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A Class of Copulas Derived from Residual Implications and Its Applications

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Abstract

Copulas are multivariate distribution functions with uniform marginal distributions. In this paper, we study a class of copulas called radial copulas, which is derived from residual implications where the extensions of level curves intersect at a point. This class of radial copulas is a comprehensive and asymmetric extension of a class of Archimedean copulas. We derive analytical formulas for key concordance measures, including Spearman's rho and Kendall's tau, and demonstrate that these formulas cover the entire range of positive and negative correlations. Furthermore, we estimate the parameters of radial copulas and evaluate their performance through a simulation study under various dependence structures. Finally, using two datasets, we compare the performance of the class of radial copulas to that of several well-known copula models.

Keywords

Copula, Residual Implication, Archimedean Copulas, Dependence Measures

1. Introduction

Copulas are multivariate distribution functions characterized by uniform marginal distributions. In 1959, Sklar [1] demonstrated that for any multivariate distribution, a copula exists that allows the joint distribution to be expressed as the copula applied to the marginal distributions. Because copulas are independent of the marginal distributions, they provide a convenient framework for addressing various modeling challenges. Copulas have been used in many fields, including finance, engineering, economics, and environmental science [2]-[5].

Fuzzy implications play a critical role in fuzzy logic, enabling logical reasoning in uncertain situations [6]. Among them, residual implications stand out as a key type and are derived from aggregation functions [7]-[10]. In recent years, many researchers have explored fuzzy implications formed by various aggregation functions [11]-[13], while others have focused on characterizing residual implications based on different aggregation functions [14] [15]. Additionally, some studies constructed different fuzzy implications from copulas, including probabilistic implications and probability *S*-implications [16]-[18].

There is a one-to-one correspondence between left-continuous semicopulas and their derived residual implications [9]. In 1999, a characterization for residual implications derived from left-continuous semicopulas was provided by Demirli *et al.* [9]. In 2007, Durante *et al.* [19] extended this work by characterizing residual implications derived from quasi-copulas. More recently, in 2022, Ji *et al.* [20] further advanced this area by characterizing residual implications derived from copulas through their level curves.

This article aims to study a class of copulas derived from residual implications, where the extensions of the level curves intersect at a point. Specifically, this class represents a comprehensive and asymmetric extension of the Archimedean copula family (4.2.7) as described in [23], providing greater flexibility for modeling real-world phenomena. The proposed copulas include two dependence parameters, a and b, and are able to capture correlation coefficients in the range $[-1,1]$, i.e. Kendall's tau and Spearman's rho. It is well known that multiparameter copulas are capable of modeling a variety of dependency structures, including non-linear, asymmetric and tail dependent relationships that one-parameter copulas cannot adequately capture. In summary, the new model allows for both positive and negative dependencies without imposing constraints on the correlation structure. These important features motivate a theoretical and practical analysis of the proposed copulas.

The remainder of the paper is structured as follows. Section 2 provides a review of the fundamental concepts and key properties related to copulas and residual implications. In section 3, we introduce radial copulas and examine some dependence concepts. Section 4 focuses on estimating the two dependence parameters of radial copulas and includes a simulation study to assess the performance of these estimators. In section 5, we perform an analysis of two real datasets to examine the applications of the proposed radial copulas. Finally, section 6 presents our conclusions.

2. Preliminaries

Copulas are multivariate functions characterized by uniform marginal distributions. Let us first review some of their definitions and properties.

Definition 1 [21] *A binary operation* $C:[0,1]^2 \to [0,1]$ *is defined as a semicopula if it fulfills the subsequent conditions: [label=(C)]*

1. $C(0,0)=0$ and $C(1,1)=1$;
2. For every $u,v \in [0,1]$, it holds that $C(u,1)=u$ and $C(1,v)=v$;
3. C is nondecreasing in both variables.

Definition 2 [22] *A binary operation* $C:[0,1]^2 \to [0,1]$ *is defined as a quasi-copula if it is a semicopula and also adheres to the 1-Lipschitz condition. Specifically, this implies that the following inequality is valid for all* $u_1,u_2,v_1,v_2 \in [0,1]$: $|C(u_1,v_1)-C(u_2,v_2)| \le |u_1-u_2|+|v_1-v_2|$.

Definition 3 [23] *A binary operation* $C:[0,1]^2 \to [0,1]$ *is defined as a copula if it is a semicopula and also fulfills the condition of being 2-increasing. Specifically, this indicates that for any* $u_1,u_2,v_1,v_2 \in [0,1]$ *where* $u_1 \le u_2$ *and* $v_1 \le v_2$, *the following inequality is valid*: $C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1) \ge 0$.

Residual implications derived from various classes of left-continuous semicopulas have been examined in [9] [19]. For any left-continuous semicopula C, the derived residual implication, denoted as $R_C:[0,1]^2 \to [0,1]$, is defined by

$$R_C(u,w)=\sup\{v\in[0,1] \mid C(u,v)\le w\}. \tag{1}$$

This function R_C is referred to as the residuum of C or the C-residuum [19].

Proposition 4 [9] *If* $C:[0,1]^2 \to [0,1]$ *is a left-continuous semicopula, then the corresponding* C *residuum* R_C *has the properties listed below*:

1. $R_C(u,w)=1$ if and only if $u \le w$;
2. $R_C(1,w)=w$ for all $w\in[0,1]$;
3. If $u_1,u_2,w\in[0,1]$ with $u_1\le u_2$, then it holds that $R_C(u_1,w)\ge R_C(u_2,w)$;
4. If $u,w_1,w_2\in[0,1]$ with $w_1\le w_2$, then it holds that $R_C(u,w_1)\le R_C(u,w_2)$;
5. The first variable of the R_C function is left-continuous;
6. The second variable of the R_C function is right continuous.

On the contrary, if a function $R:[0,1]^2 \to [0,1]$ fulfills conditions (R1) through (R6), then there exists a left-continuous semicopula C_R where $R=R_{C_R}$. Additionally, the definition of C_R is as follows:

$$C_R(u,v)=\inf\{w\in[0,1] \mid R(u,w)\ge v\}. \tag{2}$$

The set of all functions $R:[0,1]^2 \to [0,1]$ that fulfill conditions (R1) through (R6) is represented by $\mathcal{R}$. Given any $R\in\mathcal{R}$, the associated deresiduum, referred to as the R-deresiduum, is denoted by C_R [19]. For simplicity, when the context is clear, R_C and C_R are often abbreviated as R and C, respectively.

Similar to the t-norms, the adjointness condition of the

$$C(u,v)\le w \Leftrightarrow R(u,w)\ge v \tag{3}$$

applies to any left-continuous semicopula C and its residuum R [9].

Proposition 5 [19] [20] *Let* $C:[0,1]^2 \to [0,1]$ *be a quasi-copula. The* C*-residuum* R_C *fulfills conditions (R1) through (R6) and two additional properties*:

1. For all $u,w,a\in[0,1]$ such that $u+a\le 1$ and $w+a\le 1$, it holds that $R(u+a,w+a)\ge R(u,w)$;
2. For all $u,w,a\in[0,1]$ with the condition that $u\ge w+a$, it holds that $R(u,w+a)\ge R(u,w)+a$.

