

A Delayed Predator-Prey Model with Fear Effect and Cannibalism

Guoqing Li, Xiaolin Lin, Shaoyi Geng

School of Mathematics and Data Science, Shaanxi University of Science and Technology, Xi'an, China

Email: linxiaolin@sust.edu.cn

How to cite this paper: Li, G.Q., Lin, X.L. and Geng, S.Y. (2025) A Delayed Predator-Prey Model with Fear Effect and Cannibalism. *Journal of Applied Mathematics and Physics*, 13, 506-524.

<https://doi.org/10.4236/jamp.2025.132028>

Received: January 25, 2025

Accepted: February 22, 2025

Published: February 25, 2025

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Abstract

In this paper, we consider the fear effect and gestation delay, and then establish a delayed predator-prey model with cannibalism. Firstly, we prove the well-posedness of the model. Secondly, the existence and stability of all equilibriums of the system are studied. Thirdly, the Hopf bifurcation at the coexistence equilibrium is investigated, and the conditions for the occurrence of Hopf bifurcation at the unique positive equilibrium point of the system with delay are determined. Finally, the numerical simulation results show that as the time delay increases, the equilibrium loses its stability, and the system has periodic solution.

Keywords

Predator-Prey Model, Fear Effect, Cannibalism, Stability, Hopf Bifurcation

1. Introduction

Predation serves as the primary determinant of prey mortality. During the course of evolution, prey has developed a variety of sophisticated strategies, such as detecting, eluding, or combating predators, while actively searching for new food sources [1]-[3]. A growing body of research has demonstrated that the mere presence of predators can exert a substantial influence on the physiological traits and behavioral patterns of prey. In other words, the presence of a predator has a greater effect on the prey than the predator directly hunting. Zanette *et al.* [4] studied the effects of predators on the reproductive habits of songbirds and found that the presence of fear factors influenced the birth and survival rates of songbirds. Wang *et al.* [5] proposed a specific mathematical model to characterize the role of the fear effect on predator-prey systems:

$$\begin{cases} \frac{du}{dt} = f(k, v)r_0u - du - au^2 - g(u)v, \\ \frac{dv}{dt} = cg(u)v - mv, \end{cases}$$

where prey and predator density are denoted by $u \geq 0$ and $v \geq 0$, k reflects the fear level of the prey due to perceiving the risk of predation, r_0 is the natural birth rate of prey in the absence of predators, $f(k, v)$ is the cost of the antipredation defense of the prey due to fear and it is monotonically decreasing in k and v , d is the natural death rate of prey, a is the density-dependent death rate of the prey due to intraspecies competition, $g(u)$ is the functional response function which is independent of v , and m is the predator death rate. It obtained some results on how the fear effect affects the dynamic behavior of predator-prey models through mathematical analysis and numerical simulations. For more related research, see [6]-[10].

Certain predator populations are capable of consuming their own kind to gain energy, enabling them to survive in the absence of prey. And we study such models by considering cannibalism. In some primates [11], fish [12], carnivorous mammals [13] and spiders [14], cannibalism [15], also known as intraspecific predation, involves eating offspring or siblings. This phenomenon is sometimes referred to as the “lifeboat mechanism” because it prevents the extinction of predator communities. Depending on the rate of cannibalism, cannibalism can have a positive or negative impact on the population. Many scholars have been inspired to study and produce many remarkable research results. Therefore, for some predators, it makes more sense to include predators in a stage structure model as well. Deng *et al.* [16] proposed a Lotka-Volterra prey-predator with predator cannibalism:

$$\begin{cases} \frac{du}{dt} = ru \left(1 - \frac{u}{K}\right) - a_1uv, \\ \frac{dv}{dt} = a_2uv + b_1v - cv - \frac{b_2v^2}{\beta v}, \end{cases}$$

where parameter r represents the intrinsic growth rate of the prey, K represents the carrying capacity of the prey in the environment, a_1 represents the rate of predation, a_2 represents the rate at which prey biomass is converted into predator birth, b_1 represents the rate at which cannibalism is converted into predator birth, c represents the rate at which predators die, b_2 represents the rate at which cannibalism occurs within predator individuals, and β represents the cannibalism half-saturation constant. The final term and the second term in the second equation of the system represent the phenomena of cannibalism.

The existence of delay is common in ecosystem. In ecology, physics, biology and other fields, delayed differential equations are more useful than conventional equations. In ecosystem, changes in growth and development, reproduction processes, and environmental factors can produce time lags, and population conditions are affected by time lags. Considering the time delay factor in the predator model can

better reflect the actual situation of the ecosystem. In many current studies, the energy obtained by predators through food does not immediately affect the reproduction of predator groups, and this effect can be described by the Holling-type functional response function, which often displays time delay. In the delayed predator-captor model, delay may affect the stability or instability of prey density due to predation (see [17]-[20]). Due to new ecological evidence and theoretical advances, researchers are making new advances in modeling various aspects of biological interactions. Holling [21] proposed a more accurate point of view that has become one of the most widely used ecologies. Hussien *et al.* [22] considered the existence of cannibalism in the predator population, considered the pregnancy delay of the predator population, and added the predator refuge constant to obtain the following model:

$$\begin{cases} \frac{du}{dt} = ru \left(1 - \frac{u}{K} \right) - \frac{a_1 uv}{\beta_1 + u}, \\ \frac{dv}{dt} = \frac{a_2 u(t-\tau) v(t-\tau)}{\beta_2 + u(t-\tau)} + b_1 v - cv - \frac{b_2 (1-m) v^2}{\beta_2 + (1-m) v}, \end{cases} \quad (1)$$

where parameter b_2 represents the cannibalism coefficient of the predator population, β_2 represents the half-satiation constant of cannibalism, and m represents the predator refuge constant. The results show that reducing the cannibalism rate will destroy the coexistence equilibrium point and make the system close to periodic dynamics. On this basis, the Holling Type II functional response, based on the traditional Lotka-Volterra model, is a function of increases, concave, smooths out, and saturation at high prey numbers. Many authors have studied predator-prey models that use this functional response function, both with and without time delay, and even with spatial dependence (see [23] [24]).

