theory demonstrates that γ's evolution ($1 \to 0 \to -1$) corresponds to cyclic transitions between positive, zero, and negative curvature states, fully aligning with the cyclic universe paradigm. Notably, the S-shaped evolution curve $\gamma(x) = \tanh(kx)$ and virtual outer loops γ_{upper} and γ_{lower} (closed via $\operatorname{sech}(kx)$ perturbations) form a Taiji-like topological closure at $x = \pm\infty$ ($\gamma = \pm 1$), embodying philosophical circularity (**Figure 1**). This framework not only provides a purely geometric explanation for dark matter and dark energy but also reveals the universe's dynamic cyclic nature through curvature oscillations. Future observational constraints from galaxy rotation curves and supernova distance measurements will refine key parameters (k, a_0, Λ_0), enhancing the model's falsifiability.

Finally, we address a potential conceptual ambiguity: Could the introduced $\gamma(x) \in \mathbb{R}$ in the extended structure group $\tilde{G} = G \times \mathbb{R}$ originate from dark matter or dark energy rather than geometric properties? To clarify this, we systematically analyze the theoretical framework, computational logic, and physical implications to justify why $\gamma(x)$ fundamentally stems from spacetime geometry rather than external dark components.

First, within the extended principal bundle $\tilde{P} = P \times \mathbb{R}$, the additional gauge freedom $\gamma(x)$ manifests through the connection form $\omega_\gamma = d\gamma$. Crucially, the total modified connection $\tilde{\omega} = \omega_G + \omega_\gamma$ preserves the original curvature structure ($\tilde{\Omega} = \Omega_G$), as verified by the curvature calculation $\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}$. This geometric invariance demonstrates that $\gamma(x)$ does not modify spacetime curvature but dynamically scales gravitational interactions via the action principle. The field equation $\gamma G_{\mu\nu} + 2\pi A\left(\frac{1-\gamma}{\gamma}\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$ explicitly reveals $\gamma(x)$'s role as a geometric modulator acting directly on the Einstein tensor $G_{\mu\nu}$ [9], distinct from dark matter/energy contributions conventionally embedded in $T_{\mu\nu}$.

Second, the functional design $\gamma(x) = \tanh(kx)$ inherently encodes cosmic-scale gravitational behavior:

- At $\gamma \to 1$ (small scales), standard Einstein gravity is recovered;
- For $0 < \gamma < 1$ (galactic scales), amplified gravity mimics dark matter effects;
- At $\gamma \to 0^+$ (cosmological scales), divergence of $\Lambda(\gamma) = 2\pi A\left(\frac{1-\gamma}{\gamma}\right)$ drives accelerated expansion [17].

Unlike dark sector models that postulate independent energy-momentum terms $T_{\mu\nu}^{(\mathrm{DM/DE})}$ [18], our framework internalizes these phenomena through geometric renormalization of the gravitational sector, eliminating the need for ad hoc matter fields.

Finally, while our approach aligns conceptually with geometric modified gravity theories (e.g., f(R) [19], scalar-tensor [6] [20]), its uniqueness lies in unifying dark matter, dark energy, and cyclic cosmology via principal bundle extension and hyperbolic $\gamma(x)$ dynamics. The $\gamma(x)$ field emerges as a novel gauge transformation operator [10]-[14] [21], intrinsically tied to spacetime geometry rather than external matter. Observational tests—such as quantifying $\gamma(x)$'s imprint on galaxy rotation curves and large-scale structure—will further distinguish this geometric interpretation from dark matter/energy paradigms. We emphasize that $\gamma(x)$'s physical essence resides in its role as a *geometric phase* modulating gravitational interactions across cosmic scales, offering a parsimonious alternative to conventional dark sector hypotheses.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Two Perspectives on Gauge Transformation

1. Gauge transformation in physics (traditional definition)

In physics, gauge transformation usually appears in gauge field theory (such as electromagnetism, Yang-Mills theory), indicating the transformation of local symmetry of the field. It acts on gauge fields (such as A_μ) or matter fields (such as ψ), but not directly on the cross section of the principal bundle. It is usually defined as follows:

For a gauge field A_μ (local form, $= s^*\omega$), G is the gauge group (such as $U(1)$ or $SU(N)$), and the gauge transformation is:

$$A'_\mu = g(x)A_\mu g(x)^{-1} + g(x)\partial_\mu g(x)^{-1},\quad g(x) \in G \tag{A1}$$