By contrast, if $R:[0,1]^2 \to [0,1]$ fulfills conditions (R1) through (R8), then the R-deresiduum C_R is a quasi-copula.

The level set of $R\in\mathcal{R}$ for a constant $t\in[0,1]$ is defined as

$\{(u,w) \mid R(u,w)=t\}$. Let $R:[0,1]^2 \to [0,1]$ be a function that satisfies conditions (R1) through (R8). For $0 \le t < 1$, if $R(u,w_1)=R(u,w_2)=t$, then according to condition (R8), we conclude that $w_1 = w_2$. Consequently, the level set of R associated with $t \in [0,1)$ is made up of points lying on a curve, known as the level curve of R [20]. For $t=1$, the level set of R is defined as $\{(u,w) \mid u \le w \le 1, 0 \le u \le 1\}$. The boundary curve of this level set, $\{(u,w) \mid w=u, 0 \le u \le 1\}$, is referred to as the level curve of R for the constant 1. We often denote the level curve of R as $w=L_t(u)$, where $0 \le t \le 1$.

Proposition 6 [20] *Let $C:[0,1]^2 \to [0,1]$ be a quasi-copula, and let R represent its residuum. Suppose that $w=L_t(u)$ represents a level curve of R for $t \in [0,1]$. For $u_1, u_2 \in Dom(L_t(u))$ such that $u_1 \le u_2$, we have $0 \le L_t(u_2) - L_t(u_1) \le u_2 - u_1$.*

The following proposition gives a necessary and sufficient condition for the R-deresiduum C to be a copula.

Proposition 7 [20] *Let $R:[0,1]^2 \to [0,1]$ satisfy conditions (R1) through (R8). Let $\{w=L_t(u) \mid t \in [0,1]\}$ denote the set of level curves of R. The R-deresiduum C is a copula if and only if each level curve of R has a domain that is an interval, and the inequality $L_{v_2}(u_1)-L_{v_1}(u_1) \le L_{v_2}(u_2)-L_{v_1}(u_2)$ holds for $u_1,u_2,v_1,v_2 \in [0,1]$, where $u_1 \le u_2$, $v_1 \le v_2$, and $L_{v_j}(u_i)$ exists for all $i,j \in \{1,2\}$.*

The domain of any level curve is either $(a,1]$ or $[a,1]$, where $0 \le a \le 1$, for the residual implication derived from a copula. Notably, the vertical distance between two level curves is consistently greater on the right side than on the left [20].

3. Proposed Copulas

This section introduces a class of copulas derived from residual implications, which are characterized by extensions of level curves that intersect at a point. This class includes two dependence parameters, providing greater flexibility in modeling dependencies.

For the product copula $\Pi(u,v)=uv$, the corresponding residual implication is given by $R_\Pi(u,w)=\begin{cases} \frac{w}{u}, & \text{if } u > w \\ 1, & \text{if } u \le w \end{cases}$. For $0 \le v \le 1$, the level curves of R_Π can

be expressed by the equation $w=L_v(u)=vu$, where $0 < u \le 1$. Clearly, for a given value of v, the level curves of R_Π are segments of lines with a slope of v, all intersecting at the origin.

According to Proposition 6, if the level curves lie on a line, then the slope of the level curves of residual implications lies within the interval $[0,1]$. Consequently, when the level curves of a residual implication derived from a copula are straight lines intersecting at a common point, this intersection point is represented as $(-a,-b)$, where $0 \le b \le a$. Given the intersection point $(-a,-b)$ of the extended level curves and noting that they always pass through $(1,v)$, we derive the equation of the line on which these level curves lie as $\frac{w+b}{u+a}=\frac{v+b}{1+a}$. This simplifies to $w=\frac{uv+bu+av-b}{1+a}$. Thus, $R_{a,b}(u,w)=\begin{cases}\frac{aw+w-bu+b}{u+a}, & \text{if } u>w\\ 1, & \text{if } u\le w\end{cases}$.

It follows from Propositions 5 and 7 that the function $R_{a,b}$ is a residual implication derived from a copula.

For a,b in $\overline{\mathbf{R}}$ such that $0\le b\le a$, let $R_{a,b}:[0,1]^2\to[0,1]$ be a function given by

$$R_{a,b}(u,w)=\begin{cases}\frac{aw+w-bu+b}{u+a}, & \text{if } u>w,\\ 1, & \text{if } u\le w.\end{cases} \tag{4}$$

Then the corresponding copula $C_{a,b}:[0,1]^2\to[0,1]$ is given by

$$C_{a,b}(u,v)=\begin{cases}0, & \text{if } v<\frac{b(1-u)}{u+a},\\ \frac{uv+bu+av-b}{1+a}, & \text{if } \frac{b(1-u)}{u+a}\le v<\frac{(1+a-b)u+b}{u+a},\\ u, & \text{if } v\ge\frac{(1+a-b)u+b}{u+a}.\end{cases} \tag{5}$$

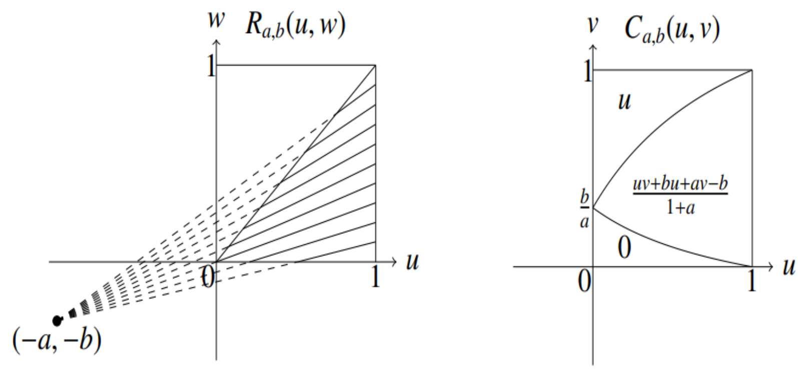


Figure 1. Graphs of $R_{a,b}$ and $C_{a,b}$ for $a=1$ and $b=0.3$.

We refer to the copula $C_{a,b}$ as a radial copula. The left side of Figure 1 shows some level curves of $R_{a,b}$, while the right side displays the graph of the radial copula $C_{a,b}$.

Remark 1 *Here are some special cases of the radial copula*: *[label = ()]*

1. If $a=0$, then $b=0$, i.e., $C_{0,0}(u,v)=\Pi(u,v)=uv$.

2. If $a=b<+\infty$, then $C_{a,a}(u,v)=\max\left((uv+au+av-a)/(1+a),0\right)$, which is equal to the Archimedean copula family (4.2.7) in [23].

