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Two Letters on Music of Giovanni Battista Benedetti to Cipriano de Rore. Part I: Introduction and Commentary

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Abstract

In two letters to the famous musician Cipriano de Rore, Giovanni Battista Benedetti discussed topics related to music theory. The letters were published in *Diversarum speculationum mathematicarum & physicarum liber* in 1585. In them, Benedetti clearly proved that the pitch of a note in a polyphonic piece can rise or fall by a syntonic comma when performed with just intonation; he also proposed a form of temperament to avoid the problem, formulated the law of vibrating strings, and suggested that the pitches of notes can be related to the frequency of their air pulses. The letters were read and commented upon by scholars immediately following Benedetti, and continue to be the subject of in-depth studies by historians of music and science. Despite their relevance, there is no systematic study of them. Only some parts have been commented on by various historians, following their interests but without a unified view. Most importantly, no complete English translation has been proposed with a faithful transcription of the Latin text. The present work, divided into two parts, the first a commentary on all the salient aspects, the second the translation and semi-diplomatic transcription, intends to fill the gap.

Keywords

Theory of Music, History of Acoustic, Vibrations of Strings, Migration of Pitches

1. Introduction

In 1585, the mathematician and natural philosopher Giovanni Battista Benedetti (1530-1590) made a significant contribution to the ongoing discussion of

Renaissance music about the structure of scales with his famous treatise *Diversarum speculationum mathematicarum & physicarum liber* (Benedetti, 1585). The last part of the work consists of a large collection of letters (epistulae), arranged according to the importance of the addressees, including some who were already deceased. Among the letters are two written to Cipriano de Rore (c. 1515-1565), both of which concern music. While it is possible that these two letters were written before De Rore's death, whether they were actually intended to be sent is a difficult question to settle. Benedetti and de Rore were both members of the Farnese court in Parma for a number of years (Benedetti was in Parma for about eight years from at least 1560; de Rore, a generation older, was at the same court from at least 1561 until his death, except for a brief interlude in Venice (Owens, & Schiltz, 1985). It is therefore reasonable to assume that the two were acquainted). Beyond any possible personal connection or direct esteem, however, Benedetti may have had an additional motive for his choice of addressee. De Rore was one of the most famous musicians of his time, an innovator: his madrigals had influenced Monteverdi and the Florentine camerata around Giovanni de Bardi; for many he was the foremost exponent of the *seconda pratica*, the art of text and music reflecting and influencing each other (Walker, 1978: p. 79).

Very little is known about Benedetti's life, and only a little more about his scientific works (for a biography see Bordiga, 1985 and Field, 1987; Choen, 1987; Roero, 1997). Beyond the letters discussed in the present paper, there is no evidence that Benedetti had any particular interest in music. We can reasonably assume that he was in the mainstream of Renaissance music, after Zarlino, Fogliano and de Rore. However, the two letters to De Rore show that Benedetti must have had some training and knowledge of the art, to the point that he may have been a composer himself, which would not be too surprising given Benedetti's status and education and the value placed on the discipline of music at the time.

Benedetti was a creative mathematician, belonging to the generation that had rediscovered and appreciated Archimedes. He believed that mathematics was a branch of philosophy and a superior form of knowledge. This epistemological position is clearly testified in a letter to Domenico Pisani (Bordiga, 1985: p. 635), and from his own comments in the *Diversarum speculationum mathematicarum & physicarum liber*:

> I am amazed that, however well versed you are in Aristotelian philosophy, you nevertheless in your writings make a distinction between philosopher and mathematician, as if the mathematician were not as much a philosopher as the natural one and the metaphysician. Actually, as far as the certainty of conclusions is concerned, he deserves the title of philosopher much more than them (Benedetti, 1585: p. 298).

Benedetti's letters, were read and commented on by scholars immediately following him; it is certain, for example, that they were known to Isaac Beeckman (Beeckman quoted Benedetti in his diaries and explicitly mentioned the letters, Beeckman, 1939-1953: vol. 3, p. 275), but they have also recently been the subject of significant study by music historians (see, for example, Duffin, 2006; Duffin, 2017; Lindley, 2001; Palisca, 1985; Palisca, 1994), because they were written by a professional mathematician and, moreover, a leading figure in the study of the new mathematics. Other students of the new mathematics had previously written about music, such as Gerolamo Cardano (1501-1576), a generation before Benedetti. However, the content of Cardano's musical writings is largely independent of his mathematical thought, but rather representative of the broad tradition of musical treatise writing and subject matter that predates Zarlino, see *De musica* in (Cardano, 1568: vol. 10). On the other hand, the letters have received little attention from historians of science, probably because they deal with issues of musical theory, far removed from the disciplines of mechanics and mathematics in which Benedetti is a leading figure.

Comments and translations of certain parts of the letters considered relevant have been published (see Palisca, 1994: pp. 213-223; Palisca, 1985: pp. 257-264; Duffin, 2006; Duffin, 2017). The attention of music historians tends to focus on the second letter, especially the passages that illustrate critical issues in the adoption of just intonation by singers. Historians of science are also aware of the letters to de Rore because of their inclusion in a treatise fundamental to the development of mechanics, and because the last lines of the second letter deal with questions of acoustics and dynamics (Capecchi & Capecchi, 2022).

