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A Nonlinear Micropolar Continuum Theory for Thermoviscoelastic Solid Medium Based on Classical Rotations

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Abstract

This paper presents a nonlinear micropolar nonclassical mathematical continuum theory for finite deformation/finite strain deformation physics of compressible thermoviscoelastic solids based on classical rotations ${}_c\boldsymbol{\Theta}$ and its rates. Stress and moment measures for finite deformation/finite strain physics are utilized in conjunction with the finite deformation/finite strain measures presented in ref. [1] to derive conservation and the balance law as well as the constitutive theories using conjugate pairs in the entropy inequality and the representation theorem. The nonlinear micropolar nonclassical continuum theory presented in this paper for thermoviscoelastic solid: (1) incorporates nonlinear ordered rate dissipation mechanism for the viscous medium based on rates of Green's strain tensor up to order n. This is usual viscous dissipation (macrodissipation) in the solid medium due to the viscosity of the medium. (2) Also incorporates additional ordered rate dissipation mechanism due to microconstituents and the viscosity of medium, which depends upon rates of the symmetric part of the rotation gradient tensor up to order $\underset{\sim}{n}$. We refer to this dissipation mechanism as microdissipation or microviscous dissipation. This dissipation mechanism is consistent with the deformation measure derived in ref. [1] for nonlinear micropolar nonclassical continuum theory. (3) With the assumption of small deformation, small strain, the nonlinear micropolar nonclassical continuum theory presented here reduces to a consistent linear micropolar nonclassical continuum theory with both mechanisms of dissipation. (4) In the absence of micropolar physics, the theory reduces to finite deformation/finite strain classical continuum theory for compressible thermoviscoelastic solid medium. The complete mathematical model consisting of the conservation and balance laws and the constitutive theories has closure without the conservation of micro inertia law needed in the micropolar theories of Eringen for closure. It has been shown that the balance of moment of moments is an essential balance law in all micropolar theories for achieving thermodynamic and mathematical consistency of the resulting linear micropolar theory. The balance of moment of moments balance law is necessary and has been successfully used in ref. [2] to derive a nonlinear micropolar theory for thermoelastic solid and is essential in this nonlinear micropolar nonclassical continuum theory for thermoviscoelastic solid based on classical rotations ${}_c\boldsymbol{\Theta}$ presented in this paper. The nonlinear micropolar nonclassical continuum theory based on rotations ${}_c\boldsymbol{\Theta}, {}_\alpha\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ (neglecting ${}_c\boldsymbol{\Theta}$) is not considered in the present work due to the fact that the linear micropolar nonclassical continuum theory based on these rotations is thermodynamically and mathematically inconsistent [1].

Keywords

Nonclassical, Micropolar, Dissipation, Ordered Rate, Conservation and Balance Laws, Representation Theorem, Microviscous Dissipation, Microdissipation, Ordered Rate, Finite Deformation Theories, Finite Strain

1. Introduction, Literature Review

In the comprehensive works published by Eringen and Eringen *et al.* [3]-[21] and references [22]-[24] on 3M nonclassical continuum theories, it is relatively easy to conclude that whatever needs to be done regarding the theoretical foundations of 3M linear and nonlinear nonclassical continuum theories has already been done and that what remains are probably the applications. However, upon closer examination, we find that this indeed is not the case. In the works published by Eringen and Eringen *et al.*, as well as by many other researchers who follow the approaches introduced by Eringen and Eringen *et al.*, there are some serious shortcomings, omissions, and the use of mathematically unjustifiable approaches that have led to linear and nonlinear micropolar nonclassical continuum theories that are thermodynamically and/or mathematically inconsistent; hence, they are not valid nonclassical micropolar continuum theories. We will list some of the concerns and issues below and discuss their consequences on the resulting nonclassical continuum theories.

(a) Yang *et al.* [25] and Surana *et al.* [26] [27] have shown that balance of moment of moments is an essential balance law in all micropolar nonclassical continuum theories. When this balance law is used, the Cauchy moment tensor becomes symmetric. The constitutive theory for the nonsymmetric moment tensor has been addressed using two approaches: (1) in the first approach, the nonsymmetric tensor is considered as a constitutive tensor with nonsymmetric tensors as its argument tensors (amongst others). This approach is used almost exclusively in all published works of Eringen and Eringen *et al.* and by those that follow Eringen. It is now well established due to works of Zhang, Wang, Spencer and Smith

[28]-[39], that for a nonsymmetric constitutive tensor, the basis of the space of the constitutive tensor cannot be established. Thus, nonsymmetric constitutive tensors lead to constitutive theories (derived in published works by using potentials or polynomial approach) that are in violation of basic principles of mathematics, hence cannot be valid constitutive theories. (2) In the second approach, the nonsymmetric Cauchy moment tensor is additively decomposed into symmetric and skew symmetric tensors (Surana *et al.* [26] [27]) followed by derivation of constitutive theories for each using representation theorem. This approach is mathematically consistent based on the works of Zhang, Wang, Spencer and Smith [28]-[39]. Surana *et al.* [26] [27] have shown using simple 2D micropolar physics that this approach leads to nonphysical constitutive theories. Thus, in the absence of balance of moment of moments balance law, there are no valid means of deriving constitutive theory for the Cauchy moment tensor. This is a major problem in all 3M nonclassical continuum theories of Eringen and Eringen *et al.* This is significant enough to question the validity of the published works on 3M.

(b) The strain measures presented by Eringen [8] are in fact deformation measures. These are derived using expressions that are not dimensionless and hence fail to yield the simple linear strain measure (change in length per unit length) for 1D case. These measures in their original form or in the modified form may be viewed as strain measures if they appear in the rate of work conjugate pairs in the entropy inequality.

(c) In any deforming solid continua, the deformation consists of elongation of the material lines and change in angle between them and rigid rotation of the material lines. Additive decomposition of the displacement gradient tensor ${}^{d}\boldsymbol{J}$ (linear elasticity) into symmetric ${}^{d}_{s}\boldsymbol{J} = \boldsymbol{\varepsilon}$ and skew symmetric tensors ${}^{d}_{a}\boldsymbol{J}$ allows us to separate strains ($\boldsymbol{\varepsilon}$) and the rotations ${}^{d}_{a}\boldsymbol{J}$. Energy equation and the entropy inequality establish rate of work conjugate pairs that enable determination of constitutive tensors and their argument tensors. We remark that choice of ${}^{d}\boldsymbol{J}$ or ${}^{d}_{a}\boldsymbol{J}$ as argument tensors of the stress tensor is invalid. Important point to note is that rigid rotations (as in ${}^{d}_{a}\boldsymbol{J}$) or strain plus rigid rotations (as in ${}^{d}\boldsymbol{J}$) cannot be argument tensor of the stress constitutive tensor. The deformation measures [1] clearly show the strain measure for rigid microconstituents in micropolar nonclassical continuum theory to be zero. Thus, equations (20.1) and (20.6) in ref. [8], which consider strain measure as ${}^{d}\boldsymbol{J}$ plus rotations of microconstituents and then use this measure as an argument of stress tensor, have no meaning. A constitutive theory for stress tensor based on this strain measure is obviously in violation of thermodynamics (second law) and is bound to be erroneous.

(d) In micropolar nonclassical continuum theory, balance of angular momenta defines antisymmetric Cauchy stress tensor in terms of the gradients of the Cauchy moment tensor. If we have constitutive theory for Cauchy moment tensor (which we do), then the antisymmetric Cauchy stress tensor is defined, hence cannot be part of the constitutive theory in the stress tensor. This requires additive decomposition of $\boldsymbol{\sigma}^{(0)}$ *i.e.*, $\boldsymbol{\sigma}^{(0)} = {}_{s}\boldsymbol{\sigma}^{(0)} + {}_{a}\boldsymbol{\sigma}^{(0)}$ in which only ${}_{s}\boldsymbol{\sigma}^{(0)}$, the symmetric part of contravariant Cauchy stress tensor $\boldsymbol{\sigma}^{(0)}$, can be a constitutive tensor. A single constitutive theory for ${}_s\boldsymbol{\sigma}^{(0)}$ must address volumetric as well as distortional deformation that are mutually exclusive; this is obviously not possible. Further additive decomposition of ${}_s\boldsymbol{\sigma}^{(0)} = {}_s^e\boldsymbol{\sigma}^{(0)} + {}_s^d\boldsymbol{\sigma}^{(0)}$ is needed to describe volumetric (constitutive theory for equilibrium tensor, ${}_s^e\boldsymbol{\sigma}^{(0)}$) and distortional (constitutive theory for deviatoric ${}_s^d\boldsymbol{\sigma}^{(0)}$ tensor) deformation physics that are mutually exclusive. None of these aspects are discussed in the literature on micropolar nonclassical continuum theories. In our view, the published work in this area lacks clarity, consistent use of concepts related to physics, and of course, the derivations of the constitutive theories are in total violation of the representation theorem.

(e) In case of linear micropolar nonclassical continuum theory, the deformation measures of references [1] clearly show that the theory must only consider rigid rotations of the microconstituents. If we assume that rigid rotations ${}_\alpha\boldsymbol{\Theta}$ of the microconstituents are unknown degrees of freedom at a material point (Eringen, Eringen *et al.* and others), then a material point has ${}_c\boldsymbol{\Theta}$ classical rotations (known in terms of displacement gradients) and unknown rotations ${}_\alpha\boldsymbol{\Theta}$. In the published works, linear micropolar nonclassical continuum theory based on ${}_c\boldsymbol{\Theta} + {}_\alpha\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ (neglecting ${}_c\boldsymbol{\Theta}$) have been reported by many researchers. Surana *et al.* [40] [41] have shown that linear micropolar nonclassical continuum theory based on these two rotation considerations lead to spurious conjugate pairs in the entropy inequality that necessitate constitutive theory for ${}_a\boldsymbol{\sigma}^{(0)}$, which of course is not physical as ${}_a\boldsymbol{\sigma}^{(0)}$ are defined by balance of angular momentum equations. In our view [1], all published works on linear micropolar nonclassical continuum theory based on ${}_c\boldsymbol{\Theta} + {}_\alpha\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ are in violation of physics, hence result in invalid micropolar nonclassical continuum theory.

(f) Principle of equipresence used almost exclusively in published micropolar nonclassical continuum theory is not supported by second law of thermodynamics. It creates nonphysical coupling between classical and nonclassical physics in the micropolar nonclassical continuum theory.