3. If $a=+\infty$ and $0\le b<+\infty$, then $\lim_{a\to+\infty}C_{a,b}(u,v)=M(u,v)$.

4. If $k\in[0,1]$ and $b=ka$, then

$$\lim_{a\to+\infty} C_{a,ka}(u,v)=\begin{cases}0, & \text{if } v<k(1-u),\\ ku+v-k, & \text{if } k(1-u)\le v<(1-k)u+k,\\ u, & \text{if } v\ge(1-k)u+k.\end{cases}$$

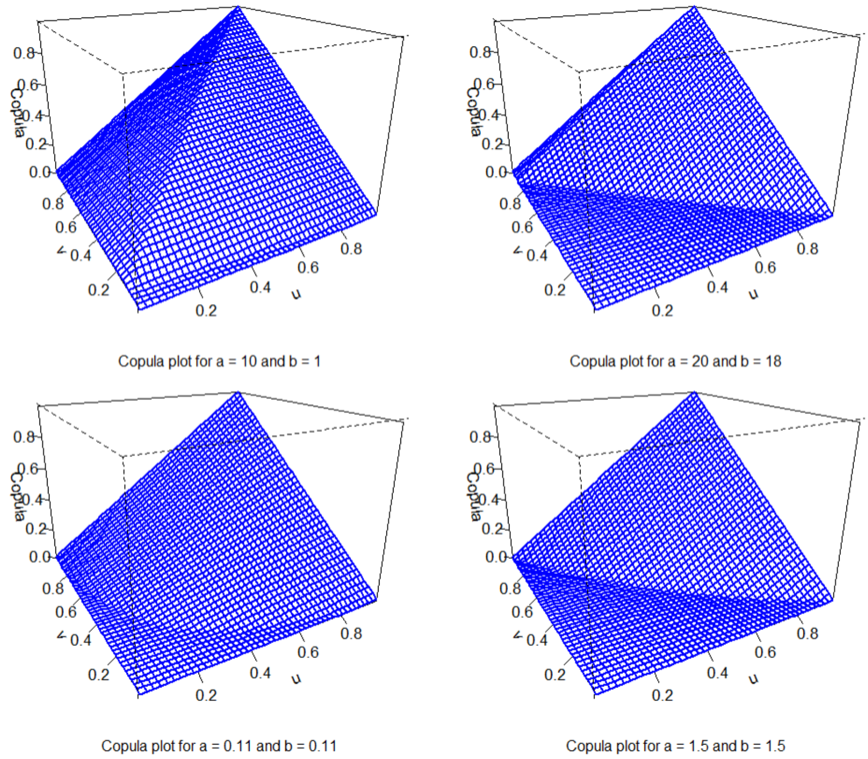


Figure 2. Graphical plots of $C_{a,b}$ for various parameter sets a and b.

The latter part of the formula is equal to the copula C_θ studied in [20, Example 5.14]. In particular, if $k=0$, then $\lim_{a\to+\infty} C_{a,0}(u,v)=M(u,v)$. If $k=1$, then $\lim_{a\to+\infty} C_{a,a}(u,v)=W(u,v)$.

The class of radial copulas is obviously an asymmetric extension of the Archimedean copula family (4.2.7), as presented in [23]. Notably, the Archimedean copula family (4.2.7) does not include the Frechet-Hoeffding upper bound M. As a result, the class of radial copulas provides a comprehensive extension of the Archimedean copula family (4.2.7). In **Figure 2** and **Figure 3**, we show the radial copula plots and their contour plots for four different parameter value sets a and b.

There are various methods employed to generate observations (x,y) for a pair of random variables (X,Y) characterized by a joint distribution function H. One such method involves utilizing the copula function and the concept of conditional probability, referred to as the conditional distribution method [23]. This approach proposes an algorithm for generating random samples from the copula $C_{a,b}$ using the conditional copula V given $U=u$. The algorithm is as follows:

1. Generate two independent uniform random variables u_i and t_i from the interval $[0,1]$.

2. Define $v_i = c_{u_i}^{(-1)}(t_i)$, where $c_{u_i}^{(-1)}$ represents a quasi-inverse of the function c_{u_i}, and also $c_{u_i}(v_i) = \frac{\partial C_{a,b}(u_i, v_i)}{\partial u_i}$. Specifically, the conditional distribution function $c_{u_i}(v_i)$ is given by

$$c_{u_i}(v_i) = \frac{\partial C_{a,b}(u_i, v_i)}{\partial u_i} = \begin{cases} 0, & \text{if } v_i < \dfrac{b(1-u_i)}{u_i + a}, \\ \dfrac{v_i + b}{1+a}, & \text{if } \dfrac{b(1-u_i)}{u_i + a} \le v_i < \dfrac{(1+a-b)u_i + b}{u_i + a}, \\ 1, & \text{if } v_i \ge \dfrac{(1+a-b)u_i + b}{u_i + a}, \end{cases}$$

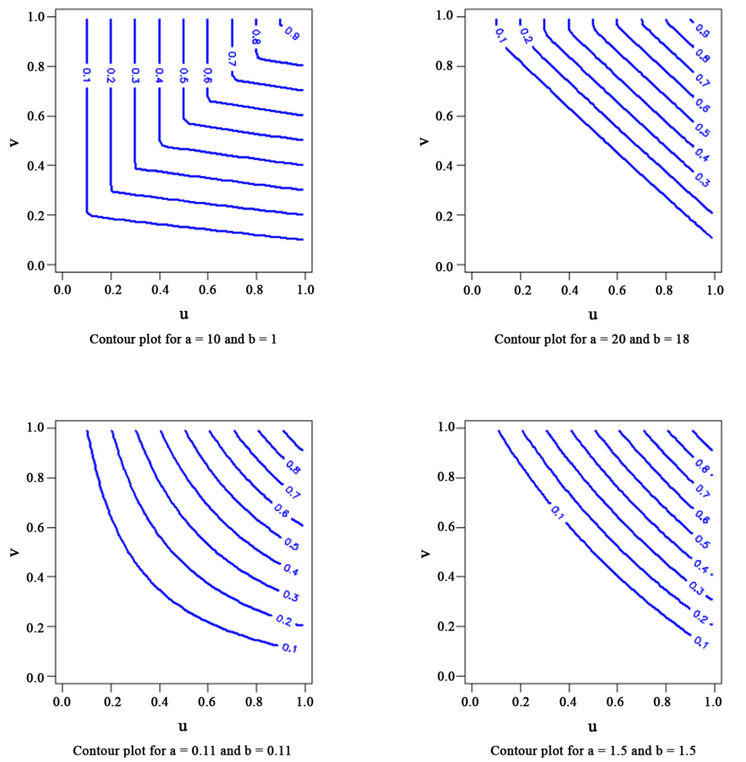


Figure 3. Contour plots of $C_{a,b}$ for various parameter sets a and b.

and its quasi-inverse $c_{u_i}^{(-1)}(t_i)$ is given by

$$c_{u_i}^{(-1)}(t_i) = \begin{cases} \dfrac{b(1-u_i)}{u_i + a}, & \text{if } t_i < \dfrac{b}{u_i + a}, \\ (1+a)t_i - b, & \text{if } \dfrac{b}{u_i + a} \le t_i < \dfrac{u_i + b}{u_i + a}, \\ \dfrac{(1+a-b)u_i + b}{u_i + a}, & \text{if } t_i \ge \dfrac{u_i + b}{u_i + a}. \end{cases}$$

3. Repeat steps i and ii n times to generate a set of independent and identically distributed pairs (u_i, v_i) from the copula $C_{a,b}$.