In natural ecosystems, long-term monitoring of predator populations can reveal patterns related to cannibalism and density. Rudolf [25] studied dragonfly predation models, providing the first experimental evidence for the indirect effects of cannibalism behavior and density constraints on prey. The study of biological populations and the analysis of related ecological phenomena have certain relevance to the study of the relationship between cannibalism and density in fish populations. To some extent, it shows the importance and significance of long-term monitoring of predator populations in natural ecosystems. Meanwhile, prey animals are not only affected by the immediate presence of predators (fear effect), but predators also sometimes resort to cannibalism when resources are scarce. Moreover, biological processes such as reproduction and development take time, which is captured by the time-delay factor. The previous study did not couple these parts in the same model to accurately describe how these factors interact over time and affect the stability and dynamics of the predator-prey system. Therefore, assuming that young individuals have density limitations due to their dependence on nutritional or biological resources. We propose the following model:

$$\begin{cases} \frac{dx}{dt} = \frac{ax}{1+by} - cx^2 - \frac{\varphi_1 xy}{x+m}, \\ \frac{dy}{dt} = \frac{\varphi_2 x(t-\tau)y(t-\tau)}{x(t-\tau)+m} + (r-d)y - \frac{\sigma y^2}{y+n}, \end{cases} \quad (2)$$

with the initial conditions

$$x(\theta) = x_0(\theta) \geq 0, y(\theta) = y_0(\theta) \geq 0, \theta \in [-\tau, 0]. \quad (3)$$

The rest of this paper is organized as follows. In Section 2, we establish the well-posedness of solutions of system (2). In Section 3, we investigate the existence and stability of equilibrium and the existence of Hopf bifurcation are investigated. In Section 4, we analyze the stability of Hopf bifurcation. In Section 5, we conducted some numerical simulations. Finally, a brief conclusion is given in Section 6.

2. Well-Posedness

In this section, we will prove the well-posedness of model (2) and obtain the following theorem.

Theorem 1. For any initial value $(x, y) \in \mathbb{R}_+^2$, system (2) has a unique solution satisfying (3), which exists globally in $(0, +\infty)$, is nonnegative, and remains bounded. Moreover, the region

$$\Pi := \left\{ (x, y) \in \mathbb{R}_+^2, 0 \leq H(t) \leq \frac{F}{G} \right\}$$

is both positively invariant and attractive for (3), where

$$H(t) := \varphi_2 x + \varphi_1 y, \quad F := \frac{\varphi_2(a+G)^2}{4c}, \quad 0 < G < r-d.$$

Proof. According to [26] (Lemma 4), we derive that every solution of (2) with the initial condition (3) is positive, that is, any solutions remain positive when $\forall t > 0$ in the region $\mathbb{R}_+^2$. The following is a proof of well-posedness.

Firstly, similar to the Lyapunov function as in [22], let

$$H(t, \tau) = g_1 x(t-\tau) + g_2 y(t),$$

where g_1, g_2 are positive constants. After calculation, the

$$\begin{aligned} H' &< \frac{g_1 ax(t-\tau)}{1+by} - g_1 cx^2(t-\tau) + \frac{(g_2 \varphi_2 - g_1 \varphi_1)x(t-\tau)y(t-\tau)}{x(t-\tau)+m} + g_2(r-d)y \\ &< g_1 ax(t-\tau) - g_1 cx^2(t-\tau) + \frac{(g_2 \varphi_2 - g_1 \varphi_1)x(t-\tau)y(t-\tau)}{x(t-\tau)+m} + g_2(r-d)y. \end{aligned}$$

Set $g_1 = \varphi_2, g_2 = \varphi_1$, then

$$H' < \varphi_2 ax(t-\tau) - \varphi_2 cx^2(t-\tau) + \varphi_1(r-d)y.$$

Choose a constant $G > 0$ such that

$$\begin{aligned} H' + GH &< \varphi_2 ax(t-\tau) - \varphi_2 cx^2(t-\tau) + \varphi_1(r-d)y + G\varphi_2 x(t-\tau) + G\varphi_1 y(t) \\ &= \varphi_2(a+G)x(t-\tau) - \varphi_2 cx^2(t-\tau) + \varphi_1(r-d+G)y, \end{aligned}$$

take $G < d - r$, then $H' + GH < \varphi_2(a + G)x(t - \tau) - \varphi_2cx^2(t - \tau) = F(x(t - \tau))$. Since $F''(x(t - \tau)) = -2c\varphi_2 < 0$, so that $F(x(t - \tau))$ has a maximum value of $\frac{\varphi_2(a + G)^2}{4c}$.

Applying differential inequality theory results, we obtain

$$0 \leq H(t) \leq \frac{F}{G}(1 - e^{-Gt}) + H(x(0), y(0))e^{-Gt}, \quad t \geq 0.$$

As $t \rightarrow +\infty$, we have $0 < H(t) \leq \frac{F}{G}$. Therefore, all the solutions of system (2) are confined in the region

$$\Pi := \left\{ (x, y) \in \mathbb{R}_+^2, 0 \leq H(t) \leq \frac{F}{G} \right\}.$$

This completes the proof.

3. Equilibria and Its Stability

In this section, we will show all equilibrium of the model (2) and analyze its stability.

