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Estimation of Electromagnetic Form Factor of σ_T and Its Correction of σ_L for Charged Pion

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Abstract

We estimate the electromagnetic form factor of the transverse part of cross section σ_T and provide a correction for the electromagnetic form factor of the longitudinal component of cross section σ_L for the charged pion within the frame work of hadronic operator. To achieve this, we consider a slightly deformed curve deviating from a straight line and construct a set of differential equations by comparing them to the equation determining charged pion wave function in a straight line case. By solving these equations, we employ the Fourier transform of these wave functions.

Keywords

Transverse, Longitudinal, Cross Section, Electromagnetic Form Factor

1. Introduction

Recently, surprising experimental results have been reported in hadron Physics. One notable finding pertains to the electromagnetic form factor of the neutral pion, π^0. Typically, the t dependence of neutral pion, is parametrized by Regge-like function. However, this parametrization is no longer valid at $-t > 1\,\text{GeV}^2$, where a Gaussian-like behavior as published by Dlamini *et al.* [1] is observed. These results seem to be confirmed by the electromagnetic form factor of the charged pion π^+, reported by Huber and Horn [2] [3], whose results remain Regge-like despite of using the same technique, specifically, error analysis, as in Ref. [1]. In both sets of experiments, their technique seems to perform well for the Longitudinal-Transverse separation of a differential cross section. In the differential cross section of t-dependent generalized parton distribution (GPD), σ_L, longitudinal part, reflects t-channel contribution, that is, quasi-elastic mechanism so that it reflects strong interaction between quark and antiquark due to exchanging gluons. Meanwhile, the transverse component σ_T, reflects contributions beyond the *t*-channel, highlighting non-gluon exchange interaction [4]. Therefore, precise L-T separation is crucial for accurately determining *t*-channel contributions. In this regard, techniques used by Ref. [1]-[3] seem reliable based on their results. Although past Rosenbluth separation (L-T separation) was inadequate [5]-[8], we can expect to have reliable σ_T data near future. Thus, it is meaningful to estimate σ_T by various ways. In this paper, we show an attempt to estimate σ_T in the frame work of hadronic operator proposed by Suura [9].

2. Formulation and Evaluation

Previously we derived the pion electromagnetic form factor from straight string that corresponds to the electric field line of color neutral charge, similar in shape to the electric field line of an electric dipole. The key difference is that gluons select only one path at a time, whereas in the electric dipole scenario, the entire electric field line exists simultaneously. Additionally, the probability of selecting a path becomes a weighting factor and only a slightly deformed line, deviating from a straight line, can be chosen. We point that, in most cases, gluons opt for a straight line path.

The most important assumption is that this curve corresponds to a state whose energy eigenvalue is pion mass $\left(E=\frac{g^2 L_1}{2}\delta(0)\right)$. We showed that the pion wave function is obtained from the equation for χ_3, which is a solution for the vector component as described in Ref. [10]. Furthermore, the kinetic term of χ_3 has exact same form as in the two-dimensional case, meaning that the correction explore pertains solely to χ_3, the charged pion (vector channel solution).

By considering the energy eigenvalue as the pion mass squared, that is $P_0=\frac{g^2|L_1|}{2}\delta(0)$, the equation for χ_3 from Ref. [10] becomes as

$$\frac{\partial^2\chi_3}{\partial r^2}+\frac{1}{r}\frac{\partial\chi_3}{\partial r}-\frac{1}{r^2}\chi_3-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\chi_3=0 \tag{1}$$

Note that L_1 is replaced by $|L_1|$ following to erratum [11].

In Equation (1), we can consider the term $-\frac{1}{r^2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2$ as potential term. Actuary $-\frac{1}{r^2}$ term comes from kinetic term and $-\left(\frac{g^2|L_1|}{2}\right)^2 r^2$ term comes from interaction potential due to gluon exchange. However, in following process, we will treat both terms as potential terms for the time being. Although an electric field line is different from a part of arc, we are considering only a slight deviation from a straight line, utilizing an arc segment with a large radius ρ_2 instead of the exact electric field line. In this context, r is described as $r=\rho_2\varphi$. The important point here is that this coordinate system differs coordinate system, where $\rho_1=\sqrt{x_1^2+y_1^2}$. Thus, we describe ρ_2 as $\rho_2=\sqrt{x_2^2+y_2^2}$. Under our assumption, the equation for the curved line is described as follows.

$$\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_1}\frac{\partial}{\partial\rho_2}+\frac{1}{\rho_2^2}\frac{\partial^2}{\partial\varphi_2^2}-\frac{1}{r^2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\right]X(\rho_2,\varphi_2)=0 \tag{2}$$

The description of r-terms serves as potentials through which we obtained the eigenvalue corresponding to the pion mass squared. This is because we are considering correction to the electromagnetic form factor within the same eigenvalue framework. Thus, in this equation, we consider r-terms as constants while keeping in mind the relationship of $r=\rho_2\varphi_2$.