Figure 4 displays scatter plots derived from 500 simulated data points, which were generated using the previously described algorithm with four distinct sets of parameter values for a and b. Figures 4(c) and 4(d) correspond to the scatter plots of copulas in [25] with $\theta = 0.9$ (right) and $\theta = 0.4$ (left), respectively.

For any copula C, we have the decomposition

$C(u,v) = A_C(u,v) + S_C(u,v),$

where

$$A_C(u,v) = \int_0^u \int_0^v \frac{\partial^2}{\partial s \partial t} C(s,t)\,\mathrm{d}t\mathrm{d}s \text{ and } S_C(u,v) = C(u,v) - A_C(u,v), \quad (6)$$

with A_C representing the absolutely continuous component and S_C the singular component of the copula C.

If a copula C satisfies $C = A_C$ on $[0,1]^2$, that is, if C can be represented as a joint distribution with density $\frac{\partial^2 C(u,v)}{\partial u \partial v}$, then C is absolutely continuous. If $C = S_C$ on $[0,1]^2$, meaning that $\frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$ almost everywhere, then C is classified as singular [23]. Conversely, if this condition does not hold, C can be decomposed into an absolutely continuous component A_C and a singular component S_C.

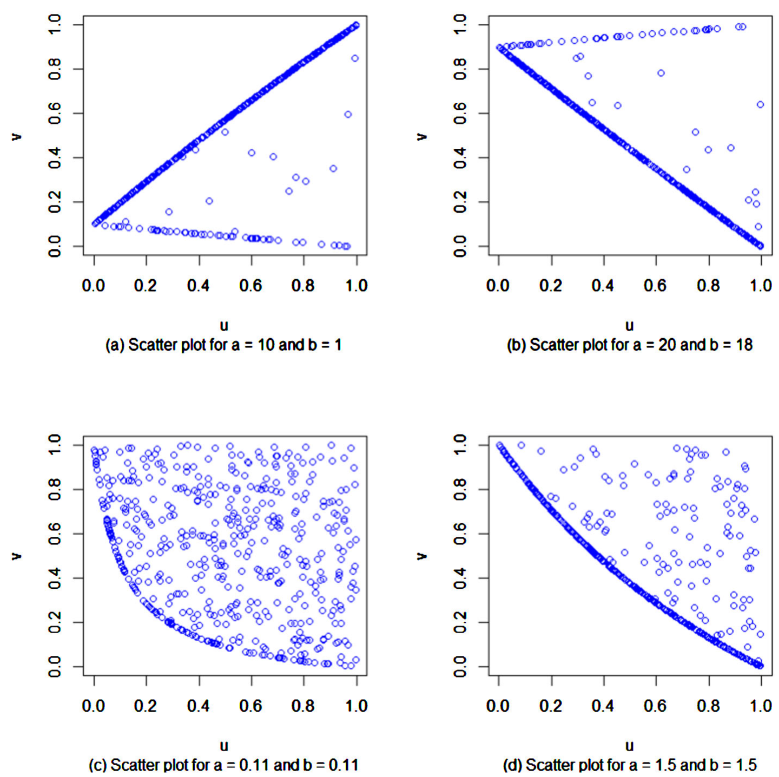


Figure 4. The scatter plots are derived from 500 simulated observations of $C_{a,b}$.

Proposition 8 *The radial copula contains an absolutely continuous* $A_{C_{a,b}}$ *and singular components* $S_{C_{a,b}}$. *Thus we have*

$$A_C(u,v)=\begin{cases}\dfrac{uv}{1+a}, & \text{if } \dfrac{b(1-u)}{u+a}\le v<\dfrac{(1+a-b)u+b}{u+a},\\ 0, & \text{otherwise,}\end{cases}$$

and consequently

$$S_C(u,v)=\begin{cases}0, & \text{if } v<\dfrac{b(1-u)}{u+a},\\ \dfrac{bu+av-b}{1+a}, & \text{if } \dfrac{b(1-u)}{u+a}\le v<\dfrac{(1+a-b)u+b}{u+a},\\ u, & \text{if } v\ge\dfrac{(1+a-b)u+b}{u+a}.\end{cases}$$

Proof. The proof is immediate, so it is omitted.

Thus, the $C_{a,b}$-measure of the singular component of $C_{a,b}$ is expressed as $S_{C_{a,b}}(1,1)=1$. In other words, if U and V are uniformly distributed random variables across the interval [0, 1], and their joint distribution is characterized by the copula $C_{a,b}$, then it follows that $P[U=V]=1$.

In the following proposition, we explore the statistical properties of the copula $C_{a,b}$. Specifically, we examine the behavior of classical concordance measures, including Kendall's τ and Spearman's ρ, under the copula $C_{a,b}$.

Let X and Y be continuous random variables associated with the copula C, where F and G represent their respective marginal distribution functions. Using the probability transformations $u=F(x)$ and $v=G(y)$, Kendall's τ for X and Y can be expressed as [23]:

$$\tau_{X,Y}=1-4\iint_{\mathbf{I}^2}\frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)\,\mathrm{d}u\mathrm{d}v, \tag{7}$$

and Spearman's ρ as [23]:

$$\rho_{X,Y}=12\iint_{\mathbf{I}^2}C(u,v)\,\mathrm{d}u\mathrm{d}v-3. \tag{8}$$

Proposition 9 *Let* X *and* Y *be continuous random variables of copula* $C_{a,b}$, *where* $0\le b\le a$. *Then*

1. Kendall's τ for the random variables X and Y is given by

$$\tau_{a,b}=(2a-4b)\left(1-a\ln\frac{1+a}{a}\right), \tag{9}$$

which takes values between −1 and 1.

2. Spearman's ρ for the random variables X and Y is given by

$$\rho_{a,b}=3\left(2a^2+a-2b-4ab\right)+6\left(2a^2b+2ab-a^2-a^3\right)\ln\frac{1+a}{a}, \tag{10}$$

which also takes values between −1 and 1.

Proof.

1. Given that the copula $C_{a,b}$ contains both an absolutely continuous and a

singular component, we apply formula 7 to compute Kendall's $\tau_{a,b}$. Then

$$\frac{\partial}{\partial u}C_{a,b}(u,v)\frac{\partial}{\partial v}C_{a,b}(u,v)=\begin{cases}\frac{(u+a)(v+b)}{(1+a)^2}, & \text{if } \frac{b(1-u)}{u+a}\le v<\frac{(1+a-b)u+b}{u+a},\\ 0, & \text{otherwise,}\end{cases}$$

and hence

$$\begin{aligned}\tau_{a,b}&=1-4\iint_{\mathbf{I}^2}\frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)\mathrm{d}u\mathrm{d}v\\ &=1-4\int_0^1\left(\int_{\frac{b(1-u)}{u+a}}^{\frac{(1+a-b)u+b}{u+a}}\frac{(u+a)(v+b)}{(1+a)^2}\mathrm{d}v\right)\mathrm{d}u\\ &=1-4\left(\frac{1}{2(1+a)}\int_0^1\frac{(1+a-2b)u^2+2bu}{u+a}\mathrm{d}u+\frac{b}{1+a}\int_0^1 u\mathrm{d}u\right)\\ &=1-4\left(\frac{1}{4}-\frac{1}{2}a+\frac{1}{2}a^2\ln\frac{1+a}{a}+b\left(1-a\ln\frac{1+a}{a}\right)\right)\\ &=(2a-4b)\left(1-a\ln\frac{1+a}{a}\right).\end{aligned}$$