As far as the content of the letters is concerned, it can be said that Benedetti's work was original for his time, and many historians believe that it led directly to the revolutionize and overthrow of the then dominant system of just intonation by clearly showing its main drawbacks for vocal music and keyboard tuning. In truth, it has to be said that no convincing argument has ever been put forward by the historians, as to the role of Benedetti's letters in the abandonment of just intonation. Apparently, the letters were largely ignored by contemporary musicians, while they were of interest to the mathematicians of the day.

The major objective of this study is to present a comprehensive English translation of the letters, which have previously only been fully accessible in the original Latin. Nevertheless, commentary will be provided to facilitate a more comprehensive understanding of the content. While some elements, such as the migration of pitches, may require minimal clarification and have been extensively illustrated in other literature, there remain other aspects that can still be elucidated. A further rationale for providing a commentary is that, hitherto, there has been no comprehensive framing of the letters, given that only parts of them have been commented on by various historians, following their respective areas of interest but without a unified view.

2. Comments to the Letters on Music

The first letter is concerned with just intonation in polyphonic vocal music, and the measure of the size of tones and semitones in this repertoire. The second letter discusses the migration of pitches in a just intonation system, the tuning of keyboards, the law of vibrating strings, and puts forward an explanation for the nature of the pitches of notes.

2.1. First Letter

At first glance to a modern reader, the first letter to de Rore may seem to have been written by a teacher to a student or amateur with the intention of communicating with people who had little musical skill or knowledge, as its contents include some very basic information about music theory that a renowned musician would surely be expected to know. However, the context of the letter must be taken into account; it was written at a time when music theory and practice had just embraced the just intonation rather than the Pythagorean intonation. In this transitional period, music theorists (such as Ramos, Gaffurius, Spataro, Fogliano, and Zarlino) had not yet reached an agreement on the formation of a musical scale and how the various tones and semitones should be positioned and adjusted. In this context, the first letter is of great value.

It is not easy to give a reason for this letter, as well as for the other, considering that there are no documents written by Benedetti from which we can know his ideas on musical matters. For example, Palisca argues that the first letter shows how non-consonant intervals are created by introducing chromatic changes in a diatonic system (Palisca, 1994: p. 214), while Duffin thinks that Benedetti's goal is to illustrate the different sizes of intervals that are part of the just intonation system (Duffin, 2017: p. 240). To us, Duffin's hypothesis seems more likely, but we think it is not entirely satisfactory. If Benedetti's goal was to find out which intervals are used in the just intonation, he would have chosen more examples, perhaps even from different authors, to find other intervals as well, such as a fourth of a semitone. We propose a different interpretation that accounts that Benedetti was one of the promoters of the new science, where great attention was paid to the empirical data that should be the basis of any scientific analysis, even in the field of music. It is therefore probable that the first letter simply serves to illustrate the approach he intended to follow in the analysis of the second letter, for example.

This approach consists in examining the musical production of his time, not so much that of the theorists, but that of the practitioners, who compose a musical piece with more regard for the pleasantness of the music than for the rules of just intonation of the theorists, and who probably do not even know the exact sizes of the intervals they use in their compositions. Unlike the music theorists, who promoted a prescriptive approach to define the sizes of whole tones and semitones (usually by defining them as harmonic or arithmetic means of other intervals), Benedetti used a more observational approach (that of Archimedes' mixed mathematics), starting from a limited number of physical principles arrived at through acoustic experience and considered certain. A similar conclusion was reached by Beeckman, for whom the intervals of any scale are derived from the difference between consonances, and not from any mathematical principle assumed a priori, as in Descartes, for example (Benedetti, 1585: vol. 3, p. 136).

In music, Benedetti derived his principles from aural experience: the recognition of some musical intervals as consonances. These consonances are the octave and the fifth (perfect consonances), the fourth, the major and minor third, and the major and minor sixth, represented by the following ratios: (2:1), (3:2), (4:3), (5:4) and (6:5), (5:3) and (8:5). In analyzing certain pieces, Benedetti emphasizes some non-consonant intervals, in particular the major (9:8) and the minor (10:9) whole tones, and what he called the *maius* semitone (27:25), the *minor* semitone (16:15), and the *minimum* semitone (25:24) (modern nomenclature changes for semitones, (16:15) is called the major semitone and (25:24) the minor semitone; (27:25) is sometimes called the maximum semitone). These values and nomenclature are also those adopted by Fogliano in his 1525 *Musica theorica* (Fogliano, 1529, ff. 18v-19r, and f. 31v). Benedetti's results could thus be seen as a confirmation of Fogliano's approach to just intonation.

Figure 1. Benedetti's examples, first letter, above; a piece from Hellas comment, below.