To analyze Equation (2), we divide the domain into two regions: (i) the region where $r<\sqrt{\frac{2}{g^2|L_1|}}=r_B$ and (ii) $r>r_B$. In region (i), we approximate the potential as $-\frac{1}{r^2}$ while in region (ii), the potential is $-\left(\frac{g^2|L_1|}{2}\right)^2 r^2$. Thus, the equations we need to address become

Region (i)

$$\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}+\frac{1}{\rho_2^2}\frac{\partial^2}{\partial\varphi_2^2}-\frac{1}{r^2}\right]X_1(\rho_2,\varphi_2)=0 \tag{3}$$

Region (ii)

$$\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}+\frac{1}{\rho_2^2}\frac{\partial^2}{\partial\varphi_2^2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\right]X_2(\rho_2,\varphi_2)=0 \tag{4}$$

First, we solve Equation (4) to derive the dimensionless electromagnetic form factor.

Equation (4) can be transformed into the following form.

$$\left[\rho_2^2\left(\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\right)+\frac{\partial^2}{\partial\varphi_2^2}\right]X_2(\rho_2,\varphi_2)=0 \tag{5}$$

In this form, we can use the method of separation of valuables. By setting $X_2(\rho_2,\varphi_2)=X_2(\rho_2)X_2(\varphi_2)$, Equation (5) becomes as

$$\begin{aligned}&\frac{\rho_2^2}{X_2(\rho_2)}\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\right]X_2(\rho_2)\\&=\frac{1}{X_2(\varphi_2)}\left[-\frac{\partial^2}{\partial\varphi_2^2}\right]X_2(\varphi_2)=l_2^2\end{aligned} \tag{6}$$

where l_2^2 is arbitrary constant.

Equation (6) is written as the set of following equations.

$$\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2-\frac{{l_2}^2}{\rho_2^2}\right]X_2(\rho_2)=0 \tag{7}$$

$$\left[\frac{\partial^2}{\partial\varphi_2^2}+l_2^2\right]X_2(\varphi_2)=0 \tag{8}$$

Taking $X_2=\rho_2^{-\frac{1}{2}}\bar{X}_2$, Equation (7) becomes

$$\left[\frac{\partial^2}{\partial\rho_2^2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2-\frac{l_2^2-\frac{1}{4}}{\rho_2^2}\right]\bar{X}_2(\rho_2)=0 \tag{9}$$

Changing valuable as $\rho_2=\alpha\bar{\rho}_2$, Equation (9) becomes

$$\left[\frac{\partial^2}{\partial\rho_2^2}-\left(\frac{g^2|L_1|}{2}\right)^2 r^2\alpha^2-\frac{l_2^2-\frac{1}{4}}{\bar{\rho}_2^2}\right]X_2=0 \tag{10}$$

Taking $\alpha=\dfrac{1}{g^2|L_1|r}$, Equation (10) becomes

$$\left[\frac{\partial^2}{\partial\bar{\rho}_2^2}-\frac{1}{4}-\frac{l_2^2-\frac{1}{4}}{\bar{\rho}_2^2}\right]\bar{X}_2(\bar{\rho}_2)=0 \tag{11}$$

Equation (11) is in the standard form of the Whittaker equation with $\kappa=0$ and $\mu=l_2$ [12].

Then we can take solutions of Equation (11) as

$W_{0,l_2}(\bar{\rho}_2)=W_{0,l_2}\left(\dfrac{\rho_2}{\alpha}\right)=W_{0,l}\left(g^2|L_1|r\rho_2\right)$.

Rewriting $g^2|L_1|r\rho_2$ as $\sqrt{g^2|L_1|}\rho_2\sqrt{g^2|L_1|}r$, we observe that both $\sqrt{g^2|L_1|}\rho_2$ and $\sqrt{g^2|L_1|}r$ are dimensionless. For three-dimensional Fourier transform the term $\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})$ can be rewritten as $\exp\left(\mathrm{i}\dfrac{\boldsymbol{q}}{\sqrt{g^2|L_1|}}\cdot\sqrt{g^2|L_1|}\boldsymbol{r}\right)$. In this rewritten form, both momentum and configuration space are dimensionless. This suggests that Fourier transform should be taken with a scale transformation in spherical coordinates such as $r\to\sqrt{g^2|L_1|}r$ to maintain a dimensionless form.