In particular, if $a=+\infty$ and $b=0$, then we have

$$\lim_{a\to+\infty}\tau_{a,0}=2a\left(1-a\ln\frac{1+a}{a}\right),$$

which can be written as

$$\begin{aligned}\lim_{t\to0}\tau_{\frac{1}{t},0}&=\frac{2t-2\ln(1+t)}{t^2}\\ &=\frac{2t-2\left(t-\frac{1}{2}t^2+O(t^3)\right)}{t^2}\\ &=1.\end{aligned}$$

Similarly, if $a=b=+\infty$, then we obtain $\lim_{a\to+\infty}\tau_{a,a}=-2a\left(1-a\ln\frac{1+a}{a}\right)=-1$.

2. We use formula 8 and the copula $C_{a,b}$ to derive Spearman's $\rho_{a,b}$. Then we have

$$\begin{aligned}\rho_{a,b}&=12\iint_{\mathbf{I}^2}C(u,v)\mathrm{d}u\mathrm{d}v-3\\ &=12\left(\int_0^1\left(\int_{\frac{b(1-u)}{u+a}}^{\frac{(1+a-b)u+b}{u+a}}\frac{uv+bu+av-b}{1+a}\mathrm{d}v+\int_{\frac{(1+a-b)u+b}{u+a}}^1 u\mathrm{d}v\right)\mathrm{d}u\right)-3\\ &=12\left(\int_0^1\left(\frac{(1-a+2b)u^2}{2(u+a)}+\frac{(a-b)u}{u+a}\right)\mathrm{d}u\right)-3\\ &=12\left(\int_0^1\left(\frac{1-a+2b}{2}(u-a)+\frac{a^2}{2(u+a)}+a-b+\frac{a}{u+a}\right)\mathrm{d}u\right)-3\\ &=12\left(\frac{2a^2+a-2b-4ab+1}{4}+\frac{1}{2}\left(2a^2b+2ab-a^2-a^3\right)\ln\frac{1+a}{a}\right)-3\end{aligned}$$

$$=3\left(2a^2+a-2b-4ab\right)+6\left(2a^2b+2ab-a^2-a^3\right)\ln\frac{1+a}{a}.$$

In particular, if $a=+\infty$ and $b=0$, then we have

$$\lim_{a\to+\infty}\rho_{a,0}=3\left(2a^2+a\right)+6\left(-a^2-a^3\right)\ln\frac{1+a}{a},$$

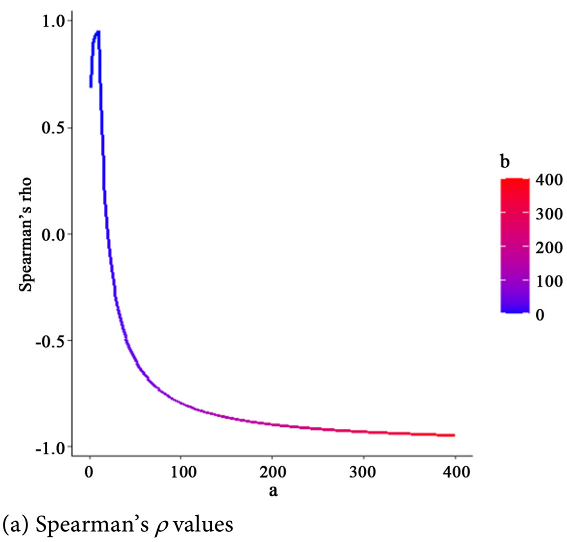
which can be written as

$$\begin{aligned}\lim_{t\to 0}\rho_{\frac{1}{t},0} &= 3\left(\frac{2}{t^2}+\frac{1}{t}\right)+6\left(-\frac{1}{t^2}-\frac{1}{t^3}\right)\ln(1+t)\\ &= 3\left(\frac{2}{t^2}+\frac{1}{t}\right)+6\left(-\frac{1}{t^2}-\frac{1}{t^3}\right)\left(t-\frac{1}{2}t^2+\frac{1}{3}t^3+O\left(t^4\right)\right)\\ &= \frac{t^2-2t^3}{t^2}\\ &= 1.\end{aligned}$$

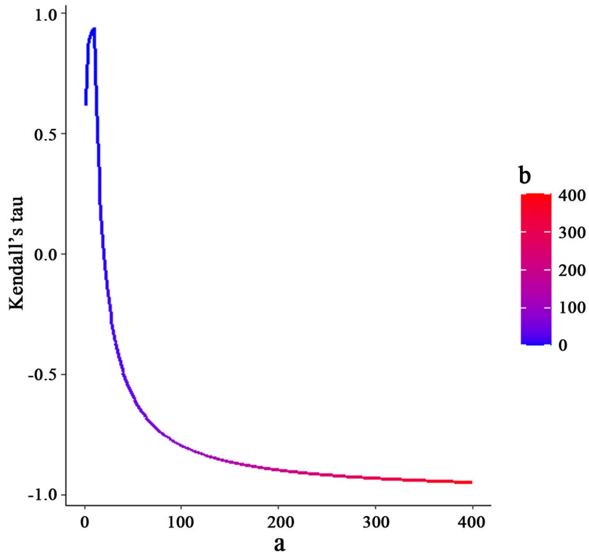
Similarly, if $a=b=+\infty$, then we get

$$\lim_{a\to+\infty}\rho_{a,a}=3\left(-2a^2-a\right)+6\left(a^2+a^3\right)\ln\frac{1+a}{a}=-1.$$

The full $[-1,1]$ range for Kendall's $\tau_{a,b}$ and Spearman's $\rho_{a,b}$ is theoretically proven in Proposition 9, which establishes that both $\tau_{a,b}$ and $\rho_{a,b}$ are constrained within this range for all radial copulas. In practice, the a parameter controls the overall strength of the dependence between the variables, with larger values indicating stronger dependence. The b parameter, on the other hand, primarily affects the tail dependence, with larger values leading to greater correlation at the extreme values. Together, a and b provide flexibility in modeling both general dependence and extreme comovements. In **Figure 5**, we plot Spearman's ρ and Kendall's τ as functions of the dependency parameters a and b, where b is constrained by the condition $0\le b\le a$. Specifically, it is set as $b=\min(a-10,a)$ and further adjusted to 0 if $b<0$.



(a) Spearman's ρ values



(b) Kendall's τ values

Figure 5. Spearman's ρ and Kendall's τ plotted against dependence parameters a and b.

4. Parameter Estimation of Radial Copulas: A Simulation Study

In this section, we estimate the two dependence parameters (a,b) of the copula $C_{a,b}$ using the (τ,ρ)-inversion method and bivariate *L*-moments. Furthermore, a simulation study is performed to measure the performance of these two estimation methods.