The field strength here is

$$F'_{\mu\nu} = g(x)F_{\mu\nu}g(x)^{-1} \tag{A2}$$

For matter field ψ (such as electron field):

$$\psi' = g(x)\psi \tag{A3}$$

The characteristics of this construction are: the transformation is local $g(x)$ dependent $x \in M$; the field strength $F_{\mu\nu}$ is transformed in the adjoint representation, not directly a scalar scaling. For example, in electromagnetism ($G = U(1)$):

$$A'_\mu = A_\mu + \partial_\mu \Lambda(x) \tag{A4}$$

$$F'_{\mu\nu} = F_{\mu\nu} \quad \text{(unchanging)} \tag{A5}$$

2. Gauge transformation in our model

First, we define: principal bundle $\tilde{P} = P \times \mathbb{R}$, structure group $\tilde{G} = G \times \mathbb{R}$; primitive section $\tilde{s}(x) = (s(x), \gamma(x))$; then the gauge transformation is $U = (h(x), \tilde{\gamma}(x)) \in G \times \mathbb{R}$:

$$\begin{aligned} \tilde{s}'(x) &= \tilde{s}(x)\cdot(h(x),\tilde{\gamma}(x)) \\ &= (s(x),\gamma(x))(h(x),\tilde{\gamma}(x)) \\ &= (s(x)h(x),\gamma(x)+\tilde{\gamma}(x)) \end{aligned} \tag{A6}$$

where $\tilde{\gamma}(x)$ represents an additional scalar degree of freedom, potentially related to cosmic dynamics, and the connection transformation is:

$$\omega'_G = Ad_{h^{-1}}\omega_G + h^{-1}dh \tag{A7}$$

$$\omega'_\gamma = d(\gamma + \tilde{\gamma}) = d\gamma + d\tilde{\gamma} \tag{A8}$$

Field strength can be scaled:

$$\hat{F}'_{\mu\nu} = \gamma(x)\hat{F}_{\mu\nu} \quad \text{(Direct scalar scaling)} \tag{A9}$$

Note that (A9) is not directly derived from $\tilde{\Omega} = \Omega_G$, but introduced via the action S in (20), reflecting a physical gauge scaling. The difference and connection between the physical gauge transformation defined above and the gauge transformation we defined are:

1) The difference is manifested in the different objects of action. Physical gauge

transformations act directly on the gauge field A_μ (gauge potential) or the matter field ψ; the representation of the field is changed through the local group element $g(x)$. In our model, the gauge transformation acts on the principal bundle cross section $\tilde{s}(x)$; the new cross section $\tilde{s}'(x)$ is defined through the group action $U=(h(x),\tilde{\gamma}(x))$.

2) The field strength transformation takes a different form. The physical gauge transformation, $F'_{\mu\nu}=g(x)F_{\mu\nu}g(x)^{-1}$ (adjoint transformation); the eigenvalue ratio is kept unchanged (similarity transformation). In our model, $\hat{F}'_{\mu\nu}=\gamma(x)\hat{F}_{\mu\nu}$ (scalar scaling); $\gamma(x)$ can be arbitrarily changed (e.g. < 1 or < 0), beyond the adjoint transformation.

3) The group structure is different. The physical gauge transformations involve only G (e.g. $U(1)$, $SU(N)$), with no additional fibers. Our model is extended to $G\times\mathbb{R}$, introducing the scalar degrees of freedom of $\mathbb{R}$.

4) However, the two are interconnected, showing that they have a common foundation; both originate from gauge symmetry, namely local group transformation; the principal bundle P and the connection ω are the geometric sources of the gauge field.

5) The cross section corresponds to the field. Physically, $A_\mu=s^*\omega$ is the result of the cross section $s(x)$ pulling back the connection; the gauge transformation $s(x)\to s(x)g(x)$ causes the change of A_μ. Our model: $\tilde{s}'(x)=\tilde{s}(x)\cdot(h(x),\tilde{\gamma}(x))$ changes the cross section; $\omega'_\gamma=d\gamma+d\tilde{\gamma}$ reflects the dynamics of $\gamma(x)$. Both are to introduce local degrees of freedom and describe the transformation law of the physical field.

3. Mathematical basis of the gauge transformation constructed

1) Group action of the principal bundle: The definition of the principal bundle $\tilde{P}$ requires the structure group $\tilde{G}=G\times\mathbb{R}$ through the right multiplication action:

$$(p,\gamma)\cdot(h,\tilde{\gamma})=(ph,\gamma+\tilde{\gamma}) \tag{A10}$$

Transformation of the local section $\tilde{s}$: $M\to\tilde{P}$:

$$\tilde{s}'(x)=\tilde{s}(x)\cdot U(x),\ U(x)=(h(x),\tilde{\gamma}(x)) \tag{A11}$$

The above is the standard gauge transformation form in principal bundle theory. The transformation properties of the connection here are: $\tilde{\omega}$ as a connection 1-form satisfies the equivariance of $G\times\mathbb{R}$:

$$R_U^*\tilde{\omega}=Ad_{U^{-1}}\tilde{\omega}+U^{-1}dU \tag{A12}$$

$$U=(h,\tilde{\gamma}),\ U^{-1}=(h^{-1},-\tilde{\gamma}) \tag{A13}$$

Here $\tilde{\omega}$ can be decomposed into:

$$\omega'_G=Ad_{h^{-1}}\omega_G+h^{-1}dh \tag{A14}$$

$$\omega'_\gamma=d\gamma+d\tilde{\gamma} \tag{A15}$$

This is the mathematical definition of a gauge transformation.