To illustrate Benedetti's approach, it is sufficient to consider two of the seven three-part musical examples (separated from each other by vertical lines, which does not identify bars but simply indicate the end of an example) presented in his letter (Benedetti, 1585: p. 278), which are transcribed and shown in **Figure 1**. These two examples are inspired by a passage from Cipriano de Rore's madrigal *Hellas comment* (De Rore, 1569, **Figure 1**, below). In accordance with the common practice of the time, Benedetti assumed the vertical coincidence of the three voices, bass (B, lower voice), tenor (T, middle voice), and superius (S, higher voice); and then deduced the consequences of their performance in just intonation.

Benedetti's choice of the madrigal *Hellas comment* as a reference seems to be caused by having the E-E♭ chromaticism in the higher voice (B-B♭) in Benedetti's version), partly to illustrate the different sizes of semitone intervals, and also because it was a characteristic example of new music, chromatic and with a false harmonic relationship (see the clash between the G major and E♭ major chords in the second Benedetti's example, as well as the similarly striking movement from A minor to C minor chords in de Rore).

In the first example, one can see two intervals of a tenth between the lower voice (bass) and the upper voice (superior), as the former remains at a G, while the latter drops from B to B♭, (note that in the original Benedetti's staff B is written as ♯; which when transcribed into modern notation is read as B) producing a major tenth (5:2) that passes to a minor tenth (12:5). The difference between the two intervals is given by 5/2:12/5 = 25/24, which according to Fogliano's definitions (accepted by Benedetti) is a *minimum* semitone. The same minimum semitone is also found in the second example. In analyzing the remaining five examples (shown in the letters), Benedetti emphasized the presence of the major and minor whole tones (9:8) and (10:9), as well as two other semitones (16:15) and (27:25). No other intervallic relationships are identified.

2.2. Second Letter

Benedetti's second letter deals with four main topics: the tendency of pitches to migrate up or down during vocal performance, the temperament of keyboards and fretted instruments, the law of vibrating strings and a physical explanation for the cause of consonances. The first topic has been treated extensively and generally satisfactorily (Palisca, 1985: p. 263; Duffin, 2017). The second has received less scholarly attention (Lindley, 2022: p. 15), and the third has been treated, but not in depth, by historians of music (see Palisca, 1985: p. 259) and historians of science (Truesdell, 1960: pp. 22-23). In the following, these three topics will be discussed anew, but only the basic aspects will be emphasized, leaving the reader free to read the letter to de Rore on its own merits.

2.2.1. Migration of Pitches

It was well known to Renaissance musicians that when several consonant intervals are played in succession, both in Pythagorean and just intonations, notes are obtained that cannot be contained in any scale, however rich, and therefore cannot be reproduced by fixed-pitch instruments such as keyboards. These notes can, however, be reproduced by the human voice, albeit with some inconvenience, in particular the "fixed" pitches wander throughout a piece. The consequences of this phenomenon were ignored by the theorists of the time. For example, Zarlino wrote:

> But if the voices were proportionate to each other, and well united, without any obstacle; and were uttered by the cantors with some discretion, & with good sense; so that one voice did not prevail over the other; I firmly believe

> that such intervals would be perfectly heard, & that the listeners would take no little pleasure, & satisfaction with the chants that they heard (Zarlino, 1562: p. 136).

Benedetti's goal was to fully understand this phenomenon by considering three-part musical pieces that follow just intonation; however, his considerations and conclusions apply to any polyphony performed with just intonation. To achieve his goal, Benedetti considered two pieces that he had composed specifically to highlight the critical issues of just intonation. In the first example, we can see that the pitch of the notes is gradually shifted up, while in the second example, it is shifted down.

To follow Benedetti's reasoning, it is sufficient to look at the four chords of the first two bars of the first piece which are shown in **Figure 2** for the sake of clarity. If we follow the superius S [the highest voice], we can see that it goes from a G in the first chord to an A in the second, thus rising by a major tone (9:8) as it goes from a fourth to a fifth in relation to the tenor T. The superior keeps A in the third chord, forming a fourth with the tenor, and then drops to a note X, forming a minor third with the tenor. It is not difficult to see that X is not the same as the G of the first chord, although it is very close to it. In fact, if we calculate the distance of X from A, starting from the tenor E, we can see that AX = EA (4/3): EX (6/5) = 10/9, which is a minor whole tone. Since the distance of G from A is that of a major whole tone (9/8), X is higher than G by the interval 9/8:10/9 = 81/80, known as *syntonic comma* (since the difference is small, X can still be called G, as Benedetti did; this is also natural because at the time there were no way to get the absolute pitch of a note (for a formal definition of a note as an interval of pitches not very different from each other, see Lindley and Turner-Smith (1993: p. 20)).

This upward migration is also reproduced in the following four chords, and could easily be reproduced for other subsequent chords, resulting in an upward migration of 10 commas or more, so that X eventually differs only slightly from A. This migration also affects all voices, not just the highest. As a result, the pitches of all the notes and harmonies in a piece can easily and unintentionally be raised by singing in just intonation, even though they remain unchanged in the written music.