Consequently, the three-dimensional Fourier transform integral becomes

$$\int_0^\infty \mathrm{d}\left(\sqrt{g^2|L_1|}\right)\left(\sqrt{g^2|L_1|}r\right)^2 J_{\frac{1}{2}}\left(\frac{q}{\sqrt{g^2|L_1|}}\sqrt{g^2|L_1|}r\right) \tag{12}$$

where $J_{\frac{1}{2}}(z)$ is Bessel function equal to spherical Bessel $j_0(z)$.

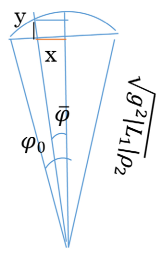
This means we work in dimensionless spherical coordinates.

Above consideration gives that we have to use scale changed ρ_2 as $\sqrt{g^2|L_1|}\rho_2$ because of keeping the relation $\rho_2\varphi=r$ such as $\sqrt{g^2|L_1|}\rho_2\varphi=\sqrt{g^2|L_1|}r$.

The solution of Equation (11) is $\dfrac{1}{\sqrt{\rho_2}}W_{0,l_2}\left(\sqrt{g^2|L_1|}\rho_2\sqrt{g^2|L_1|}r\right)$ but for solution, we can multiply arbitrary constant. Then we choose $\left(\dfrac{1}{g^2|L_1|}\right)^{\frac{1}{4}}$ as multiplying constant. However, ρ_2 of $\frac{1}{\sqrt{\rho_2}}$ part of solution is not $\sqrt{g^2|L_1|}\rho_2$ so that final integration for ρ_2 is $\int_{\rho_{20}}^{\infty}\mathrm{d}\left(\sqrt{g^2|L_1|}\rho_2\right)$.

Here we calculate an area made by a curved line.

Area made by curved line is calculated by the following formula.



$$\text{Area} = 2\int_0^{x_0}\mathrm{d}x 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

Using $\rho = \sqrt{g^2|L_1|}\rho_2$, (x, y) coordinate is described as $x = \rho\sin\bar{\varphi}$, $y = \rho\cos\bar{\varphi} - \rho\cos\varphi_0$.

Then area becomes by noticing $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan\bar{\varphi}$

$$\begin{aligned}\text{Area} &= 4\pi\rho^2\int_0^{\rho\sin\varphi_0}\mathrm{d}\bar{\varphi}\cos\bar{\varphi}\left(\cos\bar{\varphi} - \cos\varphi_0\right)\frac{1}{\cos\bar{\varphi}} \\ &= 4\pi\rho^2\left(\sin\left(\rho\sin\varphi_0\right) - \cos\varphi_0\rho\sin\varphi_0\right) = 4\pi\rho^3\frac{1}{3}\varphi_0^3 \\ &= 4\pi\rho^3\frac{1}{24}\left(2\varphi_0\right)^3\end{aligned}$$

To obtain the last line, we use the fact that under $\varphi_0 \ll 1$, $\sin\varphi_0 = \varphi_0$, $\cos\varphi_0 = 1 - \frac{\varphi_0^2}{2}$ and that $\rho\varphi_0 = \frac{r}{2}$ is small because pion is small object although ρ is large.

This shows the area calculation contributes a quantity such as

$$\left(\sqrt{g^2|L_1|}\rho_2\right)^3\left(\frac{\sqrt{g^2|L_1|}r}{\sqrt{g^2|L_1|}\rho_2}\right)^3 = \left(\sqrt{g^2|L_1|}r\right)^3.$$

The solution of Equation (8) is easily obtained as

$$X_2(\varphi) = \mathrm{e}^{il\varphi} \tag{13}$$

Thus $\left|X_2(\varphi)^2\right| = 1$.