In view of (a)-(f), the published works on 3M nonclassical continuum theory, whether linear or nonlinear, there are many concerns, hence, the theories are questionable. In published work on micropolar nonclassical continuum theory, the thermodynamic and mathematical details use many questionable and unfounded approaches that cannot be supported by thermodynamic principles or mathematics. Thus, the natural question arises: Do we have any valid linear and nonlinear micropolar nonclassical continuum theory at present that is thermodynamically and mathematically consistent? We discuss answer to this question below. Surana *et al.* [1] [2] [26] [27] [40]-[43] have shown the following in these works:

(1) Classical rotations (${}_c\boldsymbol{\Theta}$) constitute a free field in classical continuum mechanics. It is always present undisturbed, hence has no effect on the development of classical continuum theory. This implies that each material point in the de-

formed solid medium has rotations ${}_c\boldsymbol{\Theta}$ at the material points about the axes of triad, axes being parallel to x-frame. The presence of microconstituents offers resistance to the free field; as a consequence, the free field is no longer a free field, but instead defines the rigid rotations of the microconstituents *i.e.*, ${}_c\boldsymbol{\Theta}$ are the rotations of the rigid microconstituents (see [1] for explanation). Thus, we see that classical rotations ${}_c\boldsymbol{\Theta}$ are sufficient to account for the rigid rotations of the microconstituents. This eliminates the need for ${}_\alpha\boldsymbol{\Theta}$ as additional unknown dofs at a material point, hence the need for spurious and nonphysical constitutive theory for ${}_a\boldsymbol{\sigma}^{(0)}$ necessitated due to presence of ${}_\alpha\boldsymbol{\Theta}$ as unknown degrees of freedom at a material point.

(2) Balance of moment of moments must always be used as a balance law in all nonclassical continuum theory (Yang *et al.*, [25], Surana *et al.* [26] [27]). When this balance law is used, Cauchy moment tensor is symmetric and the problems associated with the constitutive theory for nonsymmetric moment tensor are eliminated.

(3) Based on (1) and (2), Surana *et al.* [1] [40] [41] have presented thermodynamically and mathematically consistent linear micropolar nonclassical continuum theory for solid and fluent media with successful model problem studies. To our knowledge, works of Surana *et al.* are the only works that contain valid linear micropolar nonclassical continuum theory that are supported by thermodynamics and mathematics. Model problem studies confirm that the micropolar nonclassical continuum theory based on ${}_c\boldsymbol{\Theta}$ contain correct micropolar physics. Thermodynamic and mathematical consistency of linear micropolar nonclassical continuum theory based on ${}_c\boldsymbol{\Theta}$ is assurance of its validity.

(4) Surana *et al.* [40] [41] addressed thermodynamic and mathematical consistency of linear micropolar theories based on rotations ${}_c\boldsymbol{\Theta}$, ${}_c\boldsymbol{\Theta}+{}_\alpha\boldsymbol{\Theta}$ and ${}_\alpha\boldsymbol{\Theta}$ with the conclusion that the only linear micropolar theory based on rotations ${}_c\boldsymbol{\Theta}$ is a valid linear micropolar theory.

(5) In references [44]-[46] authors derived conservation and balance laws for micropolar medium for finite deformation/finite strain physics using first Piola-Kirchhoff stress $\boldsymbol{\sigma}^*$, first Piola-Kirchhoff moment $\boldsymbol{m}^*$, rate of deformation gradient tensor $\dot{\boldsymbol{J}}$ and the rate of classical rotation gradient tensor, ${}^{c\Theta}\dot{\boldsymbol{J}}$, use of these measures is convenient in the derivation, but the final form of the conservation and balance laws can always be expressed in terms of contravariant second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ and contravariant second Piola-Kirchhoff moment tensor $\boldsymbol{m}^{[0]}$ and their conjugate rates as shown in this paper. These are valid measures for finite deformation/finite strain physics. Additive decomposition of $\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]} + {}_a\boldsymbol{\sigma}^{[0]}$ and ${}_s\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]} + {}_s^d\boldsymbol{\sigma}^{[0]}$ are used. The constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ in terms of thermodynamic pressure is derived using entropy inequality in Eulerian description. The final reduced form of the entropy inequality contains constitutive tensors $\boldsymbol{q}, {}_s^d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ with conjugate tensors $\boldsymbol{g}, \dot{\boldsymbol{\varepsilon}}_{[0]}$ and ${}^{c\Theta}\dot{\boldsymbol{J}}$, where $\boldsymbol{q}$ and $\boldsymbol{g}$ being tensors of rank one, and all others are symmetric tensors of rank two. Constitutive theories for ${}_s^d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{m}^{[0]}$ and $\boldsymbol{q}$ are

derived using representation theorem. This nonlinear micropolar nonclassical continuum theory is thermodynamically and mathematically consistent, and the mathematical model consisting of conservation and balance laws and the constitutive theories has closure. To our knowledge, this is the only thermodynamically and mathematically consistent nonlinear micropolar nonclassical continuum theory for thermoviscoelastic solid matter available in the published literature.

2. Scope of Work

In this paper, we present finite deformation/finite strain nonlinear micropolar nonclassical continuum theory for compressible thermoviscoelastic solids. This nonlinear micropolar nonclassical continuum theory incorporates the finite deformation/finite strain deformation physics for thermoelastic solid, but also incorporates mechanisms of dissipation that are absent in the nonlinear micropolar nonclassical continuum theory for thermoelastic solids. We begin with the conservation and balance laws in references [1] [26] [41] for finite deformation/finite strain micropolar physics expressed in terms of $\boldsymbol{\sigma}^*$, $\boldsymbol{m}^*$ and their rate of work conjugates $\dot{\boldsymbol{J}}$ and ${}^{c\Theta}\dot{\boldsymbol{J}}$. $\boldsymbol{\sigma}^*, \boldsymbol{m}^*$ are convenient to use in the derivation of the conservation and balance laws but these are not valid measures for finite deformation/finite strain physics. Expressing $\boldsymbol{\sigma}^*$ and $\boldsymbol{m}^*$ in terms of $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$, contravariant second Piola-Kirchhoff stress and moment tensors and establishing their rate of work conjugate is not straight forward due to micropolar physics. This is an important aspect of the derivation presented in this paper. Of course, without the valid rate of work conjugate, the constitutive theories cannot be derived. Thus, there are two important aspects of the work presented in this paper: (1) conservation and balance laws expressed in terms of $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ with thermodynamically valid rate of work conjugates (2) the second aspect of the work is derivation of ordered rate constitutive theories for $\boldsymbol{\sigma}^{[0]}, \boldsymbol{m}^{[0]}$ and $\boldsymbol{q}$ based on the conjugate pair in the entropy inequality and theory of isotropic tensors, thus ensuring thermodynamic and mathematical consistency of the resulting mathematical model. Theories incorporate micro as well as macro ordered rate nonlinear deformation mechanism. The additive decomposition of $\boldsymbol{\sigma}^{[0]}$; $\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]} + {}_a\boldsymbol{\sigma}^{[0]}$; ${}_s\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]} + {}_s^d\boldsymbol{\sigma}^{[0]}$ is necessary to consider various aspects of deformation physics. The constitutive theory for equilibrium stress tensor ${}_s^e\boldsymbol{\sigma}^{[0]}$ describing volumetric deformation physics remains the same as in ref. [42] [43]; hence, only the final forms of the equations related to the constitutive theory for equilibrium stress tensor are presented. Constitutive theories for ${}_s^d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ address distortional deformation physics as well as dissipation mechanisms. The rate of work conjugate pair $\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]}$ in the reduced form of the entropy inequality suggests that elasticity must be due to $\boldsymbol{\varepsilon}_{[0]}$ and dissipation mechanism is due to $\dot{\boldsymbol{\varepsilon}}_{[0]} = \boldsymbol{\varepsilon}_{[1]}$ *i.e.*, Green's strain rate of order one. In the present work, we generalize this nonlinear dissipation mechanism to be a function of Green's strain rates of up to order n *i.e.*, dependent on $\boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \cdots, n$. Thus, we have ordered rate theory for the dissipation of the medium, which is entirely due to viscous drag forces between the particles of the medium. The second nonlinear mechanism of dissipation is due to microconstituents and the viscosity of the medium, micro viscous dissipation or micro dissipation. The rigid rotations of the microconstituents must overcome the viscous drag forces due to surrounding viscous medium. This mechanism is naturally a function of the rotation rates of the microconstituents. Based on this line of reasoning and using classical rotations ${}_c\boldsymbol{\Theta}$, we have a free field ${}_c\boldsymbol{\Theta}$ and associated rotation rate ${}_c\dot{\boldsymbol{\Theta}}$ field in the absence of microconstituents. Due to the presence of microconstituents, the rotations ${}_c\boldsymbol{\Theta}$ and their rates ${}_c\dot{\boldsymbol{\Theta}}$ are in fact rotations and rotation rates of the microconstituents. The mechanical work expended due to the rotations of the microconstituents in overcoming viscous drag results in additional entropy generation that influences thermal field. This mechanism of dissipation, micro dissipation or micro viscous dissipation, is also incorporated in the present work as an ordered rate theory of up to order n. The constitutive theories for $\boldsymbol{q}, {}_s^d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ are derived using integrity (complete basis) in conjunction with representation theorem. In each case, material coefficients are derived. The material coefficients can be a function of the combined invariants of the argument tensors of each constitutive tensor and temperature θ. Simplified constitutive theories that are linear in the components of the argument tensor are also presented. Linear micropolar nonclassical continuum theory is shown to be a complete subset of nonlinear micropolar nonclassical continuum theory presented in this paper. Also, the nonlinear finite deformation/finite strain classical continuum theory remains intact when micropolar physics is discarded. It is shown that the nonlinear micropolar nonclassical continuum theory for thermoviscoelastic solid presented in this paper is thermodynamically and mathematically consistent, and that the mathematical model consisting of conservation and balance laws and constitutive theories has closure.