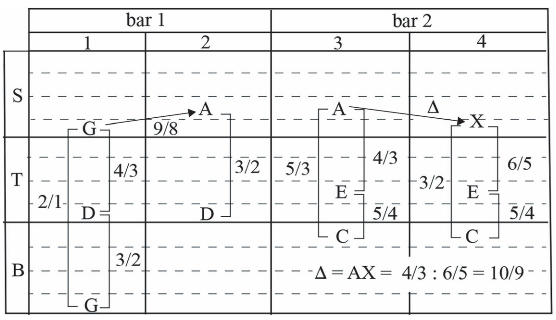

Figure 2. Intervals of the four chords of the first example. A schematization with interval ratios, adapted from (Benedetti, 1585: p. 279).

A modern reader may be surprised that the Renaissance music theorists, the proponents of just intonation, did not emphasize this migration of pitches before Benedetti, given the evidence for the phenomenon and how well known and documented it is to any modern student of music theory (see, for example, Blackwood, 1529). Note that this migration occurs not only in polyphony, but also in monophonic music, as clearly illustrated in Huygens (1888-1950a, vol. 20, p. 77), Sauveur (1707: p. 207).

There are several plausible explanations for the mystery of how it was possible to use just intonation, despite its instability and the impossibility of practicing it in principle. One, and perhaps the most convincing, is that singers instinctively made the micro-adjustments necessary to avoid migration, and thus the phenomenon was not noticed in practice. This the explanation Huygens gave in his *Cosmotheoros* (Huygens, 1888-1950b: p. 755).

Another compelling argument can be made for the possibility that singers simply did not sing with just intonation (Keller et al., 2022) In Walker (1996), it is shown that essentially just intonation can be maintained in choral music with the precise introduction of syntonic commas in a few places. Duffin (2006, 2017), in two interesting papers, suggests some strategies for solving what he calls Benedetti's conundrum. According to him in problematic passages, and even in Benedetti's examples, it is possible to devise solutions that follow the principles of just tuning, modifying only one or two pure intervals rather than the entire system, as temperament does. Moreover, many composers lacked the theoretical background necessary to recognize the problem in the first place.

2.2.2. A Tuning Procedure

In the second part of his second letter, Benedetti addressed another fault of just intonation closely related to pitch migration. To explain it, he imagined the superposition of three just fifths (3:2). The resulting interval (27:8), or (27:16), when reduced to the octave, is not just but exceeds the just major sixth by a syntonic comma (81:80), similarly to what happens in the migration of pitches for a polyphonic performance. To avoid this fault, Benedetti suggested that the above three fifths be narrowed by 1/3 of a comma to make them imperfect and make the major sixth just. Benedetti claimed to be the first to find the emergence of a syntonic comma against just intonation with a cycle of three fifths, and to suggest to narrow the fifths by 1/3 of a comma to avoid the inconvenience.

He then proposed a method for tuning keyboards based on his findings, which is illustrated by the virtual keyboard shown in **Figure 3** (with note names instead of keys). The keyboard consists of three octaves, from G_1 to a G_4 (Benedetti, 1585: p. 281). The symbol ♭ indicates a flat, and ♯ a sharp; the numbers under the note indicate the octave to which it belongs. For example, the first ♭ is for the note B_1♭ (in the first octave) while the third ♯ is for G_2♯ (in the second octave).

In any procedure of tuning based on narrowed fifths, it is impossible to know in advance how much to narrow them (unless you have an electronic tuner), so it is necessary to check tuning against intervals that are known to be just. Often the

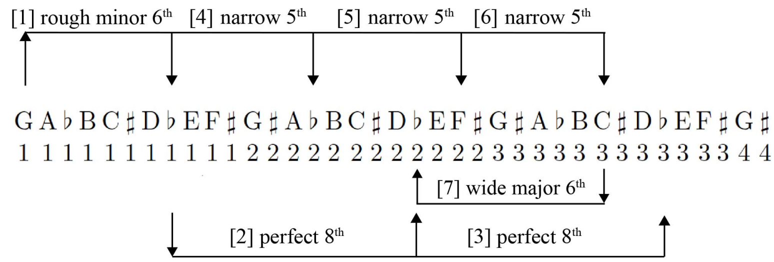

Figure 3. First temperament steps.

check is made against major or minor thirds. Benedetti proposed instead a check against the major sixth for any three cycles of fifths; this approach is not common because sixths are more difficult to recognize than thirds (at least this is the opinion of practitioner musicians; Benedetti however at the end of the second letter considered the major sixth more agreeable than major and minor thirds). However, contrary to what one might expect from his previous discussion, the check is not made with a really just sixth, but with a tolerably widened one.

In order to explain Benedetti's move, we must consider that he should have realized that his proposal would result in a 1/3 comma temperament, which practical musicians did not consider very suitable for a keyboard, because although it contains just minor thirds, it also contains exceedingly narrow major thirds (Panetta, 1987: p. 19).