Next we solve the Equation (3). In this case, recalling $r = \rho_2\varphi_2$ we change the Equation (3) as

$$\left[\rho_2^2\left(\frac{\partial^2}{\partial\rho_2^2} + \frac{1}{\rho_2}\frac{\partial}{\partial\rho_2^2}\right) + \frac{\partial^2}{\partial\varphi_2^2} - \frac{1}{\varphi_2^2}\right]X_1(\rho_2, \varphi_2) = 0 \tag{14}$$

For Equation (14) we can use separation of valuable methods to solve equations. By setting $X_1(\rho_2,\varphi_2)=X_1(\rho_2)X_1(\varphi_2)$ Equation (5) becomes

$$\frac{1}{X_1(\rho_2)}\left[\rho_2^2\left(\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}\right)\right]X_1(\rho_2)=\frac{1}{X_1(\varphi_2)}\left[-\frac{\partial^2}{\partial\varphi_2^2}+\frac{1}{\varphi_2^2}\right]X_1(\varphi_2)=-l_1^2 \quad (15)$$

where $-l_1^2$ is arbitrary constant.

Equation (15) becomes the set of the following equations.

$$\left[\frac{\partial^2}{\partial\rho_2^2}+\frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}+\frac{l_1^2}{\rho_2^2}\right]X_1(\rho)=0 \quad (16)$$

$$\left[\frac{\partial^2}{\partial\varphi_2^2}-l_1^2-\frac{1}{\varphi_2^2}\right]X_1(\varphi_2)=0 \quad (17)$$

To solve Equation (17), changing variable as $\varphi_2=\alpha\bar{\varphi}_2$ and setting $\alpha=\frac{1}{2l_1}$, Equation (17) becomes

$$\left[\frac{\partial^2}{\partial\bar{\varphi}_2^2}-\frac{1}{4}-\frac{1}{\bar{\varphi}_2^2}\right]X_1(\bar{\varphi}_2)=0 \quad (18)$$

Equation (18) is Whittaker equation with $\kappa=0,\mu=\pm\frac{\sqrt{5}}{2}$ [12]. For solution of Equation (18), we choose $M_{0,\frac{-\sqrt{5}}{2}}(\bar{\varphi}_2)=M_{0,\frac{-\sqrt{5}}{2}}(2l_1\varphi_2)\left(\mu=-\frac{\sqrt{5}}{2}\right)$. Recalling the relation that $\rho\varphi=r$, the solution of Equation (18) can be represented as $M_{l,\frac{-\sqrt{5}}{2}}\left(2l_1\frac{r}{\rho_2}\right)=M_{l,\frac{-\sqrt{5}}{2}}\left(2l_1\frac{\sqrt{g^2|L_1|}r}{\sqrt{g^2|L_1|}\rho_2}\right)$.

To find a solution of Equation (16), we set $X_1(\rho_2)=\rho_2^n$ and institute this into Equation (16). Then determining equation for n becomes

$$n(n-1)+n+l^2=0$$

This gives $n=\pm il_1$. Thus, a solution of Equation (16) is that $X_1(\rho_2)=\rho_2^{il_1}$ so that $|X_1(\rho_2)|^2=1$.

Recalling scale changed Fourier Transform of Equation (12), the electromagnetic form factor is obtained as the following equation.

$$\begin{aligned}F_\pi=&\int_{\rho_0}^{\infty}\mathrm{d}\left(\sqrt{g^2|L_1|}\rho_2\right)\\&\times\left[\int_0^{r_B}\mathrm{d}\left(\sqrt{g^2|L_1|}r\right)\left(\sqrt{g^2|L_1|}r\right)^5\left|M_{0,\frac{-\sqrt{5}}{2}}\left(2l_1\frac{\sqrt{g^2|L_1|}r}{\sqrt{g^2|L_1|}\rho_2}\right)\right|^2J_{\frac{1}{2}}\left(\frac{q}{\sqrt{g^2|L_1|}}\sqrt{g^2|L_1|}r\right)\right.\\&\left.+\int_{r_B}^{\infty}\mathrm{d}\left(\sqrt{g^2|L_1|}r\right)\left(\sqrt{g^2|L_1|}r\right)^5\left|\frac{1}{\sqrt{\sqrt{g^2|L_1|}\rho}}W_{0,l_2}\left(\sqrt{g^2|L_1|}\rho_2\sqrt{g^2|L_1|}r\right)\right|^2J_{\frac{1}{2}}\left(\frac{q}{\sqrt{g^2|L_1|}}\sqrt{g^2|L_1|}r\right)\right]\end{aligned} \quad (19)$$

Note that in Equation (19) we include the contribution of area consideration.

Recall that the wave function in region (i) originates from the potential that is in the kinetic term of the equation for χ_3 while the wave function of region (ii) arises from the interaction potential term of the equation for χ_3. According to the definition mentioned in Sec.1, the form factor for region (i) corresponds to the transverse form factor $F_{\pi T}$ and the form factor for region (ii) corresponds to the longitudinal form factor $F_{\pi L}$. To derive an approximate form of $F_{\pi T}$ and $F_{\pi L}$, we extend r_B to ∞ in region (i) and to 0 in region (ii).