So, in traditional gauge field theory, $A'_\mu=gA_\mu g^{-1}+g\partial_\mu g^{-1}$ is a transformation of G; and by introducing the scalar action of ℝ, we can make $\hat{F}_{\mu\nu}$ scalable to $\gamma(x)\hat{F}_{\mu\nu}$, which goes beyond the adjoint transformation. Further, due to the requirements of cosmic evolution, $\gamma(x) = \tanh(kx)$ needs to change dynamically (positive, negative, zero), and $\tilde{\gamma}(x)$ provides this degree of freedom; this is consistent with the gauge symmetry in physics, but expands the scope of action.

Therefore, this extension has a reasonable basis; it is manifested in geometric consistency, that is, the structure of the principal bundle $\tilde{P}$ requires group action to affect the cross section; the direct product form of $G \times \mathbb{R}$ naturally decomposes into $h(x)$ and $\tilde{\gamma}(x)$. It also has physical innovation, that is, the scalar scaling of $\gamma(x)$ explains dark matter ($0 < \gamma < 1$) and dark energy ($\gamma \to 0$), which requires an additional gauge degree of freedom of ℝ. Furthermore, the scalar $\gamma(x) = \tanh(kx)$ is chosen to match cosmic evolution, with $0 < \gamma < 1$ mimicking dark matter and $\gamma \to 0$ driving expansion, validated by the field equation.

In short, the physical gauge transformation acts directly on the field, and our transformation acts on the cross section; the traditional transformation is in adjoint form, and we introduce scalar scaling. However, both are based on local symmetry, and the principal bundle is the common foundation.

Appendix B. Derivation of $\omega_G \wedge d\gamma = 0$

The process of obtaining $[T_a, D] = 0$ and $\tilde{\Omega} = \Omega_G$ in the main text involves the calculation of the curvature of the principal bundle connection, the commutation relation of Lie algebras, and the properties of the wedge product. We give a step-by-step derivation here.

First, we set the principal bundle to be: $\tilde{P} = P \times \mathbb{R}$, the structure group $\tilde{G} = G \times R$, then the connection form is: $\tilde{\omega} = \omega_G + \omega_\gamma$, where $\omega_G \in \Omega^1(P, \mathrm{g})$ is the connection of G, g is the Lie algebra of G; $\omega_\gamma = d\gamma \in \Omega^1(\tilde{P}, \mathbb{R})$ is the connection of $\mathbb{R}$, $\mathbb{R}$ is the Lie algebra of $\mathbb{R}$. Then the curvature of the extended connection is, $\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}$; in addition, T_a: the generator (basis) of g, such as $a = 1, 2, \cdots, dim(\mathrm{g})$; D is the generator of the Lie algebra of $\mathbb{R}$, choose $D = 1$ (standard basis).

Then the derivation process is given

1) Definition of curvature form:

The curvature of the principal bundle connection is

$$\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} \tag{B1}$$

Substituting $\tilde{\omega} = \omega_G + \omega_\gamma$, we get

$$\tilde{\Omega} = d(\omega_G + \omega_\gamma) + (\omega_G + \omega_\gamma) \wedge (\omega_G + \omega_\gamma) \tag{B2}$$

2) Expand calculation

The exterior differential term in the above formula is

$$d(\omega_G + \omega_\gamma) = d\omega_G + d\omega_\gamma \tag{B3}$$

Here $d\omega_G$ is the g-valued 2-form; $d\omega_\gamma = dd\gamma = 0$ (since $d\gamma$ is a 1-form, $d^2\gamma = 0$).

The wedge product term in the above formula is

$$\left(\omega_G+\omega_\gamma\right)\wedge\left(\omega_G+\omega_\gamma\right)=\omega_G\wedge\omega_G+\omega_G\wedge\omega_\gamma+\omega_\gamma\wedge\omega_G+\omega_\gamma\wedge\omega_\gamma \quad \text{(B4)}$$

Substituting (B3) and (B4) into (B2) and combining them, we obtain:

$$\tilde{\Omega}=d\omega_G+d\omega_\gamma+\omega_G\wedge\omega_G+\omega_G\wedge\omega_\gamma+\omega_\gamma\wedge\omega_G+\omega_\gamma\wedge\omega_\gamma \quad \text{(B5)}$$