3. Consideration of Various Measures

In the following, we present a short summary of various measures considered in deriving the conservation and balance law and the constitutive theories for nonlinear micropolar nonclassical continuum theory for compressible thermoviscoelastic solid matter. In finite deformation physics, use of contravariant Cauchy stress tensor and derivation of corresponding first and second Piola-Kirchhoff stress tensors using correspondence rules is commonly used in classical continuum mechanics [42] [43]. In the case of nonlinear micropolar nonclassical continuum theory, these measures also remain valid. Additionally, we have contravariant Cauchy moment tensors and derivations of corresponding first and second Piola-Kircchoff moment tensors. The Cauchy principle holds for co- and contra-variant stress and moment tensors [42]-[46], and interrelationships between Cauchy, first and second Piola-Kirchhoff stress tensors as well as moment tensors provide flexibility in terms of a suitable choice of measure that maintains simplicity in the details of the derivations. Following references [42] [43] [45] [46], we have the following (for compressible matter):

Cauchy principle:

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \,;\, \bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{1}$$

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \,;\, \bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}_{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}} \tag{2}$$

Correspondence rules for first, second Piola-Kirchhoff stress and moment tensors:

$$\begin{gathered}\left\{d\bar{F}\right\} = \left\{dF\right\}; \left\{d\bar{M}\right\} = \left\{dM\right\} \\ \left[\sigma^{*}\right]^{\mathrm{T}} = |J|\left[\sigma^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}; \left[m^{*}\right]^{\mathrm{T}} = |J|\left[m^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}\end{gathered} \tag{3}$$

$$\begin{gathered}\left\{d\bar{F}\right\} = [J]\left\{dF\right\}; \left\{d\bar{M}\right\} = [J]\left\{dM\right\} \\ \left[\sigma^{[0]}\right]^{\mathrm{T}} = |J|[J]^{-1}\left[\sigma^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}; \left[m^{[0]}\right]^{\mathrm{T}} = |J|[J]^{-1}\left[m^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}\end{gathered} \tag{4}$$

We note that since $\boldsymbol{\sigma}^{(0)}$ is not symmetric. $\boldsymbol{\sigma}^{*}$ and $\boldsymbol{\sigma}^{[0]}$ are nonsymmetric as well. When balance of moment of moments is used as a balance law, $\boldsymbol{m}^{(0)}$ is symmetric, hence $\boldsymbol{m}^{[0]}$ is symmetric but $\boldsymbol{m}^{*}$ remains not symmetric.

4. Classical Rotations, Their Gradients and Other Considerations

In reference [1], authors have discussed that in the absence of microconstituents, the classical rotation field is a free field, hence the classical continuum theories are not affected by its presence. Based on classical continuum mechanics in every deforming solid matter, the classical rotation field exists as a free field. In the presence of microconstituents, the classical rotation field is no longer a free field, instead it describes the rigid rotations of the microconstituents. A simple example illustrates this quite well. Consider 1D axial deformation of an unconstrained rod subjected to a force at the right end. The rigid body translations of the rod is a free field that has no affect on the deformation of the rod as all points of the rod are moving in the same direction by the same amount. If we constrain the left end of the rod from moving, then the deformation field is no longer a free field and is in fact the actual deformation field of the constrained rod with load on the right end. Thus, we see that the obstruction (constrained left end in this case) changes the free field to the actual deformation field of the constrained rod. Our situation of ${}_{c}\boldsymbol{\Theta}$ as a free field and the microconstituents obstructing this free field is exactly similar to the axial rod. That is the free field ${}_{c}\boldsymbol{\Theta}$ in the presence of microconstituents becomes a rotation field ${}_{c}\boldsymbol{\Theta}$ describing the rotations of the microconstituents, meaning ${}_{c}\boldsymbol{\Theta}$ are in fact the rotations of the microconstituents.

Thus, in micropolar theory requiring rigid rotations of the microconstituents, classical rotations serve as their rigid rotations. Furthermore, it has been shown in references [40] [41] that if we consider rigid rotations of the microconstituents as additional unknown degrees of freedom at the material points, a valid linear micropolar theory that is thermodynamically and mathematically consistent is not possible. In case of linear micropolar theories [40] [41], authors have shown that a micropolar theory based on classical rotations as rigid rotations of the microconstituents is always thermodynamically and mathematically consistent provided balance of moment of moments is used as a balance law and the constitutive theories are derived using representation theorem. The work presented in this paper for nonlinear micropolar continuum theory for thermoviscoelastic solids is strictly based on classical rotations as rigid rotations of the microconstituents, thus in this micropolar theory a material point only has three translational degrees of freedom. Some details of classical rotations ${}_c\boldsymbol{\Theta}$, their gradients, stress and moment tensors are given below:

$$
\begin{aligned}
{}_c\boldsymbol{\Theta} &= \nabla \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_i}{\partial x_j} = \epsilon_{ijk} \frac{\partial u_i}{\partial x_j} \\
&= \boldsymbol{e}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \boldsymbol{e}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \boldsymbol{e}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\
&= \boldsymbol{e}_1\left({}_c\Theta_1\right) + \boldsymbol{e}_2\left({}_c\Theta_2\right) + \boldsymbol{e}_3\left({}_c\Theta_3\right)
\end{aligned}
\tag{5}
$$

$$
[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = [{}_sJ] + [{}_aJ] \tag{6}
$$

$$
[{}_sJ] = \frac{1}{2}\left([J] + [J]^{\mathrm{T}}\right); [{}_aJ] = \frac{1}{2}\left([J] - [J]^{\mathrm{T}}\right) \tag{7}
$$

$$
\left[{}^{c\Theta}J\right] = \frac{\partial\{{}_c\Theta\}}{\partial\{x\}} = \left[{}^{c\Theta}_{s}J\right] + \left[{}^{c\Theta}_{a}J\right] \tag{8}
$$

$$
\left[{}^{c\Theta}_{s}J\right] = \frac{1}{2}\left(\left[{}^{c\Theta}J\right] + \left[{}^{c\Theta}J\right]^{\mathrm{T}}\right); \left[{}^{c\Theta}_{a}J\right] = \frac{1}{2}\left(\left[{}^{c\Theta}J\right] - \left[{}^{c\Theta}J\right]^{\mathrm{T}}\right) \tag{9}
$$

(1)-(9) are basic measures, definitions and relations that are used in deriving the conservation and balance laws and the constitutive theories.

We note that $\frac{{}_c\boldsymbol{\Theta}}{2}$ appears in $[{}_aJ]$. In the conservation and balance laws, we must use $\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{m}^{[0]}$ and $\boldsymbol{\varepsilon}_{[0]}$ as these are valid measures for finite deformation/finite strain physics. However, since this is a relationship between $\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{m}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\sigma}^{*}$, $\boldsymbol{m}^{*}$ and $\boldsymbol{J}$, for the sake of simplicity in the derivation of the conservation and the balance laws, we can use $\boldsymbol{\sigma}^{*}$, $\boldsymbol{m}^{[0]}$ and $\boldsymbol{J}$ and finally express this in $\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{m}^{[0]}$ and $\boldsymbol{\varepsilon}_{[0]}$ using relationship between them. The deformation measures for nonlinear 3M theories derived in ref. [1] are utilized in conjunction with entropy inequality.

5. Conservation and Balance Laws

Consider conservation and the balance laws derived in references [1] [41] for finite deformation/finite strain physics of micropolar medium using $\boldsymbol{\sigma}^{*}, \boldsymbol{m}^{*}, \dot{\boldsymbol{J}}$ and ${}^{c\Theta}\dot{\boldsymbol{J}}$. Conservation of mass, balance of linear momenta, balance of angular momenta, first and the second law of thermodynamics are given below:

$$
\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t) \tag{10}
$$

$$\rho_0 \frac{Dv}{Dt} - \rho_0 \boldsymbol{F}^b - \nabla \cdot \boldsymbol{\sigma}^* = 0 \tag{11}$$

$$\nabla \cdot \boldsymbol{m}^* + \boldsymbol{\epsilon} : \boldsymbol{\sigma}^* + \rho_0 \boldsymbol{m}^b = 0 \tag{12}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{13}$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{m}^* : {}^{c^\Theta}\dot{\boldsymbol{J}} - {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) = 0 \tag{14}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{m}^* : {}^{c^\Theta}\dot{\boldsymbol{J}} - {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) \le 0 \tag{15}$$

This mathematical model contains: $\boldsymbol{u}(3), \boldsymbol{\sigma}(9), \boldsymbol{m}(6), \boldsymbol{q}(3), \theta(1)$, a total of 22 dependent variables, but has only balance of linear momenta (3), balance of angular momenta (3) and first law of thermodynamics (1), seven partial differential equations in total. Thus, an additional 15 equations are needed for closure. These are provided by the constitutive theories for stress tensor (6), moment tensor (6), and heat vector (3).

6. Constitutive Theories

In the derivation of the constitutive theories, the first important step is to establish constitutive tensors and their argument tensors. The entropy inequality and axioms of constitutive theory suffice for this purpose. Based on the conjugate pairs $\boldsymbol{q} : \boldsymbol{g}$, $\boldsymbol{\sigma}^* : \dot{\boldsymbol{J}}$ and $\boldsymbol{m}^* : {}^{c^\Theta}\dot{\boldsymbol{J}}$ and axioms of constitutive theory, $\boldsymbol{q}, \boldsymbol{\sigma}^*, \boldsymbol{m}^*$ are likely the initial choice for the constitutive theories. $\boldsymbol{\sigma}^*, \boldsymbol{m}^*, \boldsymbol{J}$ and ${}^{c^\Theta}\boldsymbol{J}$ are all nonsymmetric tensors of rank two, hence cannot be utilized in deriving constitutive theories using representation theorem. Thus, at this stage, we have:

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta); \boldsymbol{\sigma}^* \ne \boldsymbol{\sigma}^*(\boldsymbol{J}, \theta); \boldsymbol{m}^* \ne \boldsymbol{m}^*\left({}^{c^\Theta}\boldsymbol{J}, \theta \right) \tag{16}$$

Through additive decompositions, all nonsymmetric second rank tensors must be expressed as symmetric and skew-symmetric tensors. Additionally, the last term in (15) must also be addressed *i.e.*, either eliminated by substitution or quantified otherwise, so that entropy inequality is not effected by its arbitrary but admissible value. We present details below:

From balance of angular momenta, we have:

$$\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b = -\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \tag{17}$$

Using (17), the last term in (15) can be written as:

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) = -{}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \right) \tag{18}$$

A simple calculation shows that

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \right) = {}_a\boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} = \boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \tag{19}$$

Therefore,

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) = -\boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \tag{20}$$