Therefore $F_{\pi T}$ and $F_{\pi L}$ are obtained as follows.

$$F_{\pi T}=\int_{\sqrt{g^2|L_1|}\rho_0}^{\sqrt{g^2|L_1|}\rho_{\max}}\mathrm{d}\rho'\int_0^\infty \mathrm{d}r' r'^5\left|M_{0,\frac{-\sqrt{5}}{2}}\left(2l_1\frac{r'}{\rho'}\right)\right|^2 J_{\frac{1}{2}}\left(\frac{q}{\sqrt{g^2|L_1|}}r'\right) \tag{20}$$

$$F_{\pi L}=\int_{\sqrt{g^2|L_1|}\rho_0}^{\sqrt{g^2|L_1|}\rho_{\max}}\mathrm{d}\rho'\frac{1}{\rho'}\int_0^\infty \mathrm{d}r' r'^5\left|W_{0,l_2}\left(\rho' r'\right)\right|^2 J_{\frac{1}{2}}\left(\frac{q}{\sqrt{g^2|L_1|}}r'\right) \tag{21}$$

where $\rho'=\sqrt{g^2|L_1|}\rho_2$, $r'=\sqrt{g^2|L_1|}r$.

Note that for upper the limit of ρ_2 we use $\rho_{\max}$ instead of ∞. This reason is as follows. We consider a slightly deformed case. However, as Suura mentioned in Ref. [9], in QCD, a linear string may change shape but remains linear. This means string mostly stays as a straight line, and if it becomes a curved line, it should not be asymptotically approached a straight line. Therefore, setting $\rho_{\max}$ is reasonable.

First, we evaluate Equation (21) by using following integral formula as follows.

There is a formula as [13]

$$\begin{aligned}&\int_0^\infty \mathrm{d}x\, x^{2\rho-1}W_{\kappa,\mu}(ax)W_{-\kappa,\mu}(ax)J_\nu(yx)\\&=\frac{\Gamma(\rho+1+\mu)\Gamma(\rho+1-\mu)\Gamma(2\rho+2)}{\Gamma\left(\frac{3}{2}+\kappa+\rho\right)\Gamma\left(\frac{3}{2}-\kappa+\rho\right)\Gamma(1+\nu)}y^\nu 2^{-\nu-1}a^{-2\rho-1}\\&\times {}_4F_3\left(\rho+1,\rho+\frac{3}{2},\rho+1+\mu,\rho+1-\mu;\frac{3}{2}+\kappa+\rho,\frac{3}{2}-\kappa+\rho,1+\nu;-\frac{y^2}{a^2}\right)\end{aligned} \tag{22}$$

under the condition $y>0$, $\operatorname{Re}\rho>|\operatorname{Re}\mu|>-1$, $\operatorname{Re}a>0$.

Because our case is $\kappa=0$, we can apply this formula with $\rho=\frac{11}{4}$, $\mu=l_2=1$, $a=\rho'$, $\nu=\frac{1}{2}$, $y=\frac{q}{\sqrt{g^2|L_1|}}$. Note that we choose $l_2=1$ because l_2 is arbitrary constant. ${}_4F_3$ is a generalized hyper geometric series. Generalized hyper geometric series ${}_pF_q$ is defined as [13]

$${}_pF_q(\alpha_1,\alpha_2,\cdots,\alpha_p;\beta_1,\beta_2,\cdots,\beta_q;z)=\sum_{k=0}^\infty\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_q)_k}\frac{z^k}{k!}$$

where $(\alpha_i)_k=\alpha_i(\alpha_i+1)\cdots(\alpha_i+k-1)$.

Then $a^{-2\rho-1}{}_4F_3$ becomes

$$\left(\frac{1}{\rho'}\right)^{\frac{11}{2}} {}_4F_3\left(\frac{15}{4},\frac{17}{4},\frac{19}{4},\frac{11}{4};\frac{17}{4},\frac{17}{4},\frac{3}{2};-\frac{q'^2}{\rho'^2}\right)$$
$$=\sum_{k=0}^{\infty}\frac{\left(\frac{15}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{19}{4}\right)_k\left(\frac{11}{4}\right)_k}{\left(\frac{17}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{3}{2}\right)_k}\frac{(-1)^k\left(q'^2\right)^k}{k!}\left(\frac{1}{\rho'}\right)^{2k+\frac{11}{2}} \tag{23}$$

We denote q' as $q'=\frac{q}{\sqrt{g^2|L_1|}}$.