Substituting $d\omega_\gamma=0$, the above equation becomes

$$\tilde{\Omega}=d\omega_G+\omega_G\wedge\omega_G+\omega_G\wedge\omega_\gamma+\omega_\gamma\wedge\omega_G+\omega_\gamma\wedge\omega_\gamma \quad \text{(B6)}$$

Analyze the cross terms:

a) $\omega_G\wedge\omega_G$: ω_G is g-value 1-form:

$$\omega_G=\sum\nolimits_a\omega^aT_a \quad \text{(B17)}$$

where ω^a is a scalar 1-form, T_a is a basis of g; for g-valued forms, the definition of the wedge product requires consideration of multiplication in the Lie algebra:

$$\left(\omega_G\wedge\omega_G\right)(X,Y)=\left[\omega_G(X),\omega_G(Y)\right]-\left[\omega_G(Y),\omega_G(X)\right] \quad \text{(B8)}$$

In the above formula, because

$$\left[\omega_G(X),\omega_G(Y)\right]=\sum\nolimits_{a,b}\omega^a(X)\omega^b(Y)\left[T_a,T_b\right] \quad \text{(B9)}$$

$$\left[\omega_G(Y),\omega_G(X)\right]=\sum\nolimits_{a,b}\omega^a(Y)\omega^b(X)\left[T_a,T_b\right] \quad \text{(B10)}$$

therefore

$$\left(\omega_G\wedge\omega_G\right)(X,Y)=\sum\nolimits_{a,b}\left(\omega^a(X)\omega^b(Y)-\omega^a(Y)\omega^b(X)\right)\left[T_a,T_b\right] \quad \text{(B11)}$$

$$\omega_G\wedge\omega_G=\sum\nolimits_{a,b}\omega^a\wedge\omega^b\left[T_a,T_b\right] \quad \text{(B12)}$$

where $\left[T_a,T_b\right]=f_{ab}^cT_c$, f_{ab}^c is a structural constant.

b) $\omega_\gamma\wedge\omega_\gamma$: $\omega_\gamma=d\gamma$ is ℝ-valued 1-form (scalar 1-form); since the wedge product is antisymmetric, hence

$$\omega_\gamma\wedge\omega_\gamma=d\gamma\wedge d\gamma=0 \quad \text{(B13)}$$

c) $\omega_G\wedge\omega_\gamma$ and $\omega_\gamma\wedge\omega_G$ can be expanded into the form:

$$\omega_G\wedge\omega_\gamma=\sum\nolimits_a\omega^a\wedge d\gamma T_a \quad \text{(B14)}$$

$$\omega_\gamma\wedge\omega_G=d\gamma\wedge\sum\nolimits_a\omega^aT_a=\sum\nolimits_a d\gamma\wedge\omega^aT_a \quad \text{(B15)}$$

$$d\gamma\wedge\omega^a=-\omega^a\wedge d\gamma \quad \text{(Antisymmetry between 1-forms)} \quad \text{(B16)}$$

Then, because ω_G takes values at g, the generator is T_a; ω_γ takes values at $\mathbb{R}$, the generator is $D=1$; $G\times\mathbb{R}$ is a direct product group, and its Lie algebra is $g\oplus\mathbb{R}$, so the Lie algebra commutation relation is:

$$\left[T_a,D\right]=0 \quad \text{(B17)}$$

That is, because G and $\mathbb{R}$ are independent, the crossover is zero.

So $\omega_G\wedge\omega_\gamma$ should be taken in $g\oplus\mathbb{R}$, for the vector field $X,Y\in T_p\tilde{P}$:

$$\left(\omega_G\wedge\omega_\gamma\right)(X,Y)=\omega_G(X)\omega_\gamma(Y)-\omega_G(Y)\omega_\gamma(X) \quad \text{(B18)}$$

where, $\omega_G(X)\in g$ and $\omega_\gamma(Y)\in\mathbb{R}$. However, in direct product Lie algebras, the product of g and $\mathbb{R}$ needs to be defined. In $g \oplus \mathbb{R}$, the wedge product is usually defined as:

$$\omega_G \wedge \omega_\gamma = \left(T_a \otimes \omega^a\right) \wedge \left(D \otimes d\gamma\right) = \left[T_a, D\right] \otimes \left(\omega^a \wedge d\gamma\right) \quad \text{(B19)}$$

Since $\left[T_a, D\right] = 0$, the right side of the above equation is 0, so $\left(\omega_G \wedge \omega_\gamma\right) = 0$, $-\omega_G \wedge \omega_\gamma = \omega_\gamma \wedge \omega_G = 0$.

Finally, combining the results of a), b), and c) above, we have:

$$\tilde{\Omega} = d\omega_G + \omega_G \wedge \omega_G = \Omega_G \quad \text{(B20)}$$

That is, the connection γ of $\mathbb{R}$ does not contribute to curvature and maintains the geometric structure of G.