Let now consider in details Benedetti's tuning procedure. He started from the note G_1, whose pitch is not specified, and then tuned (roughly) the key E_1♭ for a minor sixth. This is not an arbitrary move; it is the same one carried out by practitioners; see (Panetta, 1987: p. 71). From this note he got E_2♭ and E_3♭ moving forward for octaves. Then a standard tuning approach is followed starting from E_1♭. After a cycle of three fifths he arrived to C_3 and proposed checking whether the interval between E_1♭ and C_3, is a tolerably widened major thirteenth, or which is the same, after subtraction an octave, whether the interval between E_2♭ and C_3, is a tolerably widened major sixth. If this is not the case, the narrowing of the fifth must be changed and the cycles repeated. The procedure is continued by moving forward by fifths and backward by octaves arriving to G_4♯ and assigning notes to 19 keys (over 35). The remaining keys are assigned a note moving with forward and backward octaves.

It is not possible to establish whether Benedetti was tuning his keyboard by looking at a known temperament, or whether he was proposing one of his own, possibly only on empirical basis, also because Benedetti's reading of music theory is not known. Leaving aside the 1/3 of a comma temperament, the other known temperaments at the time were the 1/4 and the 2/7 of a comma; but Benedetti did not name them. It seems reasonable to posit that the only certain assertion that can be made is that Benedetti employed a temperament with fifths that were narrowed by a value of less than 1/3 of a comma, basing on his concept of comma generation.

Although Benedetti's tuning is not commented on in detail in the literature, there are conflicting attempts to assign a label to it; for example, Palisca (1985: pp. 264-265), favors an equal temperament (possibly because the 12th fifth from E_1b, i.e. the fifth from G_4♯, may close the circle falling near E_1b, the starting note, and this is typical of equal temperament), while Lindley (2022: p. 22), Lindley, (1986: p. 61) favors a temperament of 2/7 of a comma, probably because among Zarlino's writings, Benedetti quoted only the *Istitutioni harmoniche* of 1558, where the named temperament is exclusively that of 2/7 of a comma.

2.2.3. The Law of Vibrating Strings and Consonances

One possible reading of the last part of Benedetti's second letter is that he explicitly formulated the law of vibrating strings, something that had never been done before. While this may be true, it may not be the most interesting point; in fact, certain assumptions of acoustic theory are probably more interesting (at least to a musician) than the proof of the law, which is presented below.

A modern reader is puzzled by the idea that the proof of a law concerning the dynamics of vibrating strings, which could possibly be verified without recourse to a sound, could appear in a text on music theory. For Benedetti's contemporaries, however, the *vibrating string* was representative of the monochord, the primary tool of music theory and acoustics, and the study of its vibrations was closely related to the sound it produced. Therefore, Benedetti's perspective and its use in this text could not have been so strange to them.

In approaching this study of the vibrating string, Benedetti began with a series of assumptions, not all of them made explicit. Some of them are experimental, some derived from the natural philosophy of the time (and as such are questionable), some mechanical, and the rest acoustic. They are:

a) The vibration of a string is periodic, let T be that period.

b) The pulses of air produced by the vibration of such a string follow one another at intervals equal to the period T.

c) The shorter the string, the shorter the period T.

d) The shorter the string the higher the pitch.

e) The periods T_1 and T_2 of two consonant notes should be in a simple ratio when there is consonance.

f) Pairs of vibrating strings with length ratios (2:1), (5:3), (8:5), (3:2), (4:3), (5:4), (6:5) produce consonant sounds.

Assumption (a) is a mechanical assumption, and also has an experimental character, but in Benedetti's time its verification was very difficult, especially when observing shorter strings. Although it can be said with relative certainty that Benedetti did not personally experiment with vibrating strings (at least there is no evidence of this), he had been involved in some engineering problems at the Savoy court, as well as in the construction of scientific instruments (Bordiga, 1985: p. 599), and therefore had a mechanistic view in the interpretation of physical phenomena. It is probably a sum of these experiences that led Benedetti to the statement of assumption (a). This is made explicit by him at several points in the text, with locutions of the kind "interval of oscillation" (period) or "number of intervals" (frequency) (Benedetti, 1585: p. 283). However, Benedetti was probably unaware of the fact that assumption (a) also implies the isochronism of oscillations.

Assumption (b) can easily be derived from the acoustic theories of the time. In ancient Greece and in the Renaissance, natural philosophy presented two competing explanations of the propagation of sound, associated with the disturbance of air caused by some kind of impact, such as the rubbing of bodies or the vibration of strings.

1) The theory of pulses. It is claimed that the vibrating body does not produce any displacement of the parts of air toward the ear, but that the propagation of sound is a phenomenon similar to the transmission of the ripples produced by a stone dropped on the surface of a pond. A very clear and well-known exposition of this impulse theory can be found in the musical writings of Boethius (1948: p. 292).

2) The missile theory. This theory states that the vibrating body moves particles of air to the ear, where the particles are perceived as sound.

From the Middle Ages to the Renaissance, the pulse theory was more widely known and accepted; but with the rediscovery of the Greek atomism, the theory of missiles gained greater credence. As for Benedetti, he was a proponent of the pulse theory, which he called "air waves" (Benedetti, 1585: p. 283); it is not known if he was also familiar with the missile theory. The latter was adopted by Isaac Beeckman (1588-1637), who took up Benedetti's studies a few years later (Beeckman, 1939-1953: vol. 1, p. 92). Regardless of the assumed theory, it was also generally believed that the vibrating body produced a disturbance in the air corresponding to it. Benedetti's novelty was to specify this assumption referring explicitly to the periodicity of the impulses.