Changing a variable as $\rho'=\frac{1}{\bar{\rho}}$, thus integral range becomes 0 to $\frac{1}{\sqrt{g^2|L_1|\rho_0}}=\overline{\rho_0}$ and after integration for $\bar{\rho}$ Equation (23) becomes

$$I=\sum_{k=0}^{\infty}\frac{\left(\frac{15}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{19}{4}\right)_k\left(\frac{11}{4}\right)_k}{\left(\frac{17}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{3}{2}\right)_k}\frac{(-1)^k\left(q'^2\right)^k}{k!}\frac{\overline{\rho_0}^{2k+\frac{13}{2}}}{2k+\frac{13}{2}} \tag{24}$$

Then $F_{\pi L}$ is described as

$$F_{\pi L}=const\sqrt{q'}\sum_{k=0}^{\infty}\frac{\left(\frac{15}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{19}{4}\right)_k\left(\frac{11}{4}\right)_k}{\left(\frac{17}{4}\right)_k\left(\frac{17}{4}\right)_k\left(\frac{3}{2}\right)_k}\frac{(-1)^k q'^k}{k!\left(2k+\frac{13}{2}\right)}\left[\overline{\rho_0}^{2k+\frac{13}{2}}-\bar{\rho}_{\max}^{\;2k+\frac{13}{2}}\right] \tag{25}$$

where $\overline{\rho_0}=\frac{1}{\rho_0'}$ and $\bar{\rho}_{\max}=\frac{1}{\rho'_{\max}}$.

Actuary we cannot describe Equation (24) as a function, however, we are interested in the behavior at large momentum q' case. Then after 3 times integration for $\overline{\rho_0}$ rough estimation gives the form of Equation (24) as

$$I^{(3)}\sim const\frac{1}{1+q'^2\overline{\rho_0}^2}\overline{\rho_0}^{\frac{19}{2}} \tag{26}$$

We show the rough derivation of Equation (26) in the **Appendix**.

The dependence of Equation (24) is q'^{-2} because Equation (24) is estimated at large q'.

Important point is that even differentiating 3 times with respect to $\overline{\rho_0}$, q' dependence of each appeared term is q'^{-2} as same as that of Equation (26). Thus total dependence of $F_{\pi L}$ at large q' is $q'^{-\frac{3}{2}}$ because of the term $\sqrt{q'}$ in Equation (25). Because we obtained q^{-1} behavior of $F_{\pi L}$ in the case of strait line in Ref. [10], we can consider that this result is actually correction.

Next we estimate $F_{\pi T}$ by evaluating Equation (20).

The definition of Whittaker function $M_{\kappa,\mu}(z)$ is following [12].

$$M_{\kappa,\mu}(z)=z^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right){}_1F_1\left(\mu-\kappa+\frac{1}{2},2\mu+1;z\right)$$

${}_1F_1$ is generalized hyper geometric series ${}_pF_q$ with $p=1$ and $q=1$.

In our case $z=2l_1\dfrac{r'}{\rho'}$ and ρ' is quite large so that taking only the first term of ${}_1F_1$ is sufficient to evaluate Equation (20).

Then in our case Whittaker function of $M_{0,\frac{-\sqrt{5}}{2}}\left(2l_1\dfrac{r'}{\rho'}\right)$ becomes

$$M_{0,\frac{-\sqrt{5}}{2}}\left(2l_1\frac{r'}{\rho'}\right)=\left(2l_1\frac{r'}{\rho'}\right)^{-\frac{\sqrt{5}}{2}+\frac{1}{2}}\exp\left(-l_1\frac{r'}{\rho'}\right) \tag{27}$$

Then Equation (20) gives $F_{\pi T}$ as

$$F_{\pi T}=\int_{\sqrt{g^2|L_1|}\rho_0}^{\sqrt{g^2|L_1|}\rho_{\max}}\mathrm{d}\rho'\left(\frac{2l_1}{\rho'}\right)^{-\sqrt{5}+1}\int_0^\infty \mathrm{d}r'r'^{5-\sqrt{5}+1}\exp\left(-2l_1\frac{r'}{\rho'}\right)J_{\frac{1}{2}}\left(q'r'\right) \tag{28}$$

Recalling that $J_{\frac{1}{2}}(z)=j_0=\dfrac{\sin(z)}{z}$.