Tsukahara [24] introduced the rank approximate *Z*-estimation (RAZ) estimators, which are derived from Kendall's tau (τ) and Spearman's rho (ρ). These estimators are referred to as the τ-score and ρ-score RAZ estimators. Let us assume where $r=1$, with $\hat{\tau}_n$ and $\hat{\rho}_n$ denoting the sample estimators of Kendall's tau (τ) and Spearman's rho (ρ), respectively. The τ-inverse estimator $\hat{\theta}_\tau$ and the ρ-inverse estimator $\hat{\theta}_\rho$ of θ are defined using a method that is analogous to the method of moments:

$$\hat{\theta}_\tau = \tau^{-1}\left(\hat{\tau}_n\right) \text{ and } \hat{\theta}_\rho = \rho^{-1}\left(\hat{\rho}_n\right).$$

When $r=2$, the parameters $\boldsymbol{\theta}=\left(\theta_1,\theta_2\right)$ can be estimated by solving the following system of equations:

$$\tau\left(\theta_1,\theta_2\right)=\hat{\tau}_n,$$

$$\rho\left(\theta_1,\theta_2\right)=\hat{\rho}_n.$$

We refer to the solution obtained from the preceding system as the (τ,ρ)-inversion estimator of $\boldsymbol{\theta}$ [25]. This method is employed to estimate the parameters (a,b) of the radial copula $C_{a,b}$ as shown in the formula 5. The corresponding Kendall's tau (τ) and Spearman's rho (ρ) are computed from the formulas 9 and 10, respectively:

$$\tau(a,b) = (2a-4b)\left(1 - a\ln\frac{1+a}{a}\right),$$

$$\rho(a,b) = 3\left(2a^2 + a - 2b - 4ab\right) + 6\left(2a^2b + 2ab - a^2 - a^3\right)\ln\frac{1+a}{a}.$$

The solution of the following system is defined as the (τ,ρ) inversion estimator for the parameters (a,b):

$$\begin{aligned} \tau(a,b) &= \hat{\tau}_n, \\ \rho(a,b) &= \hat{\rho}_n. \end{aligned} \tag{11}$$

Another key method for estimating bivariate copula models with multiparameter is the application of bivariate *L*-moments. To begin, we offer a brief overview of univariate *L*-moments. Hosking [26] proposed *L*-moments, denoted by λ_k, as an alternative to the traditional central moments $\mu_k = \mathbb{E}\left[(Y-\mu)^k\right]$, which are derived from the distribution function (df) F_Y of the random variable Y. Using the quantile function F_Y^{-1}, the L moment λ_k can be represented as follows:

$$\lambda_k = \int_{\mathbb{I}} F_Y^{-1}(u) P_{k-1}(u)\,\mathrm{d}u,$$

where P_k denotes the shifted Legendre polynomial [27]. In the following, we will use the first three shifted Legendre polynomials:

$$P_0(u) = 1, \quad P_1(u) = 2u - 1, \quad P_2(u) = 6u^2 - 6u + 1.$$

The bivariate *L*-moments (BLM) extend univariate *L*-moments to the multivariate setting. Next we introduce the fundamental notations and definitions associated with BLM. Let $X^{(1)}$ and $X^{(2)}$ represent two random variables with finite means, the marginal distributions F_1 and F_2, as well as the respective sequences of L moments denoted by $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$. Serfling *et al.* [28] introduced the *k*-th *L*-comoment of $X^{(1)}$ in relation to $X^{(2)}$, which corresponds to the covariance between $X^{(1)}$ and $P_k\left(F_2\left(X^{(2)}\right)\right)$ for values of $k \ge 1$. This relationship can be expressed as follows:

$$\lambda_{k[12]} = \mathrm{Cov}\left(X^{(1)}, P_k\left(F_2\left(X^{(2)}\right)\right)\right),$$

with asymmetric counterpart $\lambda_{k[21]}$. If F is assumed to be part of a parametric family of distribution functions, the *k*-th *L*-comoment $\lambda_{k[12]}$ is influenced by both the parameters of the marginal distributionsd and the dependence structure between $X^{(1)}$ and $X^{(2)}$. Given that our primary goal is to estimate the parameters of the copula, it is more practical to apply the *k*-th *L*-comoment of $F_1\left(X^{(1)}\right)$ in relation to $X^{(2)}$, rather than using $\lambda_{k[12]}$ directly. Thus, we define:

$$\delta_{k[12]} = \mathrm{Cov}\left(F_1\left(X^{(1)}\right), P_k\left(F_2\left(X^{(2)}\right)\right)\right), \quad k = 1, 3, \cdots.$$

This quantity $\delta_{k[12]}$ is referred to as the *k*-th bivariate copula *L*-moment of $X^{(1)}$ in relation to $X^{(2)}$.

Brahimi *et al.* [25] introduced the *k*-th bivariate copula *L*-moment of $X^{(1)}$ in relation to $X^{(2)}$, which can be expressed, for each $k \ge 1$, as

$$\delta_{k[12]}=\int_0^1\int_0^1\left(C\left(u,v\right)-uv\right)\mathrm{d}u\mathrm{d}P_k\left(v\right). \tag{12}$$

Using this formula, it is possible to set up a system of equations to estimate the parameters of multivariate copula models. For example, if $r=2$ and $C=C_\theta$, $\boldsymbol{\theta}=\left(\theta_1,\theta_2\right)$, then based on formula 12, the first two bivariate copula L-moments are provided by [25]

$$\delta_{1[12]}=2\int_0^1\int_0^1 C_{\boldsymbol{\theta}}\left(u,v\right)\mathrm{d}u\mathrm{d}v-\frac{1}{2}, \tag{13}$$

$$\delta_{2[12]}=6\int_0^1\int_0^1\left(2v-1\right)C_{\boldsymbol{\theta}}\left(u,v\right)\mathrm{d}u\mathrm{d}v-\frac{1}{2}. \tag{14}$$

Below is the estimation process for the copula $C_{a,b}$ using the BLM method, structured as follows

1. Step 1: Compute the empirical L-moments. Using the observed sample $\left(X_i^{(1)},X_i^{(2)}\right)_{i=1,\cdots,n}$, the empirical L-moment $\hat{\delta}_{k[12]}$ is computed as follows:

$$\hat{\delta}_{k[12]}=\frac{1}{n}\sum_{i=1}^{n}F_{1:n}^{*}\left(X_i^{(1)}\right)P_k\left(F_{2:n}^{*}\left(X_i^{(2)}\right)\right),$$

where for each $j=1,2$, $F_{j:n}^{*}$ denotes the rescaled empirical distribution function, given as $F_{j:n}^{*}=nF_{j:n}/\left(n+1\right)$, where $F_{j:n}\left(x_j\right)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\left\{X_i^{(j)}\le x_j\right\}$ is the empirical distribution function.