First, we can easily obtain the following results:

- 1) The trivial equilibrium point $A_0(0, 0)$;
- 2) The predator-free point $A_1(x_1, 0)$, where $x_1 = \frac{a}{c}$ always exist in $\mathbb{R}_+^2$;
- 3) The prey-free point $A_2(0, y_2)$, where $y_2 = \frac{(r-d)n}{\sigma-r+d}$, exists if $r - \sigma < d < r$;
- 4) The coexistence equilibrium point $A_3(x_3, y_3)$ satisfying

$$\begin{cases} \frac{ax_3}{1+by_3} - cx_3^2 - \frac{\varphi_1x_3y_3}{x_3+m} = 0, \\ \frac{\varphi_2x_3y_3}{x_3+m} + (r-d)y_3 - \frac{\sigma y_3^2}{y_3+n} = 0. \end{cases}$$

By solving the above equations, we get

$$x_3 = \frac{my_3(\sigma - r + d) + mn(d - r)}{(\varphi_2 + r - d - \sigma)y_3 + n(\varphi_2 + r - d)}.$$

The equilibrium point exists only if $r < d < \varphi_2 - \sigma + r$ and y_3 satisfies the equation

$$Ay^3 + By^2 + Cy + D = 0,$$

where

$$\begin{aligned} A &= am(d - r + \sigma)(\varphi_2 - d + r - \sigma) - cm^2[(d - r)^2 + 2\sigma] - \varphi_1\varphi_2(d - r), \\ B &= amn(d - r + \sigma)(\varphi_2 - d + r) + mn(d - r)(\varphi_2 - d + r - \sigma) \\ &\quad - 2cm^2(d - r)(d - \sigma) - \varphi_1n(d - r)(\varphi_2 - d + r), \\ C &= mn^2(d - r)(\varphi_2 - d + r) - c\sigma m^2(d - r)^2 - \varphi_1n^2(d - r)(\varphi_2 - d + r), \\ D &= \varphi_1n^2(d - r) - cm^2(d - r)^2. \end{aligned}$$

According to the Descartes rule of sign and $\frac{A}{D} < 0 (d \neq r)$, we find that it exists at least one positive root.

Next, we will discuss the stability of the equilibrium point. To study its local stability, we linearize the system at equilibrium point (x_i, y_i) to obtain the Jacobian matrix.

Let $\bar{x}(t) = x(t) - x_i, \bar{y}(t) = y(t) - y_i, (i = 0, 1, 2, 3)$, and for simplicity, we still denote them as $x(t), y(t)$. The linearized system is

$$\begin{cases} \frac{dx}{dt} = F_{10}x(t) + F_{01}y(t), \\ \frac{dy}{dt} = G_{010}x(t - \tau) + G_{100}y(t) + G_{001}y(t - \tau), \end{cases}$$

where

$$\begin{aligned} F_{10} &= \frac{a}{1 + by_i} - 2cx_i - \frac{\varphi_1 my_i}{(x_i + m)^2}, & F_{01} &= -\frac{abx_i}{(1 + by_i)^2} - \frac{\varphi_1 x_i}{x_i + m}, \\ G_{010} &= \frac{\varphi_2 my_i}{(x_i + m)^2}, & G_{100} &= r - d - \frac{\sigma y_i^2 + 2n\sigma y_i}{(y_i + n)^2}, & G_{001} &= \frac{\varphi_2 x_i}{x_i + m}. \end{aligned} \quad (4)$$

Then, the characteristic equation of the above system at (x_i, y_i) can be determined by

$$\begin{vmatrix} \lambda - \frac{a}{1 + by_i} + 2cx_i + \frac{\varphi_1 my_i}{(x_i + m)^2} & \frac{abx_i}{(1 + by_i)^2} + \frac{\varphi_1 x_i}{x_i + m} \\ -\frac{\varphi_2 my_i}{(x_i + m)^2} e^{-\lambda\tau} & \lambda + d - r + \frac{\sigma y_i^2 + 2n\sigma y_i}{(y_i + n)^2} - \frac{\varphi_2 x_i}{x_i + m} e^{-\lambda\tau} \end{vmatrix} = 0 \quad (5)$$

Theorem 2. The trivial equilibrium point $A_0(0, 0)$ is unstable for all $\tau \geq 0$.

Proof. Substitute A_0 into the characteristic Equation (5) to obtain

$$\begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda + d - r \end{vmatrix} = 0,$$

and further derive the eigenvalues $\lambda_1 = a > 0, \lambda_2 = r - d$. Then, if $r > d$ then A_0 is a source and if $r < d$ then A_0 is a saddle point. So, A_0 is always unstable for all $\tau \geq 0$.

Theorem 3. Assume that $\varphi_2 < \frac{(d-r)(a+cm)}{a}$ and

$0 \leq \tau < \tau' = \frac{1}{\sqrt{\left(\frac{\varphi_2 a}{a + cm}\right)^2 - (d-r)^2}} \arccos\left(\frac{(d-r)(a+cm)}{\varphi_2 a}\right)$. Then, the predator-

free equilibrium point $A_1(x_1, 0)$ is asymptotically stable.

Proof. Substitute A_1 into the characteristic Equation (5) to obtain

$$\begin{vmatrix} \lambda - a + 2cx_1 & abx_1 + \frac{\varphi_1 x_1}{x_1 + m} \\ 0 & \lambda + d - r - \frac{\varphi_2 x_1}{x_1 + m} e^{-\lambda\tau} \end{vmatrix} = 0,$$

where $x_1 = \frac{a}{c}$, after calculation, one eigenvalue is obtained as $\lambda_1 = -a < 0$, and the other eigenvalue is obtained from the following equation

$$\lambda + d - r - \frac{\varphi_2 a}{a + cm} e^{-\lambda \tau} = 0,$$

For $\tau = 0$, it is obtained $\lambda_2 = r - d + \frac{\varphi_2 a}{a + cm}$, if and only if $\varphi_2 < \frac{(d-r)(a+cm)}{a}$, the eigenvalue is negative, but when the previous assumption is not met, the eigenvalue λ_2 is positive and the equilibrium point A_1 is the saddle point.