Integral with respect to r' part of Equation (28) becomes

$$I_{r'}=\frac{1}{q'}\int_0^\infty \mathrm{d}r'r'^{5-\sqrt{5}}\exp\left(-2l_1\frac{r'}{\rho'}\right)\sin\left(q'r'\right) \tag{29}$$

There is a formula of this type of integral as [13]

$$\int_0^\infty \mathrm{d}x\,x^{\lambda-1}\exp(-\gamma x)\sin(\delta x)=\frac{\Gamma(\lambda)}{\left(\gamma^2+\delta^2\right)^{\frac{\lambda}{2}}}\sin\left(\lambda\,\mathrm{arctg}\left(\frac{\delta}{\gamma}\right)\right) \tag{30}$$

For integral condition, $\mathrm{Re}\,\lambda>-1$, $\mathrm{Re}\,\gamma>\mathrm{Im}\,\delta$.

Arctg denotes arctangent.

Applying the formula Equation (30) for Equation (29), Equation (29) becomes

$$I_{r'}=const\frac{1}{\left(\left(\frac{2l_1}{\rho'}\right)^2+q'^2\right)^{\frac{6-\sqrt{5}}{2}}}\sin\left(\left(6-\sqrt{5}\right)\mathrm{arctg}\left(\frac{q'\rho'}{2l_1}\right)\right) \tag{31}$$

Then we obtain for $F_{\pi T}$ as follows.

$$\begin{aligned}F_{\pi T}&=\frac{1}{q'}\int_{\sqrt{g^2|L_1|}\rho_0}^{\sqrt{g^2|L_1|}\rho_{\max}}\mathrm{d}\rho'\left(2l_1\right)^{-\sqrt{5}+1}\frac{\rho'^{\sqrt{5}-1}}{\left(\left(\frac{2l_1}{\rho'}\right)^2+q'^2\right)^{\frac{6-\sqrt{5}}{2}}}\sin\left(\left(6-\sqrt{5}\right)\mathrm{arctg}\left(\frac{q'\rho'}{2l_1}\right)\right)\\&=const\frac{1}{q'}\int_{\sqrt{g^2|L_1|}\rho_0}^{\sqrt{g^2|L_1|}\rho_{\max}}\mathrm{d}\rho'\frac{\rho'^5}{\left(4l_1^2+\left(q'\rho'\right)^2\right)^{\frac{6-\sqrt{5}}{2}}}\sin\left(\left(6-\sqrt{5}\right)\mathrm{arctg}\left(\frac{q'\rho'}{2l_1}\right)\right)\end{aligned} \tag{32}$$

Recalling the facts that $\mathrm{arctg}(\eta q')=\eta q'$ and $\sin(\eta' q')=\eta' q'$ when q' approaches 0, $F_{\pi T}$ becomes constant at $q'\to 0$. For large q', Equation (32) shows $q'^{-7+\sqrt{5}}$ dependence because of the fact that $\mathrm{arctg}(\eta q')=\dfrac{\pi}{2}$ at large q'.

Actuary we cannot evaluate exact integration for remained part of integral, however, rough estimation can be given as follows.

Changing variable as $(q'\rho')^2 = x$, integral without sin part becomes

$$\begin{aligned} &\int_{\tau(q'\rho_0)^2}^{\tau(q'\rho_{\max})^2} \frac{1}{q'} \frac{\mathrm{d}x}{2\sqrt{x}} \frac{\left(\frac{1}{q'}\right)^5 x^{\frac{5}{2}}}{\left(4l_1^2 + x\right)^{\frac{6-\sqrt{5}}{2}}} \sim \frac{1}{q'^6} \int_{\tau(q'\rho_0)^2}^{\tau(q'\rho_{\max})^2} \mathrm{d}x \frac{x^2}{x^{\frac{6-\sqrt{5}}{2}}} \\ &= \frac{1}{q'^6} \int_{\tau(q'\rho_0)^2}^{\tau(q'\rho_{\max})^2} \mathrm{d}x\, x^{-1+\frac{\sqrt{5}}{2}} = \frac{1}{q'^6} \left[x^{\frac{\sqrt{5}}{2}} \right]_{\tau(q'\rho_0)^2}^{\tau(q'\rho_{\max})^2} \propto q'^{-6+\sqrt{5}} \end{aligned} \tag{33}$$

The second line is obtained by using the condition of large q', that is, large x.

Multiply q'^{-1} to this integration result shows $F_{\pi T}$ behaves $q'^{-7+\sqrt{5}}$ at large q'.

This estimation is an approximation but we can say the result is close.