2. Step 2: Construct the system of equations. Using formula 12, we can derive a system of r equations as follows:

$$\delta_{k[12]}\left(\theta_1,\cdots,\theta_r\right)=\hat{\delta}_{k[12]},\ \ k=1,\cdots,r.$$

For the radial copula $C_{a,b}$, according to formulas 13 and 14, the two first bivariate copula L-moments are given by:

$$\delta_{1[12]}\left(a,b\right)=a^2+\frac{a}{2}-2ab-b+\left(2a^2b+2ab-a^2-a^3\right)\ln\left(\frac{1+a}{a}\right),$$

$$\begin{aligned}\delta_{2[12]}\left(a,b\right)=&\frac{a}{2}+6a^2+6a^3-15ab-18a^2b+15b^2+18ab^2\\&-3\left(1+a\right)\left(a^2+2a^3-2ab-6a^2b+2b^2+6ab^2\right)\ln\left(\frac{1+a}{a}\right).\end{aligned}$$

3. Step 3: Solve the system

$$\begin{aligned}&a^2+\frac{a}{2}-2ab-b+\left(2a^2b+2ab-a^2-a^3\right)\ln\frac{1+a}{a}=\hat{\delta}_{1[12]},\\&\frac{a}{2}+6a^2+6a^3-15ab-18a^2b+15b^2+18ab^2\\&-3\left(1+a\right)\left(a^2+2a^3-2ab-6a^2b+2b^2+6ab^2\right)\ln\frac{1+a}{a}=\hat{\delta}_{2[12]}.\end{aligned} \tag{15}$$

The solution $\left(\hat{a},\hat{b}\right)$ obtained is referred to as the BLM estimator for the parameters of the copula $C_{a,b}$.

A simulation study was conducted to assess the performance of the (τ,ρ) -inversion method in comparison with bivariate *L*-moments for estimating the dependence parameters (a,b) of the radial copula. The evaluation of performance is conducted through the assessment of bias and RMSE, which are defined below:

$$\text{Bias} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right),\ \ \text{RMSE} = \left(\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2\right)^{1/2},$$

where $\hat{\theta}_i$ represents an estimate of θ derived from the i -th sample out of N generated samples obtained from the underlying copula [25]. For this study, we set $N = 1000$. To evaluate the reduction in bias and RMSE as the sample size increases to $n = 50, 100, 200, 500$, the estimation procedures for both methods were repeated by solving the systems in 11 and 15.

The chosen values for the parameters (a,b) must be reasonable, ensuring that each pair reflects a specific level of dependence: weak, moderate, or strong. **Table 1** shows the selected values of the real parameters, while **Table 2** gives an overview of the bias and RMSE for both the (τ,ρ) -inversion estimator and the BLM estimator applied to the radial copula. The results for bias and RMSE show that the BLM estimator consistently outperforms the (τ,ρ) -inversion estimator across all sample sizes. The outperformance of the BLM estimator over the (τ,ρ) -inversion method is consistent with the findings of Brahimi *et al.* [25]. This is because the (τ,ρ) -inversion method involves solving an optimization problem under constraints, whereas the BLM estimator directly solves a system of equations, which is generally more efficient and stable for parameter estimation. Furthermore, as the sample size increases from $n = 50$ to $n = 500$, both the bias and RMSE for the parameters (a,b) typically decrease, indicating enhanced performance of the estimators with larger samples.

Table 1. Selected true parameters for the radial copula.

	τ	a	b
Weak	0.01	0.208	0.1
Moderate	0.5	2.650	0.5
Strong	0.8	11.647	0.9

Table 2. Bias and RMSE of radial copula (τ, ρ)-inversion and BLM estimators.

	τ = 0.01				τ = 0.5				τ = 0.8			
	a = 0.208		b = 0.1		a = 2.650		b = 0.5		a = 11.647		b = 0.9	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
n=50												
τ − ρ	0.0282	0.309	0.0132	0.1566	−0.053	0.1037	−0.0176	0.1986	−0.1477	0.2963	−0.1277	0.2351
BLM	−0.0575	0.1059	−0.0274	0.0664	−0.0104	0.0261	−0.0283	0.1195	−0.0202	0.0313	−0.0915	0.1455
n=100												
τ − ρ	0.0227	0.3157	0.0113	0.1596	−0.037	0.0673	−0.0113	0.1406	−0.138	0.264	−0.107	0.2053
BLM	−0.0314	0.0841	−0.0159	0.0502	−0.009	0.0185	−0.0129	0.0782	−0.0129	0.0245	−0.0524	0.1007
n=200												

Continued

τ − ρ	0.0239	0.3162	0.012	0.1584	−0.0209	0.0465	−0.0072	0.1007	−0.1063	0.212	−0.091	0.1749
BLM	−0.0149	0.0593	−0.0073	0.0347	−0.009	0.0185	−0.0129	0.0782	−0.0076	0.0183	−0.0288	0.0698
n=500												
τ − ρ	0.0543	0.317	0.0276	0.1587	−0.0104	0.0302	−0.0014	0.0666	−0.0819	0.1478	−0.0604	0.1217
BLM	−0.0084	0.0372	−0.0036	0.0223	−0.0064	0.0136	−0.0065	0.0491	−0.002	0.0086	−0.0071	0.0315

5. Applications

In this section, we investigate the goodness-of-fit performance of radial copulas under different dependence structures. We assess the fit of radial copulas to two datasets previously analyzed in the literature using different bivariate distributions, including various copulas.

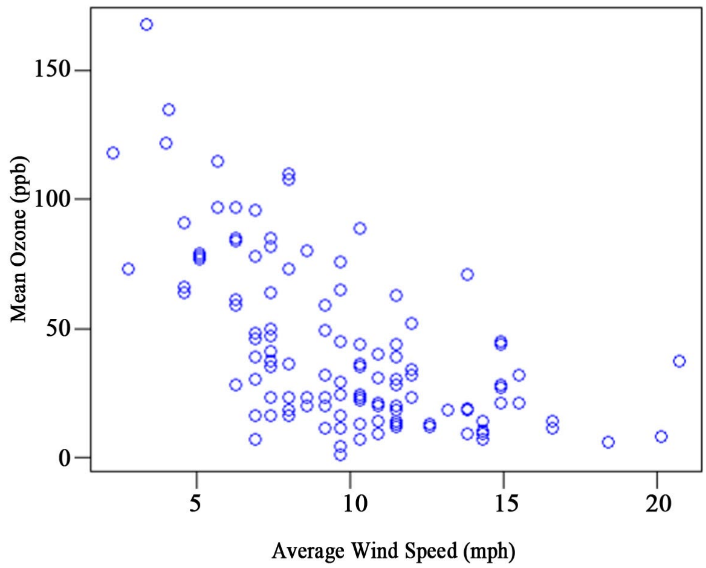


Figure 6. Scatterplot of mean wind speed versus mean ozone.

5.1. New York Air Quality Data

The dataset "air quality" contains daily air quality measurements collected over 153 consecutive days in the New York Metropolitan Area. Two variables to be examined in this study are average wind speed (measured in miles per hour) and average ozone levels (measured in parts per billion). For a comprehensive description of the data, please consult Chambers *et al.* [29] (Appendix, Dataset 2). This dataset is still accessible in the R package "datasets".