Assumption (c) is easily derived from the rather trivial (for a modern) statement that a shorter body moves faster than a longer one, as found in Aristotelian thought (*Problemata physica*, 4.23, see Barker, 1989: vol. 2, p. 94); and Aristotle's *De generatione animalium*, 786b (Barker, 1989: vol. 2, pp. 80-84). In the case of periodic motion, it can be postulated that the period is inversely proportional to the speed of the motion. Therefore, it can be inferred that a shorter string will have a lower period.

While this assumption works very well for Benedetti's arguments, it is generally incorrect:

1) It is not true that strings move more easily the smaller [shorter] they are; on the contrary, they move more easily the larger [longer] they are, as simple experiments clearly show.

2) It is not generally true that the greater the speed, the greater the frequency, because the speed depends not only on the frequency but also on the amplitude of the vibrations.

Assumption (d) is based on empirical evidence and therefore falls within the domain of musical acoustics. In conjunction with assumption (c), it suggests that the pitch of a sound produced by a vibrating string is dependent upon its period of vibration; more precisely the lower the period the higher the pitch.

Assumption (e), more commonly known as the coincidence theory, has its roots in natural philosophy; it can be found in ancient Greek texts such as the Aristotelian *Problemata physica* (Barker, 1989: vol. 2, p. 96) (for a history of the concept of coincidence see Barbieri, 2001). According to the coincidence theory, there is a consonance between two notes if and only if there are also coincidences of air pulses: "Nor can I fail to see in the established position, a way of speculating on the generation of simple consonances, that there really is some equality of percussion, or [seu] an equal concurrence of air waves, or [vel] rather of their termination" (Benedetti, 1585: p. 283) (Note that the equal concurrences of the air pulses and the equality of their moment of rest are assumed to be equivalent; this last condition leaves less ambiguities in the application than that of equal concurrence for waves of different shape).

Assumption (f) is empirical, belonging to musical acoustics. Its formulation here is evidence of Benedetti's Aristoxenian empirical approach to the explanation of consonances.

Once these assumptions have been made, it is possible to derive the law of the vibrating string (i.e. the formula relating the period to the length) in a manner that is both elegant and smart. Consider two strings of length $l_1 > l_2$, whose ratio is, for example, $l_1/l_2 = 2$. According to assumption (e), this ratio will produce a consonance, while assumption (d) implies that the periods of the air pulses T_1 and T_2 associated with the vibrations of the two strings must have a ratio expressed by simple numbers. If assumption (c) regarding $T_1 > T_2$ is allowed, then the most natural choice is $T_1/T_2 = 2$. However, for assumption (b) the period of the air pulses is equal to that of the vibrating string, for which the ratio of the period of the vibrating strings is still $T_1/T_2 = 2$, which is equal to the ratio l_1/l_2 between the lengths of the strings. This specific example is then generalized by Benedetti to the case where the ratio is any:

> So the proportion between the numbers of intervals (The number of intervals, that is the number of periods or semi-periods, in a fixed time is proportional to the frequency of vibration of the string) of the minor portion to the major and of the length of the major portion to the minor length will be the same (Benedetti, 1585: p. 283).

Thus, Benedetti demonstrated an unprecedented ['unprecedented' is used in the sense that it was a significant achievement, and to emphasize this significance] physical explanation for the Pythagorean intervals: if there is a consonance for a given ratio m/n for two lengths of string, this consonance is caused by the ratio of the frequencies of vibration of the two strings, and hence of the air pulses, being n/m.

It is usually assumed that this relationship holds for any ratio between the lengths of the strings, whether rational or irrational. Benedetti, however, did not seem to be interested in this generalization, which cannot be proved; he was content to show its validity for ratios that give rise to consonant intervals, and only for these. It should be noted that Benedetti's proof is based on a number of assumptions which, while convincing, are not necessarily true. A more convincing proof, based on the "incontrovertible" principles of mechanics and therefore valid outside a musical context, would be given later by figures such as Beeckman and Mersenne, using concepts of force and velocity typical of the new mechanics (Capecchi, 2018: pp. 196-203).

As mentioned above, Benedetti's letter contains some very interesting ideas about acoustics in addition to the law of vibrating strings. Benedetti's assumption (c) and (d), are innovative and valuable, even if their novelty is not made fully explicit in the letter.

The traditional explanation of the difference between high and low pitches, as already noted, was to associate them with higher or lower speeds of either the pulse (pulse theory) or the air particles (missile theory). There are passages in Euclid's *De sectio canonis* (Barker, 1989: vol. 2, p. 292) and in Boethius' *De istitutione musica* (Boethius 1948: p. 293), that seem to associate the peaks and tones with the transmission frequency in the modern sense, but these passages are not entirely clear. The same is true for Gerolamo Fracastoro (1476/1478-1553) (Palisca, 1985: p. 255). Benedetti made a step in favor of associating pitches with frequencies. While his may not represent a comprehensive understanding of the periodic nature of sound (notice that in assumption (a) *period* probably refers to the temporal distances between two pulses moving forward to the ear, rather than the time taken for the forward and backward movement of an oscillating pulse, as is usual in acoustics), it served as an important catalyst for further investigation.