Using (20), the entropy inequality (15) can be written as:

$$\rho_0\left(\frac{D\theta}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\boldsymbol{\sigma}^*:\dot{\boldsymbol{J}}-\boldsymbol{m}^*:{}^{c\Theta}\dot{\boldsymbol{J}}+\boldsymbol{\sigma}^*:{}_a\dot{\boldsymbol{J}}\le 0 \tag{21}$$

Consider $\boldsymbol{\sigma}^*:\dot{\boldsymbol{J}}$ term in (21)

$$\begin{aligned}\boldsymbol{\sigma}^*:\dot{\boldsymbol{J}}&=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:\left(\dot{\boldsymbol{J}}\right)^{\mathrm{T}}=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:\left(\dot{\boldsymbol{J}}\right)\\&=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:\left({}_s\dot{\boldsymbol{J}}+{}_a\dot{\boldsymbol{J}}\right)\\&=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:{}_s\dot{\boldsymbol{J}}+\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:{}_a\dot{\boldsymbol{J}}\\&=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:{}_s\dot{\boldsymbol{J}}+\left(\boldsymbol{\sigma}^*\right):{}_a\dot{\boldsymbol{J}}\end{aligned} \tag{22}$$

Consider $\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:{}_s\dot{\boldsymbol{J}}$ term in (22)

$$\begin{aligned}\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}:{}_s\dot{\boldsymbol{J}}&=\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}}\cdot\frac{1}{2}\left(\bar{\boldsymbol{L}}\cdot\boldsymbol{J}+\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{L}}^{\mathrm{T}}\right)\\&=\frac{1}{2}\boldsymbol{J}\cdot\left(\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}}:\left(\left(\bar{\boldsymbol{D}}+\bar{\boldsymbol{W}}\right)\cdot\boldsymbol{J}+\boldsymbol{J}^{\mathrm{T}}\cdot\left(\bar{\boldsymbol{D}}-\bar{\boldsymbol{W}}\right)\right)\\&=\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}-{}_a\boldsymbol{\sigma}^{[0]}\right):\left(\left(\bar{\boldsymbol{D}}+\bar{\boldsymbol{W}}\right)\cdot\boldsymbol{J}+\boldsymbol{J}^{\mathrm{T}}\cdot\left(\bar{\boldsymbol{D}}-\bar{\boldsymbol{W}}\right)\right)\\&=\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\bar{\boldsymbol{D}}\cdot\boldsymbol{J}+\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\\&\quad+\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\bar{\boldsymbol{W}}\cdot\boldsymbol{J}-\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\\&\quad-\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\bar{\boldsymbol{D}}\cdot\boldsymbol{J}-\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\\&\quad-\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\bar{\boldsymbol{W}}\cdot\boldsymbol{J}+\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\end{aligned} \tag{23}$$

We consider each term in (23)

$$\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right)\right):\left(\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]} \tag{24}$$

$$\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]} \tag{25}$$

$$\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right)\right):\bar{\boldsymbol{W}}\cdot\boldsymbol{J}=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}:\bar{\boldsymbol{W}}\cdot\boldsymbol{J}=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=0 \tag{26}$$

$$\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right)\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}=\frac{1}{2}\left({}_s\boldsymbol{\sigma}^{[0]}\right):\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=0 \tag{27}$$

$$\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right)\right):\bar{\boldsymbol{D}}\cdot\boldsymbol{J}=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}:\bar{\boldsymbol{D}}\cdot\boldsymbol{J}=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=0 \tag{28}$$

$$-\frac{1}{2}\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}=\frac{1}{2}\,{}_s\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=0 \tag{29}$$

$$-\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\left(\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=-\frac{1}{2}\boldsymbol{J}\cdot{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\right)=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\cdot\boldsymbol{J}\right) \tag{30}$$

$$\begin{aligned}\frac{1}{2}\boldsymbol{J}\cdot{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\right)&=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)^{\mathrm{T}}\\&=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\cdot\boldsymbol{J}\right)\end{aligned} \tag{31}$$

Substituting (24)-(31) in (23), we obtain

$$\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}} : {}_s\dot{\boldsymbol{J}} = {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} \tag{32}$$

Substituting from (32) into (22) we obtain

$$\left(\boldsymbol{\sigma}^*\right) : \dot{\boldsymbol{J}} = {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} + \boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \tag{33}$$

Substituting from (33) into (21), and noting that $\boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}}$ terms cancel, we obtain the following form of entropy inequality:

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \le 0 \tag{34}$$

We substitute $\left(\boldsymbol{m}^*\right)^{\mathrm{T}} = \boldsymbol{J}\cdot\left(\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}$ in the last term of (34):

$$\left(\boldsymbol{m}^*\right) : {}^{c\Theta}\dot{\boldsymbol{J}} = \boldsymbol{m}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}} : {}^{c\Theta}\dot{\boldsymbol{J}} = \boldsymbol{m}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right). \tag{35}$$

Substituting (35) in (34), we obtain the final form of entropy inequality:

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right) \le 0. \tag{36}$$

From the entropy inequality (36), we observe that $\dot{\boldsymbol{\varepsilon}}_{[0]}$ and $\left(\boldsymbol{J}\right)^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)$ are rate of work conjugate to ${}_s\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ (symmetric).

We must check the validity of these rate terms in (36) by comparing them with the nonlinear deformation measures given by:

$$[J]^{\mathrm{T}}[J] - [I]; \left[{}_aJ^{(\alpha)}\right]; 2[J]^{\mathrm{T}}\left[\frac{\partial\left[{}_aJ^{(\alpha)}\right]}{\partial\{x\}}\right] \tag{37}$$

in which $\left[J^{(\alpha)}\right]$ is the micro deformation gradient tensor. Its additive decomposition into symmetric and skew symmetric components contains microconstituent rigid rotations ${}_\alpha\boldsymbol{\Theta}$ in the skew symmetric part of tensor. Since microconstituent rigid rotations are in fact classical rotations ${}_c\boldsymbol{\Theta}$ (due to free field in the absence of microconstituents), in (37) we can replace ${}_\alpha\boldsymbol{\Theta}$ by ${}_c\boldsymbol{\Theta}$ *i.e.*, in place of $\left[{}_aJ^{(\alpha)}\right]$ by $\left[{}_aJ\right]$ and $\left[\frac{\partial\left[{}_aJ^{(\alpha)}\right]}{\partial\{x\}}\right]$ by $\left[{}^{c\Theta}J\right]$.

$$\left([J]^{\mathrm{T}}[J] - [I]\right); \left[{}_aJ\right]; 2[J]^{\mathrm{T}}\left[{}^{c\Theta}J\right] \tag{38}$$

These measures in the form listed in (38) or with some minor modifications must be utilized in the nonlinear micropolar nonclassical continuum theory for compressible thermoviscoelastic solid without memory. From entropy inequality (36), we note $\boldsymbol{q}$, ${}_s\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}$ as constitutive tensors and $\boldsymbol{g}, \dot{\boldsymbol{\varepsilon}}_{[0]}$ and $\boldsymbol{J}^{\mathrm{T}}\cdot{}^{c\Theta}\boldsymbol{J}$ as their argument tensors are valid based on axiom of constitutive theory in the nonlinear micropolar nonclassical continuum theory for thermoviscoelastic solid. Thus, we see that $[J]^{\mathrm{T}}[J] - [I]$ is a deformation measure in (37) and not a strain measure. We need to multiply by $\frac{1}{2}$ to obtain strain measure $\boldsymbol{\varepsilon}_{[0]}$, then it is valid argument tensor of ${}_s\boldsymbol{\sigma}^{[0]}$. Likewise, $2[J]^{\mathrm{T}}\left[{}^{c\Theta}_{a}J\right]$ is not conjugate to $\boldsymbol{m}^{[0]}$

either, we need to multiply by $\frac{1}{2}$ to obtain a measure conjugate to $\boldsymbol{m}^{[0]}$. We remark that as pointed out in ref. [1] and substantiated here, the derivations of strain measure initiated using $\left(d\bar{s}^{(\alpha)}\right)^2 - \left(ds^{(\alpha)}\right)^2$ only yield deformation measures and not strain measures as we have seen here. Last term in (36) requires further considerations. We note that work conjugate pair is $\boldsymbol{m}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}} \cdot {}^{c\Theta}\dot{\boldsymbol{J}}\right)$, in which $\boldsymbol{m}^{[0]}$ is symmetric, thus we must consider:

$$\boldsymbol{m}^{[0]} : \left((\boldsymbol{J})^{\mathrm{T}} \cdot {}^{c\Theta}\dot{\boldsymbol{J}}\right) = \boldsymbol{m}^{[0]} : \frac{1}{2}\left(\left(\boldsymbol{J}^{\mathrm{T}}\right)\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{\mathrm{T}} \cdot (\boldsymbol{J})\right) \tag{39}$$

We substitute (39) in (36)

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \frac{1}{2}\left(\left(\boldsymbol{J}^{\mathrm{T}}\right)\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{\mathrm{T}} \cdot (\boldsymbol{J})\right) \le 0 \tag{40}$$

We can also substitute $\left[\sigma^*\right]^{\mathrm{T}} = [J]\left[\sigma^{[0]}\right]^{\mathrm{T}}$ in the conservation and balance laws. Thus, the conservation and balance laws (10)-(15) now have the following form:

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x},t) \tag{41}$$

$$\rho_0\frac{D\boldsymbol{v}}{Dt} - \rho_0\boldsymbol{F}^b - \nabla\cdot\left(\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) = 0 \tag{42}$$

$$\nabla\cdot\left(\boldsymbol{m}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) + \epsilon : \left(\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) + \rho_0\boldsymbol{m}^b = 0 \tag{43}$$

$$\epsilon_{ijk} m_{ij}^{(0)} = 0 \tag{44}$$

$$\rho_0\frac{De}{Dt} + \nabla\cdot\boldsymbol{q} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = 0 \tag{45}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right) \le 0 \tag{46}$$

in which

$${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}^{c\Theta}\boldsymbol{\varepsilon}_{[1]} = \frac{1}{2}\left(\left(\boldsymbol{J}^{\mathrm{T}}\right)\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{\mathrm{T}} \cdot (\boldsymbol{J})\right) \tag{47}$$

Remarks

(1) ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$ is the strain rate conjugate to $\boldsymbol{m}^{[0]}$.