Now, when $\tau > 0$, by [19] proposition 1, let $u = d - r$, $v = \frac{\varphi_2 a}{a + cm}$, the eigenvalue λ_2 has a negative real part when $u < v$ and

$$\tau < \tau' = \frac{1}{\sqrt{\left(\frac{\varphi_2 a}{a + cm}\right)^2 - (d-r)^2}} \arccos\left(\frac{(d-r)(a+cm)}{\varphi_2 a}\right) \text{ is satisfied. Otherwise,}$$

the eigenvalue λ_2 has a positive real part.

Therefore, for any $\tau \geq 0$, if $\varphi_2 < \frac{(d-r)(a+cm)}{a}$, the predator-free equilibrium point is locally asymptotically stable, while it is an unstable point for $0 < \tau < \tau'$ when the assumption doesn't hold.

This proves the theorem.

Theorem 4. Assume that $b > \frac{amu^2 - \varphi_1 u}{\varphi_1 n}$ and $r > d + \frac{\sigma u}{(1+u)^2}$, where

$u = \frac{\sigma - r + d}{r - d} > 0$. Then, the prey-free equilibrium point $A_2(0, y_2)$ is locally asymptotically stable for all $\tau \geq 0$.

Proof. Substitute A_2 into the characteristic Equation (5) to obtain

$$\begin{vmatrix} \lambda - \frac{a}{1 + by_2} + \frac{\varphi_1 y_2}{m} & 0 \\ -\frac{\varphi_2 y_2}{m} e^{-\lambda \tau} & \lambda + d - r + \frac{\sigma y_2^2 + 2n\sigma y_2}{(y_2 + n)^2} \end{vmatrix} = 0, \quad (6)$$

where $y_2 = \frac{(r-d)n}{\sigma - r + d}$. The characteristic Equation (6) becomes

$$\lambda^2 + A\lambda + B = 0, \quad (7)$$

after calculation, we get

$$A = d - r + \frac{\sigma y_2^2 + 2n\sigma y_2}{(y_2 + n)^2} - \frac{a}{1 + by_2} + \frac{\varphi_1 y_2}{m},$$

$$B = \left(d - r + \frac{\sigma y_2^2 + 2n\sigma y_2}{(y_2 + n)^2} \right) \left(-\frac{a}{1 + by_2} + \frac{\varphi_1 y_2}{m} \right),$$

In order to obtain the presence of negative real parts in the eigenvalues, it holds when $A > 0$ and $B > 0$, that is

$$\begin{cases} d - r + \sigma > \frac{\sigma n^2}{(y_2 + n)^2}, \\ \frac{\varphi_1 y_2}{m} > \frac{a}{1 + by_2}, \end{cases}$$

direct computation gives the following two expressions

$$b > \frac{amu^2 - \varphi_1 u}{\varphi_1 n} \quad \text{and} \quad r > d + \frac{\sigma u}{(1 + u)^2},$$

where $u = \frac{\sigma - r + d}{r - d} > 0$.

Therefore, the prey-free equilibrium point A_2 is locally asymptotically stable if the above assumptions hold.

This proves the theorem.

Theorem 5. The following results hold:

- 1) If $C^2 + 2B < A^2$ and $B^2 > D^2$, or $(C^2 + 2B - A^2)^2 - 4(B^2 - D^2) < 0$ holds, then the equilibrium point A_3 is locally asymptotically stable for any $\tau > 0$;
- 2) If $C^2 + 2B > A^2$, $B^2 > D^2$ and $(C^2 + 2B - A^2)^2 - 4(B^2 - D^2) > 0$ holds, then the equilibrium point A_3 is locally asymptotically stable for $\tau \in [0, \tau_0]$, A_3 is unstable when $\tau > \tau_0$;
- 3) If $B^2 < D^2$ or $C^2 + 2B > A^2$ and $(C^2 + 2B - A^2)^2 = 4(B^2 - D^2)$ holds, then the equilibrium point A_3 is locally asymptotically stable for $\tau \in [0, \tau_0]$, A_3 is unstable when $\tau > \tau_0$.

Proof. Substitute A_3 into the characteristic Equation (5) to obtain

$$\begin{vmatrix} \lambda - \frac{a}{1 + by_3} + 2cx_3 + \frac{\varphi_1 my_3}{(x_3 + m)^2} & \frac{abx_3}{(1 + by_3)^2} + \frac{\varphi_1 x_3}{x_3 + m} \\ -\frac{\varphi_2 my_3}{(x_3 + m)^2} e^{-\lambda\tau} & \lambda + d - r + \frac{\sigma y_3^2 + 2n\sigma y_3}{(y_3 + n)^2} - \frac{\varphi_2 x_3}{x_3 + m} e^{-\lambda\tau} \end{vmatrix} = 0.$$

The above characteristic equation becomes

$$\lambda^2 + A\lambda + B + (C\lambda + D)e^{-\lambda\tau} = 0, \quad (8)$$

where

$$\begin{aligned} A &= d - r + \frac{\sigma y_3^2 + 2n\sigma y_3}{(y_3 + n)^2} - \frac{a}{1 + by_3} + 2cx_3 + \frac{\varphi_1 my_3}{(x_3 + m)^2}, \\ B &= \left(d - r + \frac{\sigma y_3^2 + 2n\sigma y_3}{(y_3 + n)^2} \right) \left(-\frac{a}{1 + by_3} + 2cx_3 + \frac{\varphi_1 my_3}{(x_3 + m)^2} \right), \\ C &= -\frac{\varphi_2 x_3}{x_3 + m}, \\ D &= \left(\frac{a}{1 + by_3} - 2cx_3 \right) \frac{\varphi_2 x_3}{x_3 + m} + \frac{abm\varphi_2 x_3 y_3}{(1 + by_3)^2 (x_3 + m)^2}. \end{aligned}$$

For $\tau = 0$, the characteristic equation is

$$\lambda^2 + (A + C)\lambda + B + D = 0. \quad (9)$$

According to the Routh-Hurwitz criteria, if and only if $A + C > 0$, $B + D > 0$, Equation (9) has two roots and both of its real parts are negative. Hence, A_3 is locally asymptotically stable.