This conclusion, derived from strings, can also be applied to sounds produced by other bodies, such as wind or percussion instruments. Thus, it leads to a general theory of the difference between high and low pitches and consonances: the value of pitches depends on the frequency of vibration of the air, and consonances occur when these frequencies are in simple ratios. Benedetti's implicit justification of the value of pitches from the frequency of air pulses was recognized by his immediate successors, such as Beeckman and Mersenne, and was also made explicit by Galileo Galilei in the *Discorsi e dimostrazioni matematiche sopra due nuove scienze*:

> I say that the length of the strings is not the direct and immediate cause of the sizes of the musical intervals, nor their tension, nor their thickness, but rather the ratio of the number of vibrations and impacts of the air waves that go to strike our eardrum, which also vibrates according to the same measure of time (Galilei, 1638: p. 146).

Both Beeckman/Mersenne and Galileo reflected more than Benedetti on the mechanism of wave propagation, and by assuming a constant speed for sound (this is certainly true for Mersenne) were able to associate the frequency of the cycle of rarefaction-compression of the air near the eardrum with pitches.

At the very end of his second letter, it is generally accepted that Benedetti proposed (if somewhat meekly) a criterion for judging the pleasantness of the various coincidences, based on the ratio of the lengths of the strings to the frequencies, for any given interval, "these numbers are not without an admirable regularity" (so the association of admirable regularity and pleasantness is not explicit.) (Benedetti, 1585: p. 283). The various consonances, octave, fifth, fourth, major third, minor third, etc., expressed by the ratios in radical form (2:1), (3:2), (4:3), (5:3), (5:4), (6:5), suggested to Benedetti that the product between the denominator and the numerator could be used to establish an order of pleasantness; in the cases considered above, the numerical values are 2, 6, 12, 20, 30. Note that using this criterion, the thirds would be "objectively" less pleasant than the fourths; an idea that has been questioned by most contemporary musicians.

Although Benedetti was not very clear about the "admirable regularity" of ratios, it is worth remembering that it was generally accepted by his contemporaries that the ratio of small numbers produces more pleasing consonances. Even though the orderer proposed by Benedetti, with the fourth (4:3) and the major sixth (5:3) preceding the major third (5:4), was not generally accepted, not even by Zarlino. Benedetti was probably of inspiration to Leonhard Euler who in his *Tentamen novae theoriae musicae* of 1731 (published 1739) assumed as a parameter influencing the suavitas of a ratio $(n:m)$ in its radical form the least common multiplier of m and n, i.e. $n \times m$ to which he called the *exponent* of the ratio (Euler, 1739: Chap. 4).

3. Concluding Remarks

There are several reasons why Benedetti's letters have been considered of great importance both by his contemporaries, especially mathematicians and modern historians, especially musicians. Benedetti was an esteemed representative of the new science and the new rigor of demonstrative proof. This novel method of applying mathematics and physics to music led to Benedetti's clarification of the phenomenon of the migration of pitches, resulting that could not be achieved by musicians of the time. Finally, Benedetti's demonstration of the law of vibrating strings cannot be overlooked, a truly novel scientific conclusion, presented in a clear and convincing manner, that would be referred to and reapplied in the search for an explanation of the nature of the pitches of notes for the next two centuries. The union of two different skills, scientific and musical, was undoubtedly the key to these achievements, and subsequent studies of Benedetti's letters, carried out separately by musicians and historians of science, have often failed to adequately highlight certain important features of Benedetti's work that are more closely related to the other discipline.

4. Fundamental Musical Notions

[Note]

A note is a sound that remains unchanged over time. Each note can be viewed into two different ways, either as a physical magnitude or as a feeling of sharpness. In the first view, a note is a periodic sound characterized by a constant fundamental frequency f_0. In the second view, we speak of pitch. Pitches depend on frequency, but there is no simple relation between frequency and pitch.

The notes allowed in Western music are a numerable infinity. The frequencies within which they can actually vary range from a minimum f_m to a maximum f_M.

[Octave]

An octave is a part of the space of frequencies $f_M - f_m$ such that the ratio between the final and initial values of the frequencies defining it is equal to 2; it is currently assumed that there are 10 contiguous octaves. The first denoted by C_0 conventionally starts at frequency 16.35160 Hz. Subsequent octaves are denoted by C_i, with $i = 1, 2, \cdots, 9$. To give an idea of the range of frequencies we are dealing with, consider the keyboard of a piano. The octaves that it can play are those from C_1 to C_8, with frequencies ranging from 32.7 to 8351.54 Hz (there are only a few notes in C_0 and C_9). Consider that the threshold of audibility varies from about 20 to 20,000 Hz.

Within each octave, Western music, as it has developed from ancient Greece to the 20th century, assumes only seven fundamental notes, plus other intermediate notes called *altered notes* (flats and sharps), the number of which varies according to historical period.