(2) The linear micropolar nonclassical continuum theory for infinitesimal deformation is a complete subset of the nonlinear micropolar nonclassical continuum theory described by conservation and balance laws (41)-(47). In this case,

$$\begin{gathered}\boldsymbol{J} \approx \boldsymbol{I}\,;\, \boldsymbol{\sigma}^{[0]} = \boldsymbol{\sigma}^{(0)};\, \boldsymbol{m}^{[0]} = \boldsymbol{m}^{(0)} \\ \dot{\boldsymbol{\varepsilon}}_{[0]} = \dot{\boldsymbol{\varepsilon}}\,;\, {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}\end{gathered} \tag{48}$$

in which $\boldsymbol{\varepsilon}$ is linear strain measure and ${}^{c\Theta}\boldsymbol{\varepsilon}$ is the symmetric part of the classical rotation gradient tensor. Thus, (41)-(47) reduce to the linear micropolar nonclassical continuum theory for infinitesimal deformation.

(3) When ${}_c\boldsymbol{\Theta}$ is not considered, the conservation and balance laws (41)-(47) reduce to finite deformation/finite strain classical continuum physics.

(4) When ${}_c\boldsymbol{\Theta}$ is a free field (*i.e.*, ${}_c\boldsymbol{\Theta}$ is not considered) and the deformation is infinitesimal, the conservation and balance laws (41)-(47) reduce to infinitesimal deformation classical continuum mechanics.

(5) All conjugate pairs in the entropy inequality define constitutive tensors and their argument tensors that are supported by the representation theorem; hence, they would yield thermodynamically and mathematically consistent constitutive theories.

6.1. Constitutive Theory for Equilibrium Stress ${}_s^e\boldsymbol{\sigma}^{[0]}$

Recall that the additive decomposition $\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]} + {}_a\boldsymbol{\sigma}^{[0]}$ is necessary because ${}_a\boldsymbol{\sigma}^{[0]}$ cannot be part of the constitutive theories, as it is defined by balance of angular momenta through gradients of moment tensor. To address volumetric and distortional deformation physics (mutually exclusive) in nonlinear micropolar nonclassical continuum theory, we must further additively decompose $\boldsymbol{\sigma}^{[0]}$ into equilibrium ${}_s^e\boldsymbol{\sigma}^{[0]}$ and deviatoric ${}_s^d\boldsymbol{\sigma}^{[0]}$ components. Volumetric deformation physics is associated with ${}_s^e\boldsymbol{\sigma}^{[0]}$, while distortion deformation physics is described by ${}_s^d\boldsymbol{\sigma}^{[0]}$. As discussed in ref. [1], compressibility in solids in Lagrangian description is controlled by $|J|$, and the density in the current configuration is deterministic through conservation of mass if $\boldsymbol{J}$ is known. Thus, density is not a dependent variable in the conservation and balance laws in Lagrangian description for solid matter. The equation of state in solids is a consequence of density change *i.e.*, for a density change caused due by $|J|$, there is a pressure field associated with it. The presence of this pressure field through equilibrium stress in balance of linear momenta is essential for correct force balance. We further elaborate that in compressible solids, one could determine solution for compressible case without using equation of state, but such solutions would be erroneous due to incorrect force balance in the balance of linear momenta. Since compressibility physics depends upon density and temperature, the constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ must be obtained using the constitutive theory for ${}^e\boldsymbol{\sigma}^{(0)}$. Details of the derivation of the constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ can be found in recent papers [40] [41] by the authors, including references [42] [43]. The final form of the constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ for compressible and incompressible cases are given by:

$$ {}_s^e\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|\left(\boldsymbol{J}^{-1}\right)\cdot p(\rho,\theta)\cdot\left(\boldsymbol{J}^{-1}\right)^{\mathrm{T}} = |\boldsymbol{J}|\,p(\rho,\theta)\left(\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}\right)^{-1}\ (\text{compressible}) \tag{49} $$

$$ {}_s^e\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|\left(\boldsymbol{J}^{-1}\right)\cdot p(\theta)\left(\boldsymbol{J}^{-1}\right)^{\mathrm{T}} = |\boldsymbol{J}|\,p(\theta)\left(\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}\right)^{-1}\ (\text{incompressible}) \tag{50} $$

in which, $p(\rho,\theta)$ and $p(\theta)$ are thermodynamic and mechanical pressures.

In (50), we could have used $\boldsymbol{J}\approx\boldsymbol{I}$ and $|\boldsymbol{J}|\approx 1$, but we leave the expression as it is in (50). Reduced form of entropy inequality (after considering constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$) in Lagrangian description is given by:

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s^d\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{(0)}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right) \le 0 \tag{51} $$

Constitutive Theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$

The constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ must address: (1) distortional deformation physics and (2) the macrodissipation mechanism due to viscosity of the medium. From the rate of work conjugate pairs ${}_s^d\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}$ in the reduced form of the entropy inequality, ${}_s^d\boldsymbol{\sigma}^{[0]}:\boldsymbol{\varepsilon}_{[0]}$ is the work conjugate pair, suggesting that $\boldsymbol{\varepsilon}_{[0]}$ as argument tensor of ${}_s^d\boldsymbol{\sigma}^{[0]}$ is a valid choice. From viscous fluid physics, we know that dissipation is a function of convected time derivative of the Green or Almansi strain tensor. Symmetric part of the velocity gradient tensor $\bar{\boldsymbol{D}}$ (where viscous stress is proportional to $\bar{\boldsymbol{D}}$) is the first convected time derivative of the Green and Almansi strain tensors. This holds for any viscous medium. Thus, at the very minimum, the dissipation mechanism will require ${}_s^d\boldsymbol{\sigma}^{[0]}$ to be a function of $\dot{\boldsymbol{\varepsilon}}_{[0]}$ or $\boldsymbol{\varepsilon}_{[1]}$, the first convected time derivative of the Green's strain tensor (same as ordinary time derivative in this case). We can generalize this by assuming that viscous dissipation mechanism depends on strain rates $\boldsymbol{\varepsilon}_{[i]}; i=1,2,\cdots,n$ of up to order n. This is macro dissipation. This provides ordered rate dissipation mechanism. θ is naturally an argument tensor of ${}_s^d\boldsymbol{\sigma}^{[0]}$. Thus, we can write:

$$ {}_s^d\boldsymbol{\sigma}^{[0]} = {}_s^d\boldsymbol{\sigma}^{[0]}\left(\boldsymbol{\varepsilon}_{[0]},\boldsymbol{\varepsilon}_{[i]},\theta\right); i=1,2,\cdots,n \tag{52} $$

Now we can derive constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ using representation theorem. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\,; i=1,2,\cdots,{}^{\sigma}N$ be the combined generators of the argument tensors of ${}_s^d\boldsymbol{\sigma}^{[0]}$ in (52), that are symmetric tensors of rank two. Then, $\boldsymbol{I},{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\,; i=1,2,\cdots,{}^{\sigma}N$ constitute the basis of the space of tensor ${}_s^d\boldsymbol{\sigma}^{[0]}$, referred to as integrity. Hence, we can express ${}_s^d\boldsymbol{\sigma}^{[0]}$ as a linear combination of $\boldsymbol{I},{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\,; i=1,2,\cdots,{}^{\sigma}N$ in the current configuration.

$$ {}_s^d\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\underset{\sim}{\alpha}^0\boldsymbol{I} + \sum_{i=1}^{{}^{\sigma}N} {}^{\sigma}\alpha^i\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right);\ {}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i\left({}^{\sigma}\underset{\sim}{I}^j,\theta\right); i=0,1,\cdots,{}^{\sigma}N\,;\ j=1,2,\cdots,{}^{\sigma}M \tag{53} $$

in which ${}^{\sigma}\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^{\sigma}M$ are the combined invariants of the same argument tensor of ${}_s^d\boldsymbol{\sigma}^{[0]}$ in (52).

The material coefficients in (53) are determined by expanding ${}^{\sigma}\underset{\sim}{\alpha}^i\,; i=0,1,\cdots,{}^{\sigma}N$ in Taylor series in ${}^{\sigma}\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^{\sigma}M$ and the temperature θ about a known configuration $\underline{\Omega}$ (based on principle of smooth neighborhood) and retaining only up to linear terms in ${}^{\sigma}\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^{\sigma}M$ and temperature θ (for simplicity of the resulting constitutive theory).

$$ {}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^{\sigma}M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta\big|_{\underline{\Omega}}\right); i=0,1,\cdots,{}^{\sigma}N \tag{54} $$

Substituting for ${}^{\sigma}\underset{\sim}{\alpha}^0$ and ${}^{\sigma}\underset{\sim}{\alpha}^i\,; i=1,2,\cdots,{}^{\sigma}N$ from (54) into (53)

$$ \begin{aligned} {}_s^d\boldsymbol{\sigma}^{[m]} = & \left({}^{\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^{\sigma}M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta\big|_{\underline{\Omega}}\right)\right)\boldsymbol{I} \\ & + \sum_{i=1}^{{}^{\sigma}N}\left({}^{\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^{\sigma}M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta\big|_{\underline{\Omega}}\right)\right)\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \end{aligned} \tag{55} $$

Collecting coefficients of $\boldsymbol{I},\ {}^{\sigma}\underset{\sim}{I}^j\boldsymbol{I},\ {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i,\ {}^{\sigma}\underset{\sim}{I}^j\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i, \left(\theta-\theta\big|_{\underline{\Omega}}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$ and

$\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}$ in (55)

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[0]} = & \left({}^{\sigma}\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} + \sum_{j=1}^{\sigma M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial^{\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \left(-{}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}\right)\right)\boldsymbol{I} + \sum_{j=1}^{\sigma M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial^{\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} {}^{\sigma}\underset{\sim}{I}^j\boldsymbol{I} \\
& + \sum_{i=1}^{\sigma N}\left({}^{\sigma}\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{\sigma M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial^{\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} \left(-{}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}\right)\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \\
& + \sum_{i=1}^{\sigma N}\sum_{j=1}^{\sigma M} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial^{\sigma}\underset{\sim}{I}^j}\bigg|_{\underline{\Omega}} {}^{\sigma}\underset{\sim}{I}^j\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \\
& + \sum_{i=1}^{\sigma N} \frac{\partial^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\bigg|_{\underline{\Omega}} \left(\theta-\theta|_{\underline{\Omega}}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i + \frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial\theta}\bigg|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned} \tag{56}
$$

If we define

$$
\begin{aligned}
& \sigma^0\big|_{\underline{\Omega}} = {}^{\sigma}\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} + \sum_{j=1}^{\sigma M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^0\right)}{\partial\left(-{}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\left(-{}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}\right); \quad \underset{\sim}{a}_j = \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^0\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \\
& \underset{\sim}{b}_i = {}^{\sigma}\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{\sigma M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^i\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\left(-{}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}\right); \quad \underset{\sim}{c}_{ij} = \frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^i\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \\
& {}^{\sigma}\underset{\sim}{d}_i = -\frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\theta}\Bigg|_{\underline{\Omega}}; \quad \alpha_{tm}\big|_{\underline{\Omega}} = -\frac{\partial^{\sigma}\underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}
\end{aligned} \tag{57}
$$

then, using (57) in (56), we can write (56) in a more compact form.