Now, when $\tau > 0$, assuming $\lambda = i\omega$ ($\omega > 0$) is a root of Equation (8), substituting $i\omega$ into it yields

$$(i\omega)^2 + A(i\omega) + B + (C(i\omega) + D)(\cos \omega\tau - i \sin \omega\tau) = 0,$$

then, by separating the real and imaginary parts of the above equation, it is obtained that:

$$\begin{cases} C\omega \sin \omega\tau + D \cos \omega\tau = \omega^2 - B, \\ C\omega \cos \omega\tau - D \sin \omega\tau = -A\omega. \end{cases} \quad (10)$$

By calculation, we obtain:

$$\omega^4 - E\omega^2 + F = 0, \quad (11)$$

where $E = C^2 + 2B - A^2$ and $F = B^2 - D^2$.

In the following, we will analyze the roots of Equation (11).

1) If $C^2 + 2B < A^2$ and $B^2 > D^2$, or $\Delta_1 = (C^2 + 2B - A^2)^2 - 4(B^2 - D^2) < 0$ holds, then Equation (11) has no positive roots.

2) If $C^2 + 2B > A^2$, $B^2 > D^2$ and $\Delta_2 = (C^2 + 2B - A^2)^2 - 4(B^2 - D^2) > 0$ holds, then Equation (11) has two positive roots, which are ω_+^2 and ω_-^2 , respectively, where

$$\begin{aligned} \omega_+^2 &= \frac{1}{2} \left[C^2 + 2B - A^2 + \sqrt{(C^2 + 2B - A^2)^2 - 4(B^2 - D^2)} \right], \\ \omega_-^2 &= \frac{1}{2} \left[C^2 + 2B - A^2 - \sqrt{(C^2 + 2B - A^2)^2 - 4(B^2 - D^2)} \right]; \end{aligned}$$

3) If $B^2 < D^2$ or $C^2 + 2B > A^2$ and $\Delta_3 = (C^2 + 2B - A^2)^2 - 4(B^2 - D^2) = 0$ holds, then Equation (11) has a positive root about, which is

$$\omega_+^2 = \frac{1}{2} \left[C^2 + 2B - A^2 + \sqrt{(C^2 + 2B - A^2)^2 - 4(B^2 - D^2)} \right];$$

In summary, it can be assumed that Equation (11) has two positive roots, denoted as ω_i ($i = 1, 2$), which can be substituted into the equation to obtain

$$\begin{cases} \sin \omega_i \tau = \frac{\omega_i^2 - D \cos \omega_i \tau - B}{C \omega_i}, \\ \cos \omega_i \tau = \frac{D(\omega_i - B) - C A \omega_i^2}{C^2 \omega_i + D^2}. \end{cases}$$

Then, τ_k can be expressed as

$$\tau_k = \frac{1}{\omega_i} \left(\arccos \frac{D(\omega_i - B) - C A \omega_i^2}{C^2 \omega_i + D^2} + 2k\pi \right), \quad i = 1, 2, \quad k = 0, 1, 2, \dots$$

Denote

$$\tau_0 = \min \tau_k, \quad k \geq 0.$$

This proves the theorem.

4. Hopf Bifurcation Analysis

In this section, the possibility of occurrence of Hopf bifurcation at the coexistence equilibrium point is discussed. Through the analyses in the preceding sections, we have known that Hopf bifurcation will occur under the appropriate conditions. According to Theorem (5), Equation (8) has a pair of purely imaginary roots $\pm i\omega_0$ at $\tau = \tau_0$, and all other roots have non-zero real parts. Therefore, let $\lambda(\tau) = \rho(\tau) \pm i\omega(\tau)$ be the pair of complex conjugate roots of Equation (8) at $\tau = \tau_k$, where $\rho(\tau_k) = 0, \omega(\tau_k) = \omega_i (i = 1, 2, k = 0, 1, 2, \dots)$.

According to the stability theory of time-delay differential equations, it can be concluded that when $\tau < \tau_0$, the system is stable and when $\tau > \tau_0$, the real part of the eigenvalues of the linearized system is discussed.

Theorem 6. If $\tau = \tau_0$, a Hopf bifurcation occur at the positive equilibrium point A_3 .

Proof. Firstly, we will verify the transversality conditions. Take the derivative of both sides of characteristic Equation (8) with respect to τ , we get

$$\left[2\lambda + A + Ce^{-\lambda\tau} - \tau(C\lambda + D)e^{-\lambda\tau} \right] \frac{d\lambda}{d\tau} = \lambda(C\lambda + D)e^{-\lambda\tau}, \quad (12)$$