Important point is that we need the absolute value of F_π so that we can ignore the sign of F_π.

3. Results

We obtain the following results for charged pion.

Transverse electromagnetic form factor $F_{\pi T}$ is described as Equation (32). The behavior of $F_{\pi T}$ at $q' \to 0$ becomes constant, while that of $F_{\pi T}$ at large q' becomes $\left(\frac{1}{q'}\right)^{7-\sqrt{5}}$.

Longitudinal electromagnetic form factor $F_{\pi L}$ is described as Equation (25). The behavior of $F_{\pi L}$ at $q' \to 0$ becomes 0, while that of $F_{\pi L}$ at large q' becomes $\left(\frac{1}{q'}\right)^{\frac{3}{2}}$. Note that q' denotes $\frac{q}{\sqrt{g^2 |L_1|}}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Rough estimation of Equation (24).

In order to find elementary function of ${}_4F_3$ at large q' case, for simplicity, we take ∞ for upper limit of ρ'.

Recalling the definition of $(\alpha)_k$, after cancelling one $\left(\frac{17}{4}\right)_k$, numerator of Equation (23) becomes

$$\left(\frac{15}{4}\right)_k = \left(\frac{3}{4}\right)_k \times \left(k+\frac{3}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{11}{4}\right) \Big/ \left(\frac{3}{4}\times\frac{7}{4}\times\frac{11}{4}\right)$$

$$\left(\frac{11}{4}\right)_k = \left(\frac{7}{4}\right)_k \times \left(k+\frac{7}{4}\right) \Big/ \frac{7}{4}$$

We can roughly cancel out $\left(\frac{19}{4}\right)_k$ by a remained $\left(\frac{17}{4}\right)_k$ of denominator.

After 3 times integration for $\overline{\rho_0}$ and cancellation of two $\left(\frac{17}{4}\right)_k$ terms, denominator becomes $k!\left(\frac{3}{2}\right)_k \left(2k+\frac{13}{2}\right)\left(2k+\frac{15}{2}\right)\left(2k+\frac{17}{2}\right)\left(2k+\frac{19}{2}\right) =$ $(1)_k\left(\frac{6}{4}\right)_k\left(k+\frac{13}{4}\right)\left(k+\frac{15}{4}\right)\left(k+\frac{17}{4}\right)\left(k+\frac{19}{4}\right)2^4$.

We can roughly cancel out $\left(\frac{3}{4}\right)_k$ and $\left(\frac{7}{4}\right)_k$ by $(1)_k$ and $\left(\frac{6}{4}\right)_k$, respectively. Then summation becomes

$$contant\sum\frac{\left(k+\frac{3}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{11}{4}\right)}{\left(k+\frac{13}{4}\right)\left(k+\frac{15}{4}\right)\left(k+\frac{17}{4}\right)\left(k+\frac{19}{4}\right)}(-1)^k\left(q'^2\overline{\rho_0}^2\right)^k\overline{\rho_0}^{\frac{19}{2}}$$

We are interested in large q' case. In this case, large k part of summation contributes mainly to this summation. For the large k part of summation, we can approximate

$$\frac{\left(k+\frac{3}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{11}{4}\right)}{\left(k+\frac{13}{4}\right)\left(k+\frac{15}{4}\right)\left(k+\frac{17}{4}\right)\left(k+\frac{19}{4}\right)}\sim 1$$

Also recalling the fact that series is infinite, we can approximate that the main contributing part of summation is described as

$$\sum_{k=\bar{k}}^{\infty}(-1)^k\left(q'^2\overline{\rho_0}^2\right)^k$$

Finally, we can say for small $\bar{k}$ part as

$$\text{absolute value of }\sum_{k=0}^{\bar{k}}\frac{\left(k+\frac{3}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{11}{4}\right)}{\left(k+\frac{13}{4}\right)\left(k+\frac{15}{4}\right)\left(k+\frac{17}{4}\right)\left(k+\frac{19}{4}\right)}(-1)^k\left(q'^2\overline{\rho_0}^2\right)^k$$

$$<\text{absolute value of }\sum_{k=0}^{\bar{k}}(-1)^k\left(q'^2\overline{\rho_0}^2\right)^k$$

Again we insist that this part is not main in large q' case. Thus, we can describe approximated form of summation at large q' case after 3 times integration as

$$\text{constant}\frac{1}{1+q'^2\overline{\rho_0}^2}\overline{\rho_0}^{\frac{19}{2}}$$

To obtain this form, we use the formula as

$$\sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$$