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[0]} = & \sigma_0\big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{i=1}^{\sigma N}{}^{\sigma}\underset{\sim}{a}_i\left({}^{\sigma}\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{j=1}^{\sigma M}{}^{\sigma}\underset{\sim}{b}_j\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{j=1}^{\sigma M}\sum_{i=1}^{\sigma N}{}^{\sigma}\underset{\sim}{c}_{ij}\left({}^{\sigma}\underset{\sim}{I}^j\right)\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \\
& -\sum_{j=1}^{\sigma N}{}^{\sigma}\underset{\sim}{d}^i\left(\theta-\theta|_{\underline{\Omega}}\right)\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) - \left(\alpha_{tm}\right)_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned} \tag{58}
$$

Equation (58) defines material coefficients, that can be functions of $\theta|_{\underline{\Omega}}$ and ${}^{\sigma}\underset{\sim}{I}^j\big|_{\underline{\Omega}}$; $j = 1, 2, \cdots, {}^{\sigma}M$.

This constitutive theory is based on integrity (complete basis) and requires $\left(2\left({}^{\sigma}N\right)+\left({}^{\sigma}M\right)+\left({}^{\sigma}N\right)\left({}^{\sigma}M\right)+1\right)$ material coefficients. Various simplified forms of the constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ can be obtained from (58) by choosing desired generators and invariants. The most simplified yet some what general constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ is the one in which ${}_s^d\boldsymbol{\sigma}^{[0]}$ is a linear function of the components of the argument tensor in (52). This is given by, after redefining material coefficients:

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma} = & \sigma^0\big|_{\underline{\Omega}}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}_{[0]} + \lambda\left(\text{tr}\boldsymbol{\varepsilon}_{[0]}\right)I + \sum_{i=1}^{n}2\eta_i\boldsymbol{\varepsilon}_{[i]} \\
& + \sum_{i=1}^{n}\kappa_i\left(\text{tr}\boldsymbol{\varepsilon}_{[i]}\right)I - \alpha_{tm}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned} \tag{59}
$$

in which μ and λ are similar to Lames coefficients in linear elasticity. η_i and κ_i are damping material coefficients associated with strain rate $\boldsymbol{\varepsilon}_{[i]}$. This constitutive theory is nonlinear in the components of the displacement gradient tensor. The dissipation mechanism is nonlinear and is defined by strain rates $\boldsymbol{\varepsilon}_{[i]}; i=1,2,\cdots,n$ of up to order n.

6.2. Constitutive Theory for $\boldsymbol{m}^{[0]}$

Just like the constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ that addresses macro elasticity and macro dissipation due to viscosity of the medium, the constitutive theory for $\boldsymbol{m}^{[0]}$ must also address: (1) the macro distortional deformation physics (macro elasticity) and (2) the dissipation mechanism between the microconstituents and the viscosity of the medium, micro dissipation or micro viscous dissipation mechanism. From the reduced form of the entropy inequality, we consider the rate of work conjugate pair $\boldsymbol{m}^{[0]} : \left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)$ in which:

$$\begin{aligned} {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} &= \frac{1}{2}\left((\boldsymbol{J})^{\mathrm{T}} \cdot \left({}^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{\mathrm{T}} \cdot (\boldsymbol{J})\right) \\ &= \frac{1}{2}\left(\left({}_s\boldsymbol{J} + {}_a\boldsymbol{J}\right)^{\mathrm{T}} \cdot \left({}_s^{c\Theta}\dot{\boldsymbol{J}} + {}_a^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}_s^{c\Theta}\dot{\boldsymbol{J}} - {}_a^{c\Theta}\dot{\boldsymbol{J}}\right) \cdot \left({}_s\boldsymbol{J} + {}_a\boldsymbol{J}\right)\right) \\ &= \frac{1}{2}\left({}_s\boldsymbol{J} \cdot \left({}_s^{c\Theta}\dot{\boldsymbol{J}}\right) + \left({}_s^{c\Theta}\dot{\boldsymbol{J}}\right) \cdot {}_s\boldsymbol{J} - \left({}_a\boldsymbol{J} \cdot \left({}_a^{c\Theta}\dot{\boldsymbol{J}}\right) + {}_a^{c\Theta}\dot{\boldsymbol{J}} \cdot {}_a\boldsymbol{J}\right)\right) \\ &= {}_s\boldsymbol{J} \cdot {}_s^{c\Theta}\dot{\boldsymbol{J}} - {}_a\boldsymbol{J} \cdot {}_a^{c\Theta}\dot{\boldsymbol{J}} \end{aligned} \tag{60}$$

Thus, we can write

$$ {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}^{c\Theta}\boldsymbol{\varepsilon}_{[1]} = {}_s\boldsymbol{J} \cdot \left({}_s^{c\Theta}\boldsymbol{J}_{[1]}\right) - {}_a\boldsymbol{J} \cdot \left({}_a^{c\Theta}\boldsymbol{J}_{[1]}\right) \tag{61}$$

Generalizing (61) to include rates of ${}_s^{c\Theta}\boldsymbol{J}$ and ${}_a^{c\Theta}\boldsymbol{J}$ up to orders n, we can write:

$$ {}^{c\Theta}\boldsymbol{\varepsilon}_{[i]} = {}_s\boldsymbol{J} \cdot \left({}_s^{c\Theta}\boldsymbol{J}_{[i]}\right) - {}_a\boldsymbol{J} \cdot \left({}_a^{c\Theta}\boldsymbol{J}_{[i]}\right); i=1,2,\cdots,\underset{\sim}{n} \tag{62}$$

and

$$ {}^{c\Theta}\boldsymbol{\varepsilon}_{[0]} = {}_s\boldsymbol{J} \cdot \left({}_s^{c\Theta}\boldsymbol{J}\right) - {}_a\boldsymbol{J} \cdot \left({}_a^{c\Theta}\boldsymbol{J}\right) \tag{63}$$

Thus, based on rate of work conjugate pair $\boldsymbol{m}^{[0]} : \left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)$, ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$ in (63) must be an argument tensor of $\boldsymbol{m}^{[0]}$. If we assume that dissipation between the microconstituents and the viscous medium (micro viscous dissipation or microdissipation mechanism) depends on rate of ${}^{c\Theta}\boldsymbol{J}$, then ${}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}; i=1,2,\cdots,\underset{\sim}{n}$ in (62) must also be argument tensors of $\boldsymbol{m}^{[0]}$. Choice of θ as an argument tensor of $\boldsymbol{m}^{[0]}$ is obvious. Thus, we have:

$$ \boldsymbol{m}^{[0]} = \boldsymbol{m}^{[0]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}, {}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}, \theta\right); i=1,2,\cdots,\underset{\sim}{n} \tag{64}$$

in which ${}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}$ and ${}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}; i=1,2,\cdots,\underset{\sim}{n}$ are defined by (63) and (62). Now, we can derive constitutive theory for $\boldsymbol{m}^{[0]}$ using representation theorem [28]-[39]. Let ${}^m\underset{\sim}{\boldsymbol{G}}^i; i=1,2,\cdots,{}^mN$ and ${}^m\underset{\sim}{I}^j; j=1,2,\cdots,{}^mM$ be the combined generators and combined invariants of the argument tensors of $\boldsymbol{m}^{[0]}$ in (64) that are symmetric tensors of rank two. Then $\boldsymbol{I}, {}^m\underset{\sim}{\boldsymbol{G}}^i; i=1,2,\cdots,{}^mN$ constitute the basis of the space of constitutive tensor $\boldsymbol{m}^{[0]}$ (integrity), hence we can express $\boldsymbol{m}^{[0]}$ as a linear combination of the basis in the current configuration.

$$ \boldsymbol{m}^{[0]} = {}^m\underset{\sim}{\alpha}^0 + \sum_{i=1}^{{}^mN} {}^m\underset{\sim}{\alpha}^i \left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) \tag{65}$$

The coefficients ${}^m\underset{\sim}{\alpha}^i\,; i=0,1,\cdots,{}^mN$ in the linear combination can be functions of ${}^m\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^mM$ and θ. We remark that coefficients ${}^m\underset{\sim}{\alpha}^i\,; i=0,1,\cdots,{}^mN$ are not material coefficients. Material coefficients in (65) are determined by considering Taylor series expansion of ${}^m\underset{\sim}{\alpha}^i\,; i=0,1,\cdots,{}^mN$ (based on axioms of smooth neighborhood [42] [43]) in ${}^m\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^mM$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^m\underset{\sim}{I}^j\,; j=1,2,\cdots,{}^mM$ and θ (for simplicity)

$$ {}^m\underset{\sim}{\alpha}^i = {}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}}\left({}^m\underset{\sim}{I}^j - {}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right); i=0,1,\cdots,{}^mN \tag{66} $$

Substituting ${}^m\underset{\sim}{\alpha}^0$ and ${}^m\underset{\sim}{\alpha}^i; i=1,2,\cdots,{}^m\underset{\sim}{N}$ from (66) in (65), we get:

$$ \begin{aligned} \boldsymbol{m}^{[0]} &= \left({}^m\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}}\left({}^m\underset{\sim}{I}^j - {}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\right)\boldsymbol{I} \\ &+ \sum_{i=1}^{{}^mN}\left({}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}}\left({}^m\underset{\sim}{I}^j - {}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\right){}^m\underset{\sim}{\boldsymbol{G}}^i \end{aligned} \tag{67} $$