[Scale]

A scale is a set of notes ordered according to their pitches. The fundamental notes are denoted by the *seven* symbols C D E F G A B, which are derived from the notation established in the 10th century CE; alternatively, in neo-Latin-speaking countries, the symbols Do Re Mi Fa Sol La Si are used. A scale is thus the sequence of the fundamental notes C, D, E, F, G, A, B and the altered notes delimited by them.

In the international system of symbol notation (IPN), notes of the same octave are denoted by the same subscript, so it is C_0 D_0 E_0 F_0 G_0 A_0 B_0 for the octave beginning with C_0. Particularly interesting is the middle octave beginning with C_4: C_4 D_4 E_4 F_4 G_4 A_4 B_4, where $A_4 = 440$ Hz.

The notes are represented by dots or circles arranged on a series of parallel lines from bottom to top to give a more or less sharp image. However, there are too many notes in the interval C_0-C_9, and too many lines (70) would be needed to represent them all. The problem is solved by dividing the entire C_0-C_9 space into windows of five lines, called staffs, whose position within the entire frequency space is defined by symbols called clefs. The most important of these today are the violin or G clef and the bass or F clef, as illustrated in Figure 4.

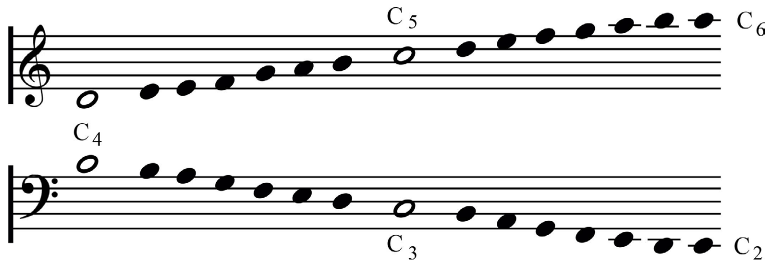

Figure 4. Clefs of G (above) and F (below).

As clear from the figure, each key by extending the staff rows above and below can represent up to a maximum of two octaves. Figure 5 reports all octaves to give the whole picture.

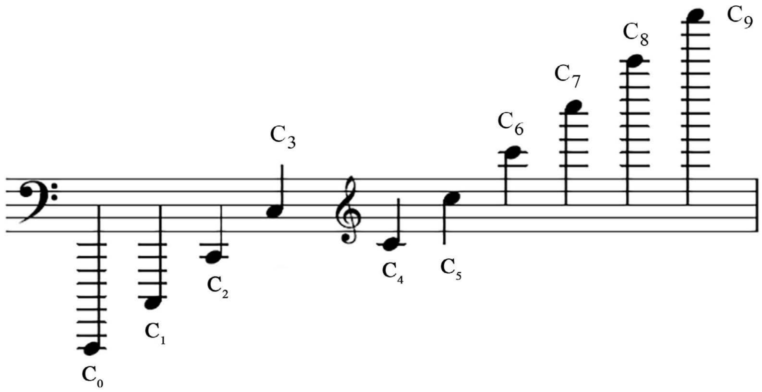

Figure 5. The whole system of octaves.

[Intervals]

An interval is the distance between (two notes or) two pitches p_1 and p_2. These distances can be given by ordinal numbers: second, third, fourth, fifth, etc., which indicate the distance between two notes in the staff. Two consecutive notes are separated by an interval of a second (or more simply by a second), such as C and D; they form an interval of a fifth (or more simply a fifth) if they are separated by five degrees on the staff, such as C and G.

The distance can also be measured basing on the numerical value of frequencies of the notes, and this is a quite common interpretation in modern theory of music. Depending on the measure of pitches the intervals can be represented by a ratio $(p_2 : p_1)$ between the extreme pitches if they are measured with frequencies, or by the difference $p_2 - p_1$ if they are measured with logarithms of frequencies.

The most common intervals are the octave, the major sixth, the minor sixth, the fifth, the fourth, the major third, the minor third, the major whole tone, the minor whole tone. Their values correspond to ratios of frequencies that have varied from time to time. Those that have gained the most acceptance in the recent past are (2:1), (5:3), (8:5), (3:2), (4:3), (5:4), (6:5), (9:8) and (10:9). Notes separated by the above intervals form what is called a just intonation scale.

Characteristic small intervals:

1) Tone. Major whole tone (9:8), minor whole tone (10:9).

2) Semitone. Less than half a tone. Major or diatonic semitone, (16:15); minor or chromatic semitone (25:24).

3) Syntonic comma. Difference between major and minor tone, (81:80).

4) Pythagorean (or ditonic) comma. The falling to the closure of the circle of 12 fifths, (531441:524288).

Intervals can be composed or decomposed. The composition, or sum, of two intervals has a ratio given by the product of their ratios. So the sum of a fifth (3:2) and a fourth (4:3), is given by $3/2 \times 4/3 = 2/1$, that is an octave. The decomposition, or difference, of two intervals has a ratio given by the division of their ratios. So the difference of a fifth (3:2) and a fourth (4:3), is given by $3/2:4/3 = 9/8$, that is a major whole tone.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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