Next,

$$\begin{aligned} \left(\frac{d\lambda}{d\tau} \right)^{-1} &= \frac{2\lambda + A + Ce^{-\lambda\tau} - \tau(C\lambda + D)e^{-\lambda\tau}}{\lambda(C\lambda + D)e^{-\lambda\tau}} \\ &= -\frac{2\lambda + A}{\lambda(\lambda^2 + A\lambda + B)} + \frac{C}{(C\lambda + D)\lambda} - \frac{\tau}{\lambda} \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Re} \left[\left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} \right]_{\tau=\tau_0} &= \operatorname{Re} \left[-\frac{2\lambda + A}{\lambda(\lambda^2 + A\lambda + B)} + \frac{C}{(C\lambda + D)\lambda} - \frac{\tau}{\lambda} \right]_{\lambda=i\omega_0} \\ &= \operatorname{Re} \left[\frac{C^2\omega_0^4 + (A^2C^2 + 2D^2)\omega_0^2 + (A^2 - 2B)D^2}{[(A^2 - 2B)\omega_0^4 + B^2\omega_0 + \omega_0^6](C^2\omega^2 + D^2)} \right] \\ &\quad - \operatorname{Re} \left[\frac{(A^2 + B^2)C^2}{[(A^2 - 2B)\omega_0^4 + B^2\omega_0 + \omega_0^6](C^2\omega^2 + D^2)} \right] \end{aligned}$$

$$\text{Clearly, } \operatorname{Re} \left[\left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} \right]_{\tau=\tau_0} \neq 0.$$

Therefore, the transversality conditions are satisfied, that is, the system has Hopf bifurcation at the positive equilibrium point A_3 .

This proves the theorem.

Next, we will explore the properties of the Hopf bifurcation, including the direction and stability. The direction of the periodic trajectory of the bifurcation of the positive equilibrium point A_3 at the critical of delay τ_0 and the stable periodic solution will be given through the center manifold and normal form theory.

Theorem 7. The following statements hold:

- 1) If $\mu_2 > 0$ ($\mu_2 < 0$), then a supercritical (subcritical) Hopf bifurcation occurs at τ_0 ;
- 2) If $\beta_2 > 0$ ($\beta_2 < 0$), then the bifurcating periodic solution is asymptotically stable (unstable);
- 3) If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solution increases (decreases).

Proof. Linearizing the system (2) in $C = C([0, 1], \mathbb{R}_+^2)$, yields the following time-delay differential equation:

$$U_t' = L_\mu(u_t) + F(\mu, u_t), \quad (13)$$

where $u(t) = (u_1(t), u_2(t))^T$, $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$ and $L_\mu : C \rightarrow \mathbb{R}_+^2$, $F : \mathbb{R} \times C \rightarrow \mathbb{R}_+^2$, with

$$L_\mu(\phi) = (\tau_0 + \mu) \begin{pmatrix} F_{10} & F_{01} \\ 0 & G_{100} \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_0 + \mu) \begin{pmatrix} 0 & 0 \\ G_{010} & G_{001} \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix},$$

$$F(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} F_{11}\phi_1^2(0) + F_{12}\phi_1(0)\phi_2(0) + F_{22}\phi_2^2(0) \\ G_{011}\phi_1^2(-1) + G_{101}\phi_2^2(0) + G_{111}\phi_1(-1)\phi_2(-1) \end{pmatrix},$$

where $F_{10}, F_{01}, G_{100}, G_{010}, G_{001}$ are given in Equation (4),

$$F_{11} = \frac{2\varphi_1 m y}{(x+m)^3} - 2C, \quad F_{12} = -\frac{ab}{(1+by)^2} - \frac{\varphi_1 m}{(x+m)^2},$$

$$F_{22} = \frac{2ab^2 x}{(1+by)^3}, \quad G_{011} = -\frac{\varphi_2 m y}{(x+m)^3},$$

$$G_{101} = -\frac{2\sigma}{y+n} + \frac{2\sigma y(y+2n)}{(y+n)^3},$$

$$G_{111} = \frac{\varphi_2 m}{(x+m)^2}, \quad G_{112} = 0 = G_{211} = 0.$$

According to the Riesz representation theorem, there exists a bounded variation function $\varpi(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^0 d\varpi(\theta, \mu) \phi(\theta), \quad \phi \in C([-1, 0], \mathbb{R}_+^2),$$

In fact, it can be chosen

$$\varpi(\theta, \mu) = (\tau_0 + \mu) \left[\begin{pmatrix} F_{10} & F_{01} \\ 0 & G_{100} \end{pmatrix} \delta(\theta) - \begin{pmatrix} 0 & 0 \\ G_{010} & G_{001} \end{pmatrix} \delta(\theta+1) \right],$$

where $\delta(\theta)$ is Dirac delta function and is defined as

$$\delta(\theta) = \begin{cases} 0, & \theta = 0, \\ 1, & \theta \neq 0. \end{cases}$$

For $\phi \in C([-1, 0], \mathbb{R}_+^2)$, define

$$J(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\varpi(\mu, \theta)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$K(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then, system (13) is equivalent to the following abstract operator equation:

$$u'_t = J(\mu)u_t + K(\mu)u_t. \quad (14)$$

For $\psi \in C([-1, 0], (\mathbb{R}^2_+)^*)$, the adjoint operator J^* of $J(\mu)$ is defined as

$$J^*\psi(\xi) = \begin{cases} -\frac{d\psi(\xi)}{d\xi}, & \xi \in [-1, 0), \\ \int_{-1}^0 d\varpi^T(\mu, t)\psi(-t), & \xi = 0, \end{cases}$$

and define the bilinear inner product as

$$\langle \psi(\xi), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_0^\theta \bar{\psi}(\xi - \theta) d\varpi(\theta)\phi(\xi) d\xi, \quad (15)$$

where $\varpi(\theta) = \varpi(\theta, 0)$ and $\langle \psi, J\phi \rangle = \langle J^*\psi, \phi \rangle$, $J = J(0)$ and $J^* = J^*(0)$ are adjoint operators. The eigenvalue corresponding to J is $\pm i\omega_0\tau_0$. It is easy to know that $\pm i\omega_0\tau_0$ is also the eigenvalue of the conjugate operator J^* .