Collecting coefficients (defined in $\underline{\Omega}$) of $\boldsymbol{I}$, ${}^m\underset{\sim}{I}^j\boldsymbol{I}$, ${}^m\underset{\sim}{\boldsymbol{G}}^i$, ${}^m\boldsymbol{I}^j\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right)$, $\left(\theta-\theta|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i$, $\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}$ in (67), we can write (67) as follow:

$$ \begin{aligned} \boldsymbol{m}^{[0]} &= \left({}^m\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^0\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\right)\right)\boldsymbol{I} + \sum_{j=1}^{{}^mM}\left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^0\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}{}^m\underset{\sim}{I}^j\boldsymbol{I} \\ &+ \sum_{i=1}^{{}^mN}\left({}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\right)\Big|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i \\ &+ \sum_{i=1}^{{}^mN}\sum_{j=1}^{{}^mM}\left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}\left({}^m\underset{\sim}{I}^j\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) \\ &+ \sum_{i=1}^{{}^mN}\left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i + \left.\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{68} $$

If we define

$$ \begin{aligned} m^0\Big|_{\underline{\Omega}} &= {}^m\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) \\ {}^m\underset{\sim}{a}_j &= \left.\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial^m\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \\ {}^m\underset{\sim}{b}_i &= {}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m I^j}\right|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\right)\Big|_{\underline{\Omega}} \\ {}^m\underset{\sim}{c}_{ij} &= \left.\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \\ {}^m\underset{\sim}{d}_i &= -\left.\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}\,;\ {}^m\alpha_{tm} = -\left.\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}} \end{aligned} \tag{69} $$

then, using (69) in (68), we can write (68) in more compact form.

$$\begin{aligned} \boldsymbol{m}^{[0]} = {} & m^0\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{{}^mM} {}^m\underset{\sim}{a}_j\, {}^m\underset{\sim}{I}^j \boldsymbol{I} + \sum_{i=1}^{{}^mN} {}^m\underset{\sim}{b}_i\, {}^m\underset{\sim}{\boldsymbol{G}}^i + \sum_{i=1}^{{}^mN}\sum_{j=1}^{{}^mM} {}^m\underset{\sim}{c}_{ij}\, {}^m\underset{\sim}{I}^j\, {}^m\underset{\sim}{\boldsymbol{G}}^i \\ & - \sum_{i=1}^{{}^mN} {}^m\underset{\sim}{d}_i \left(\theta - \theta\big|_{\underline{\Omega}}\right) {}^m\underset{\sim}{\boldsymbol{G}}^i - {}^m\alpha_{tm}\left(\theta - \theta\big|_{\underline{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{70}$$

In this constitutive theory (70) for $\boldsymbol{m}^{[0]}$, the material coefficients are defined in (69). The material coefficients can be functions of $\rho\big|_{\underline{\Omega}}, \theta\big|_{\underline{\Omega}}$, and ${}^m\underset{\sim}{I}^j\big|_{\underline{\Omega}}$; $j = 1,2,\cdots,{}^mM$. This constitutive theory requires $\left(2\left({}^mN\right) + \left({}^mM\right)\left({}^mN\right) + \left({}^mM\right) + 1\right)$ material coefficients. Simplified forms of the constitutive theories for $\boldsymbol{m}^{[0]}$ can be obtained from (70) by choosing only the desired generators and invariants. The most simplified, yet general constitutive theory for $\boldsymbol{m}^{[0]}$ is one in which $\boldsymbol{m}^{[0]}$ is a linear function of the components of the argument tensors.

$$\begin{aligned} \boldsymbol{m}^{[0]} = {} & m^0\big|_{\underline{\Omega}} \boldsymbol{I} + 2\underset{\sim}{\mu}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right) + \underset{\sim}{\lambda}\operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)\boldsymbol{I} + \sum_{i=1}^{n} 2\underset{\sim}{\eta}_i\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right) \\ & + \sum_{i=1}^{n} \underset{\sim}{\kappa}_i \operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right)\boldsymbol{I} - {}^m\alpha_{tm}\left(\theta - \theta\big|_{\underline{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{71}$$

in which $\underset{\sim}{\mu}$ and $\underset{\sim}{\lambda}$ are similar to Lame's coefficients and the terms containing these are causing distortion of the matter. $\underset{\sim}{\eta}_i, \underset{\sim}{\kappa}_i$ are damping coefficients corresponding to the rate ${}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}$ for the i^{th} rate of the rotation gradients.

6.3. Constitutive Theory for *q*

Considering

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{72}$$

and following references [42] [43] we can derive the following constitutive theory for $\boldsymbol{q}$ using representation theorem.

$$\boldsymbol{q} = -\kappa\boldsymbol{g} - \kappa_1(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \kappa_2\left(\theta - \theta\big|_{\underline{\Omega}}\right)\boldsymbol{g} \tag{73}$$

κ, κ_1 and κ_2 are material coefficients. These can be functions of $(\boldsymbol{g}\cdot\boldsymbol{g})\big|_{\underline{\Omega}}$ and $\theta\big|_{\underline{\Omega}}$. $\boldsymbol{g}\cdot\boldsymbol{g}$ is invariant of argument tensor $\boldsymbol{g}$. Simplified form of (73), the Fourier heat conduction law is given by

$$\boldsymbol{q} = -\kappa\boldsymbol{g} \tag{74}$$

7. Complete Mathematical Model

In the following, we present complete mathematical model consisting of conservation and balance laws of nonclassical continuum mechanics and the constitutive theories for finite deformation/finite strain nonlinear micropolar nonclassical continuum theory based on classical rotation ${}_c\boldsymbol{\Theta}$ for thermoviscoelastic compressible solid medium. The mathematical model consists of (41)-(47), (59), (71) and (74). We have the following (using non-reduced forms of first law of thermodynamics and second law of thermodynamics)

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x},t) \tag{75}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \nabla\cdot\left(\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) = 0 \tag{76}$$

$$\nabla\cdot\left(\boldsymbol{m}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) + \epsilon : \left(\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}\right) + \rho_0 \boldsymbol{m}^b = 0 \tag{77}$$

$$\epsilon_{ijk} m_{ij}^{(0)} = 0 \tag{78}$$

$$\rho_0 \frac{De}{Dt} + \nabla\cdot\boldsymbol{q} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = 0 \tag{79}$$

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right) \le 0 \ \ (\text{reduced form}) \tag{80}$$

$$\begin{aligned} {}_s^d\boldsymbol{\sigma} = \sigma^0\big|_{\underset{\sim}{\Omega}} \boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}_{[0]} + \lambda\left(\mathrm{tr}\boldsymbol{\varepsilon}_{[0]}\right)I + \sum_{i=1}^{n} 2\eta_i \boldsymbol{\varepsilon}_{[i]} \\ + \sum_{i=1}^{n}\kappa_i\left(\mathrm{tr}\boldsymbol{\varepsilon}_{[i]}\right)I - \alpha_{tm}\left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{81}$$

$$\begin{aligned} \boldsymbol{m}^{[0]} = m^0\big|_{\underset{\sim}{\Omega}} \boldsymbol{I} + 2\underset{\sim}{\mu}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right) + \underset{\sim}{\lambda}\,\mathrm{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)\boldsymbol{I} + \sum_{i=1}^{\underset{\sim}{n}} 2\underset{\sim}{\eta}_i\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right) \\ + \sum_{i=1}^{\underset{\sim}{n}} \underset{\sim}{\kappa}_i\,\mathrm{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right)\boldsymbol{I} - {}^m\alpha_{tm}\left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{82}$$

$$\boldsymbol{q} = -\kappa\boldsymbol{g} \tag{83}$$

in which

$$\boldsymbol{\varepsilon}_{[0]} = \frac{1}{2}\left(\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J} - \boldsymbol{I}\right) \tag{84}$$

$$\boldsymbol{\varepsilon}_{[1]} = \boldsymbol{J}_{[1]}^{\mathrm{T}}\cdot\boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}_{[1]} \tag{85}$$

$$\boldsymbol{\varepsilon}_{[2]} = \boldsymbol{J}_{[2]}^{\mathrm{T}}\cdot\boldsymbol{J} + \boldsymbol{J}_{[1]}\cdot\boldsymbol{J}_{[1]} + \boldsymbol{J}_{[1]}^{\mathrm{T}}\cdot\boldsymbol{J}_{[1]} + \boldsymbol{J}_{[1]}^{\mathrm{T}}\cdot\boldsymbol{J}_{[2]} \tag{86}$$

$${}^{c\Theta}\boldsymbol{\varepsilon}_{[i]} = {}_s\boldsymbol{J}\cdot\left({}_s^{\Theta}\boldsymbol{J}_{[i]}\right) - {}_a\boldsymbol{J}\cdot\left({}_a^{\Theta}\boldsymbol{J}_{[i]}\right);\ i = 1,2,\cdots,\underset{\sim}{n} \tag{87}$$

$$\boldsymbol{g} = \nabla\cdot\theta \tag{88}$$

$$\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt} \tag{89}$$

in which ${}_s^{c\Theta}\boldsymbol{J}_{[0]} = {}_s^{c\Theta}\boldsymbol{J}$ and ${}_a^{c\Theta}\boldsymbol{J}_{[0]} = {}_a^{c\Theta}\boldsymbol{J}$. The mathematical model (75)-(89) consists of 22 equations: balance of linear momenta(3), balance of angular momenta(3), first law of thermodynamics(1), constitutive theories for ${}_s^d\boldsymbol{\sigma}^{[0]}(6)$, $\boldsymbol{m}^{[0]}(6)$, $\boldsymbol{q}(3)$ in twenty two variables: $\boldsymbol{u}(3)$, ${}_s^d\boldsymbol{\sigma}^{[0]}(6)$, ${}_a\boldsymbol{\sigma}^{[0]}(3)$, $\boldsymbol{m}^{[0]}(6)$, $\theta(1)$ and $\boldsymbol{q}(3)$. Thus, this mathematical model consisting of conservation and balance laws for nonlinear micropolar nonclassical continuum theory and constitutive theories for compressible thermoviscoelastic solid without memory has closure.

8. Summary and Conclusions

We have presented a finite deformation/finite strain nonlinear micropolar nonclassical continuum theory for compressible thermoviscoelastic solid continua without rheology based on classical rotations ${}_{c}\boldsymbol{\Theta}$ of the microconstituents. This theory consists of conservation and balance laws including balance of moment of moments, which is a new necessary balance law in all 3M nonclassical continuum theories. The constitutive theories are derived using representation theorem. A summary of the work presented in the paper and some conclusions drawn from it are given below.