In our study, we analyze 116 observations after excluding missing data. Figure 6 illustrates a negative correlation between average wind speed and mean ozone levels. This is further confirmed by the negative empirical values of Spearman's rho (−0.590) and Kendall's tau (−0.428). To further explore this correlation, we use the bivariate *L*-moments estimator to model their dependence structure through the radial copula, obtaining the dependence parameters $(\hat{a},\hat{b})=(0.563,0.538)$. We then use formulas (9) and (10) to calculate the concordance measures for the radial copula: Spearman's rho is $\rho_{\hat{a},\hat{b}} = -0.506$ and Kendall's tau is $\tau_{\hat{a},\hat{b}} = -0.436$.

The goodness-of-fit of the radial copula $C_{a,b}$ is assessed using the Kolmogorov-Smirnov (KS) and Cramer-von Mises (CVM) statistics, applying the bootstrap algorithm presented by Genest *et al.* [30]. These two tests are related to the empirical process $\mathbb{C}_n = \sqrt{n}\left(C_n - C_{\theta_n}\right)$, and their corresponding statistics are given by

$$T_n = \sup_{(u,v)\in[0,1]^2} \left|\mathbb{C}_n(u,v)\right| \text{ and } S_n = \int_0^1\int_0^1 \mathbb{C}_n(u,v)^2 \, \mathrm{d}C_n(u,v),$$

respectively. Furthermore, various well-known copula families commonly used to model negative dependence are applied to the New York air quality data; however, none yield significant results. Therefore, our results are primarily comparable with those previously described by Ghosh *et al.* [31] and El Ktaibi *et al.* [32], which are based on different negatively dependent copulas. Specifically, the copulas in these studies have the following definitions:

$$C_\alpha(u,v) = \begin{cases} v-(1-u)+\dfrac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}(1-u)^{1+\alpha}v^{-\alpha}, 0<v\le\dfrac{\alpha}{1+\alpha}, 1-\dfrac{(1+\alpha)v}{\alpha}<u<1 \\ u-(1-v)\left[1-(1-u)^{1+\alpha}\right], 0<u<1, \dfrac{\alpha}{1+\alpha}<v<1, \end{cases}$$

and

$$C_\theta(u,v) = u^{1-\theta}v^{1-\theta}W\left(u^\theta, v^\theta\right),$$

where $W(u,v) = \max(u+v-1,0)$ and $(\alpha,\theta)\in(0,1)^2$.

Table 3 shows that our model exhibits an excellent fit to the data, presenting the highest p-values among the models considered. It also outperforms the models presented by Ghosh *et al.* [31] and El Ktaibi *et al.* [32]

Table 3. Goodness-of-fit test for air quality data.

	KS		CVM	
	Test statistic	p-value	Test statistic	p-value
$C_{a,b}$	$T_n = 0.077$	0.585	$S_n = 0.099$	0.269
C_θ	$T_n = 0.078$	0.584	$S_n = 0.120$	0.219
C_α	$T_n = 0.099$	0.232	$S_n = 0.224$	0.039

5.2. Twins Data

This dataset, as summarized by Ashenfelter *et al.* [33], contains hourly wages for 149 pairs of identical twins in the United States, each with varying levels of education. The study focuses on individuals aged 18 and older. The two variables under examination are the hourly wages (measured in log U.S. dollars) of each twin sibling. This dataset is also available in the supplementary material provided by Tang *et al.* [34]. A plot of these data is shown in **Figure 7**, where the observations tend to cluster around the major diagonal of the unit square. The calculated values of Spearman's rho and Kendall's tau for the hourly wages of each twin pair are

0.558 and 0.421, respectively, indicating a positive correlation between these variables.

Next, we fit the data using the bivariate *L*-moments estimator to model their dependence structure through the radial copula, obtaining the dependence parameters $(\hat{a},\hat{b})=(0.765,0.003)$. We then apply formulas (9) and (10) to compute the concordance measures for the radial copula. Specifically, Spearman's rho is $\rho_{\hat{a},\hat{b}}=0.571$, and Kendall's tau is $\tau_{\hat{a},\hat{b}}=0.504$. To assess the goodness-of-fit of the radial copula $C_{a,b}$, we use the KS and CVM statistics with the bootstrap algorithm from Genest *et al.* [30]. When fitting the twins dataset, various classical copula families commonly used to model positive dependence are considered, including Plackett, Clayton, and Gumbel copulas [35] [36]. As shown in **Table 4**, our model demonstrates superior goodness-of-fit to the twins data, as indicated by its relatively high p-value compared to other models. The improved performance of our model can be attributed to the increased flexibility of radial copulas in capturing complex nonlinear dependencies. In contrast, Archimedean copulas, while useful in certain contexts, may face limitations when attempting to capture such complex relationships due to their more structured and simpler forms.

Table 4. Goodness-of-fit test for twins data.

	KS		CVM	
	Test statistic	p-value	Test statistic	p-value
$C_{a,b}$	$T_n = 0.052$	0.854	$S_n = 0.102$	0.638
Gumble	$T_n = 0.056$	0.864	$S_n = 0.119$	0.527
Plackett	$T_n = 0.062$	0.777	$S_n = 0.104$	0.630
Clayton	$T_n = 0.078$	0.492	$S_n = 0.112$	0.620

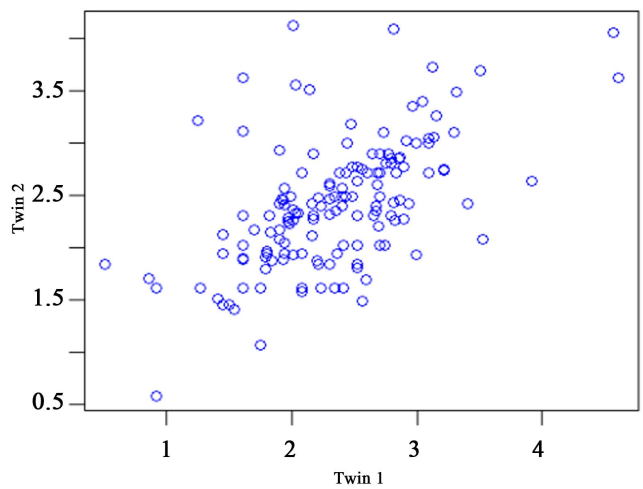


Figure 7. Scatter plot of hourly wages of identical twins.

6. Conclusions

In this paper, we have examined a class of copulas called radial copulas, which is derived from residual implications where the extensions of level curves intersect at a point. The class of radial copulas is a comprehensive and asymmetric extension of the Archimedean copula family (4.2.7) presented in [23]. Notably, radial copulas can capture both positive and negative dependencies. A simulation analysis is performed to evaluate the effectiveness of the (τ, ρ) -inversion and bivariate *L*-moments estimations methods in different modes. Furthermore, the efficacy of radial copulas is demonstrated through simulations and real case studies on two datasets (New York air quality dataset and twins dataset). We suggest that the radial copula based approach is straightforward in simulation and explanation and will be an important addition to copula theory. While applying radial copulas to high-dimensional datasets presensts challenges, such as increased computational complexity and difficulties in parameter estimation, these issues are acknowledged and will be addressed in future research.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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