Let

$$q(\theta) = (1, \eta)^T e^{i\omega_0\tau_0\theta} \quad \text{and} \quad q^*(\xi) = (1, \eta^*)^T e^{i\omega_0\tau_0\xi}, \quad \theta, \xi \in [-1, 0],$$

where $\eta = \frac{G_{010}e^{-i\omega_0\tau_0}}{G_{100} + G_{001}e^{i\omega_0\tau_0}}$, $\eta^* = -\frac{F_{01}}{G_{100} + G_{001}e^{-i\omega_0\tau_0}}$. $q(\theta)$ and $q^*(\xi)$ are the eigenvectors of $J(0)$ and $J^*(0)$ to the eigenvalues $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. Moreover, determine the parameter value of D , such that:

$$\langle q^*(\eta), q(\theta) \rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q} \rangle = 0. \quad (16)$$

According to Equation (15), we have

$$\begin{aligned} & \langle q^*(\eta), q(\theta) \rangle \\ &= \bar{D}(1, \bar{\eta}^*)(1, \eta)^T - \int_{-1}^0 \int_0^\theta \bar{D}(1, \bar{\eta}^*)e^{i\omega_0\tau_0(\xi-\theta)} d\varpi(1, \eta)^T e^{i\omega_0\tau_0\xi} d\xi \\ &= \bar{D}[1 + \eta\bar{\eta}^* + \tau_0\bar{\eta}^*(G_{010} + \eta G_{001})e^{i\omega_0\tau_0}]. \end{aligned}$$

Therefore, due to (16), it is obtained that

$$\begin{aligned} \bar{D} &= [1 + \eta\bar{\eta}^* + \tau_0\bar{\eta}^*(G_{010} + \eta G_{001})e^{i\omega_0\tau_0}]^{-1}, \\ D &= [1 + \bar{\eta}\eta^* + \tau_0\eta^*(G_{010} + \bar{\eta} G_{001})e^{i\omega_0\tau_0}]^{-1}. \end{aligned}$$

According to the theory in [27], we can obtain the property of Hopf bifurcation. Calculating the direction of the Hopf bifurcation and the stability coefficient of the periodic solution at the positive equilibrium point A_3 :

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_0[F_{11} + \eta F_{12} + \bar{\eta}^*(\eta e^{-2i\omega_0\tau_0}G_{111} + \eta^2 G_{101} + e^{-2i\omega_0\tau_0}G_{011})], \\ g_{11} &= \bar{D}\tau_0[2F_{11} + (\eta + \bar{\eta})F_{12} + \bar{\eta}^*(2G_{011} + 2\eta\bar{\eta}G_{101} + (\eta + \bar{\eta})G_{111})], \end{aligned}$$

$$g_{02} = 2\bar{D}\tau_0 \left[F_{11} + \bar{\eta}F_{12} + \bar{\eta}^* \left(\bar{\eta}e^{2i\omega_0\tau_0}G_{111} + \bar{\eta}^2G_{101} + e^{2i\omega_0\tau_0}G_{011} \right) \right],$$

$$g_{21} = 2\bar{D}\tau_0 \left[F_{12} \left(\eta H_{11}(0) + \frac{1}{2}\bar{\eta}H_{20}(0) + \frac{1}{2}H_{20}(0) + H_{11}(0) \right) \right]$$

$$+ 2\bar{D}\tau_0 \left[F_{11} \left(H_{20}(0) + 2H_{11}(0) \right) + \bar{\eta}^* (Q_1G_{111} + Q_2G_{101} + Q_3G_{011}) \right],$$

where

$$Q_1 = \frac{1}{2}\bar{\eta}H_{20}(-1)e^{i\omega_0\tau_0} + \eta H_{11}(-1)e^{-i\omega_0\tau_0} + H_{11}(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}H_{20}(-1)e^{i\omega_0\tau_0},$$

$$Q_2 = \bar{\eta}H_{20}(0) + 2\eta H_{11}(0),$$

$$Q_3 = H_{20}(-1)e^{i\omega_0\tau_0} + 2H_{11}(-1)e^{-i\omega_0\tau_0},$$

with

$$H_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}\eta(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{\eta}(0)e^{i\omega_0\tau_0\theta} + M_1e^{2i\omega_0\tau_0\theta},$$

$$H_{11}(\theta) = -\frac{ig_{11}}{\theta\tau_0}\eta(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{\eta}(0)e^{-i\omega_0\tau_0\theta} + M_2,$$

and

$$M_1 = 2 \begin{pmatrix} 2i\omega_0 - F_{10} & -F_{01} \\ -G_{010}e^{2i\omega_0\tau_0} & 2i\omega_0 - (G_{100} + G_{001}e^{i\omega_0\tau_0}) \end{pmatrix}^{-1} \begin{pmatrix} \eta F_{12} + F_{11} \\ \eta e^{-2i\omega_0\tau_0}G_{111} + \eta^2G_{101} + e^{-2i\omega_0\tau_0}G_{011} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -F_{10} & -F_{01} \\ -G_{010} & -(G_{100} + G_{001}) \end{pmatrix}^{-1} \begin{pmatrix} 2F_{11} + (\eta + \bar{\eta})F_{12} \\ 2G_{011} + 2\eta\bar{\eta}G_{101} + (\eta + \bar{\eta})G_{111} \end{pmatrix}.$$

Based on the above analysis and the expressions for g_{20}, g_{11}, g_{02} and g_{21} , we get the following values:

$$C_1(0) = \frac{i}{2\omega_0\tau_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2\operatorname{Re}\{C_1(0)\}, \quad T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}.$$

The above parameters determine the Hopf bifurcation.

5. Numerical Simulation

In this section, we will show some numerical simulation results to support the above research.

For this purpose, we choose the initial value condition $(x_0, y_0) = (1, 1.5)$ and refer to some parameters in [22] to obtain the following parameters

$$a = 0.6, b = 0.05, c = 0.5, \varphi_1 = 0.35, m = 0.5,$$

$$\varphi_2 = 0.3, r = 0.2, d = 0.3, \sigma^* = 0.1, n^* = 2. \quad (17)$$

In this case, there is a unique coexistence equilibrium point $(0.4056, 1.0468)$, and $\tau_0 = 1.402$.