(1) Finite deformation/finite strain measures have been utilized in the derivation of the theory based on the conjugate pairs in the entropy inequality.

(2) We distinguish between the strain measures and the rigid rotations in the derivation of the constitutive theories. In micropolar theories, the microconstituents can only experience rigid rotations. These rotations cannot be added to the strain measures as done in ref. [8]. This approach leads to incorrect definitions of strain tensor and consequently, erroneous constitutive theories that are based on this measure.

(3) The importance of additive stress decompositions: $\boldsymbol{\sigma}^{[0]} = {}_{s}\boldsymbol{\sigma}^{[0]} + {}_{a}\boldsymbol{\sigma}^{[0]}$; ${}_{s}\boldsymbol{\sigma}^{[0]} = {}_{s}^{e}\boldsymbol{\sigma}^{[0]} + {}_{s}^{d}\boldsymbol{\sigma}^{[0]}$ are necessary to ensure that ${}_{a}\boldsymbol{\sigma}^{[0]}$ is not part of constitutive tensors and to ensure that mutually exclusive volumetric and distortional deformations are addressed correctly in the constitutive theories.

(4) In micropolar theories, rigid rotations of the microconstituents must be incorporated in the development of the theory. We must keep in mind that classical rotations ${}_{c}\boldsymbol{\Theta}$ already exist at the material and constitute a free field in the absence of microconstituents. In the presence of microconstituents, classical rotations in fact are the rotations of the microconstituents. Thus, ${}_{c}\boldsymbol{\Theta}$ is a measure of rigid rotations of the microconstituents. This theory allows rigid rotations of microconstituents without considering ${}_{\alpha}\boldsymbol{\Theta}$ as additional unknown rotations of the microconstituents at the material points. The micropolar theory that uses unknown ${}_{\alpha}\boldsymbol{\Theta}$ at the material points is thermodynamically and mathematically consistent and the mathematical model lacks closure. (b) The linear micropolar theories based on rotations (${}_{c}\boldsymbol{\Theta} + {}_{\alpha}\boldsymbol{\Theta}$) at the material points and the linear micropolar theories based on ${}_{\alpha}\boldsymbol{\Theta}$ at the material point (ignoring ${}_{c}\boldsymbol{\Theta}$) have been shown to be thermodynamically and mathematically inconsistent and suffer from lack of closure.

(5) Use of balance of moment of moments balance law (Yang *et al.* [25], Surana *et al.* [26] [27] [41]) is shown to be essential in linear micropolar theories. This is also true in the case of nonlinear micropolar theories. In the absence of this balance law, the nonlinear micropolar theories are also nonphysical and the mathematical model suffers from lack of closure.

(6) In the derivation of the conservation and balance laws, $\boldsymbol{\sigma}^{*}$, $\dot{\boldsymbol{J}}$, and $\boldsymbol{m}^{*}$, ${}^{c\Theta}\dot{\boldsymbol{J}}$, are used for simplicity. However, in the final form of the conservation and balance laws, as well as in the constitutive theories, these are substituted in terms of $\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{m}^{[0]}$, ${}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}$ as they are the true measures for finite deformation/finite strain. $\boldsymbol{\sigma}^{*}$ and $\boldsymbol{m}^{[0]}$ and their conjugates are supported by theory of isotropic tensors. Determination of these conjugate pairs is an important aspect

of the work presented in the paper.

(7) All constitutive theories are derived using representation theorem to ensure that they are mathematically consistent.

(8) The nonlinear micropolar nonclassical continuum theory for thermoviscoelastic solid presented here incorporates two nonlinear mechanisms of dissipation, micro dissipation and macro dissipation. Both mechanisms use ordered rates of work conjugate quantities that appear in the entropy inequality. This is also an important and unique aspect of the work presented in this paper.

(i) The first viscous dissipation mechanism is due to ${}_s^d\boldsymbol{\sigma}^{[0]}$ and Green's strain rates $\boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \cdots, n$, standard nonlinear viscous drag forces between the material particles.

(ii) The second mechanism of nonlinear viscous dissipation mechanism is due to rotation rates of the microconstituents in the viscous medium, hence due to viscous drag forces between the microconstituents and the viscous medium. This mechanism depends upon the rotation rates of the gradients of microconstituent rotations ${}_c\boldsymbol{\Theta}$ up to order n, hence is also an ordered rate mechanism. Both nonlinear dissipation mechanisms reduce to linear dissipation for linear micropolar theories in which $\boldsymbol{J} \approx \boldsymbol{I}$ and $|J| \approx 1$.

(iii) The linear micropolar theory for small deformation, small strain physics for thermoviscoelastic solid is a complete subset of the nonlinear micropolar theory for thermoviscoelastic solid matter presented in this paper.

(iv) Model problem studies using this nonlinear micropolar theories and comparison with finite deformation/finite strain classical continuum theory will be presented in a follow up paper.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Nomenclature

$\bar{\boldsymbol{x}}$, $\bar{x}_i$, $\{\bar{x}\}$	deformed Coordinates
$\boldsymbol{x}$, x_i, $\{x\}$	undeformed Coordinates
ρ_0	reference density
ρ	density in Lagrangian description
$\bar{\rho}$	density in Eulerian description
η	specific entropy in Lagrangian description
$\bar{\eta}$	specific entropy in Eulerian description
e	specific internal energy in Lagrangian description
$\bar{e}$	specific internal energy in Eulerian description
${}_c\boldsymbol{\Theta}$, ${}_c\Theta_i$, $\{{}_c\Theta\}$	internal or classical rotations in Lagrangian description
${}_\alpha\boldsymbol{\Theta}$, ${}_\alpha\Theta_i$, $\{{}_a\Theta\}$	rigid rotations of the microconstituents
${}_t\boldsymbol{\Theta}$, ${}_t\Theta_i$, $\{{}_t\Theta\}$	total rotations in Lagrangian description
$\boldsymbol{J}$	deformation gradient tensor in Lagrangian description
${}_s\boldsymbol{J}$	symmetric part of deformation gradient tensor in Lagrangian description
${}_a\boldsymbol{J}$	skew-symmetric part of deformation gradient tensor in Lagrangian description
${}^d\boldsymbol{J}$	displacement gradient tensor in Lagrangian description
${}^d_s\boldsymbol{J}$	symmetric part of displacement gradient tensor in Lagrangian description
${}^d_a\boldsymbol{J}$	skew-symmetric part of displacement gradient tensor in Lagrangian description
${}^\Theta\boldsymbol{J}$	rotation gradient tensor in Lagrangian description
${}_c{}^\Theta_s\boldsymbol{J}$	symmetric part of classical rotation gradient tensor in Lagrangian description
${}_c{}^\Theta_a\boldsymbol{J}$	skew-symmetric part of classical rotation gradient tensor in Lagrangian description
${}^r_c{}^{\bar{\Theta}}_s\bar{\boldsymbol{J}}$	symmetric part of gradient of classical rotation rate tensor in Eulerian description
${}_c{}^\Theta_s\dot{\boldsymbol{J}}$	rate of symmetric part of gradient of classical rotation tensor in Lagrangian description
${}^r_c\bar{\boldsymbol{\Theta}}$	classical rotation rate tensor in Eulerian description
${}_c\dot{\boldsymbol{\Theta}}$	classical rotation rate tensor in Lagrangian description
$\boldsymbol{q}$, q_i,	heat vector in Lagrangian description
$\bar{\boldsymbol{q}}$, $\bar{q}_i$, $\{\bar{q}\}$	heat vector in Eulerian description
$\boldsymbol{v}$, v_i, $\{v\}$	velocities in Lagrangian description
$\bar{\boldsymbol{v}}$, $\bar{v}_i$, $\{\bar{v}\}$	velocities in Eulerian description
$\boldsymbol{u}$, u_i, $\{u\}$	displacements in Lagrangian description

$\bar{\boldsymbol{u}}$, $\bar{u}_i$, $\{\bar{u}\}$	displacements in Eulerian description
$\boldsymbol{P}$	average stress in Lagrangian description on the oblique plane of elementary tetrahedron
$\bar{\boldsymbol{P}}$	average stress in Eulerian description on the oblique plane of elementary tetrahedron
$\boldsymbol{M}$	average moment in Lagrangian description on the oblique plane of elementary tetrahedron
$\bar{\boldsymbol{M}}$	average moment in Eulerian description on the oblique plane of elementary tetrahedron
$\boldsymbol{\sigma}^{(0)}$, $\sigma_{ij}^{(0)}$, $\left[\sigma^{(0)}\right]$	Contravariant Cauchy stress tensor in Lagrangian description
$\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\sigma}_{ij}^{(0)}$, $\left[\bar{\sigma}^{(0)}\right]$	Contravariant Cauchy stress tensor in Eulerian description
${}_s\boldsymbol{\sigma}^{(0)}$	symmetric part of Contravariant Cauchy stress tensor tensor
${}_a\boldsymbol{\sigma}^{(0)}$	anti-symmetric part of Contravariant Cauchy stress tensor tensor
${}_s^d\boldsymbol{\sigma}^{(0)}$	deviatoric part of the symmetric Contravariant Cauchy stress tensor tensor
${}_s^e\boldsymbol{\sigma}^{(0)}$	equilibrium part of the symmetric Contravariant Cauchy stress tensor tensor
θ	temperature in Lagrangian description
$\bar{\theta}$	temperature in Eulerian description
k	thermal conductivity in Lagrangian
p	thermodynamic or Mechanical Pressure in Lagrangian description
$\bar{p}$	thermodynamic or Mechanical Pressure in Eulerian description
$\boldsymbol{g}$, g_i, $\{g\}$	temperature gradient tensor in Lagrangian description
$\bar{\boldsymbol{g}}$, $\bar{g}_i$, $\{\bar{g}\}$	temperature gradient tensor in Eulerian description
$\bar{\boldsymbol{L}}$	velocity gradient tensor in Eulerian description
$\bar{\boldsymbol{D}}$	symmetric part of the velocity gradient tensor in Eulerian description
$\bar{\boldsymbol{W}}$	Skew symmetric part of the velocity gradient tensor in Eulerian description