Firstly, fixed parameter (17), we obtained the solution and phase diagram of the

positive equilibrium point, as well as the Hopf bifurcation diagram, as shown below.

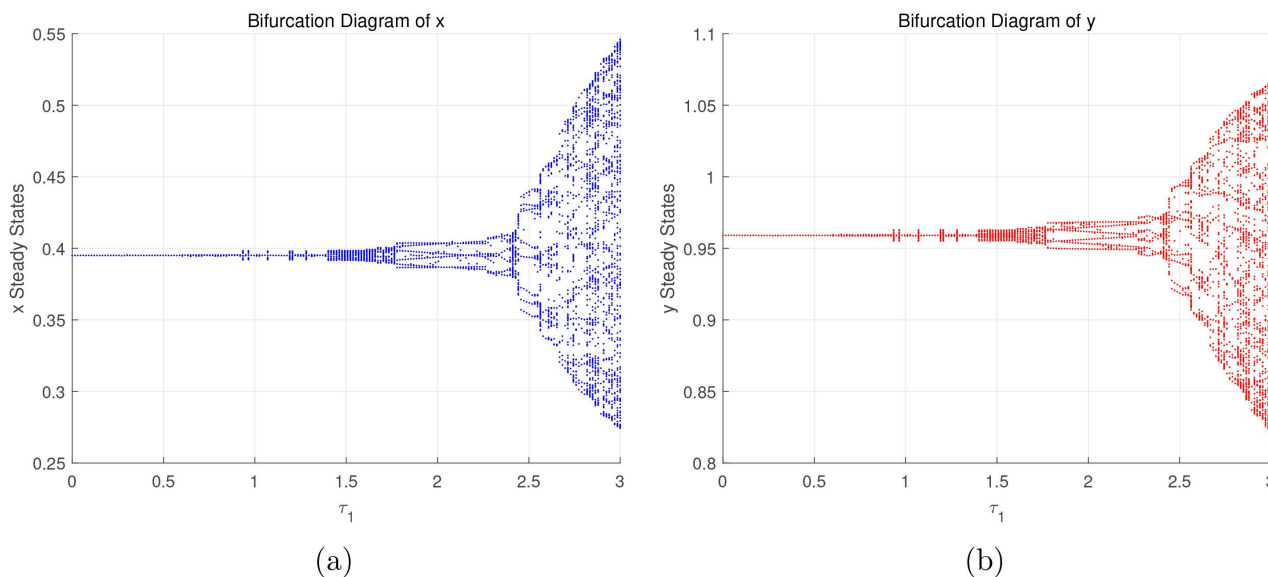


Figure 1. The Hopf bifurcation diagram.

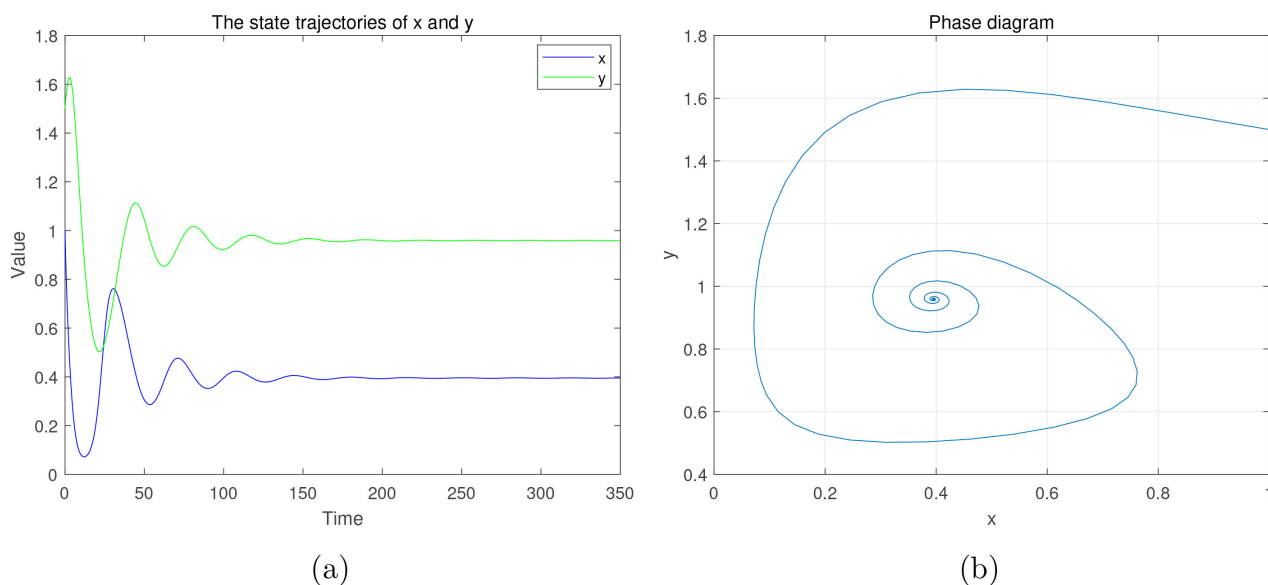


Figure 2. Solution and phase diagram of system.

From **Figure 1**, we can find that the system has a Hopf bifurcation as the parameter τ passing the bifurcation point $\tau_0 = 1.402$ (see **Figure 1**). From **Figure 2**, the solution of the system is approached to the coexistence equilibrium point $A_3 = (0.4056, 1.0468)$ starting from the initial point (see **Figure 2**).

Next, we analyze the impact of the parameters σ and n on the system.

1) Taking $\sigma = 0.05 < \sigma^*$, other parameters are the same as in (17), the numerical simulations are as follows.

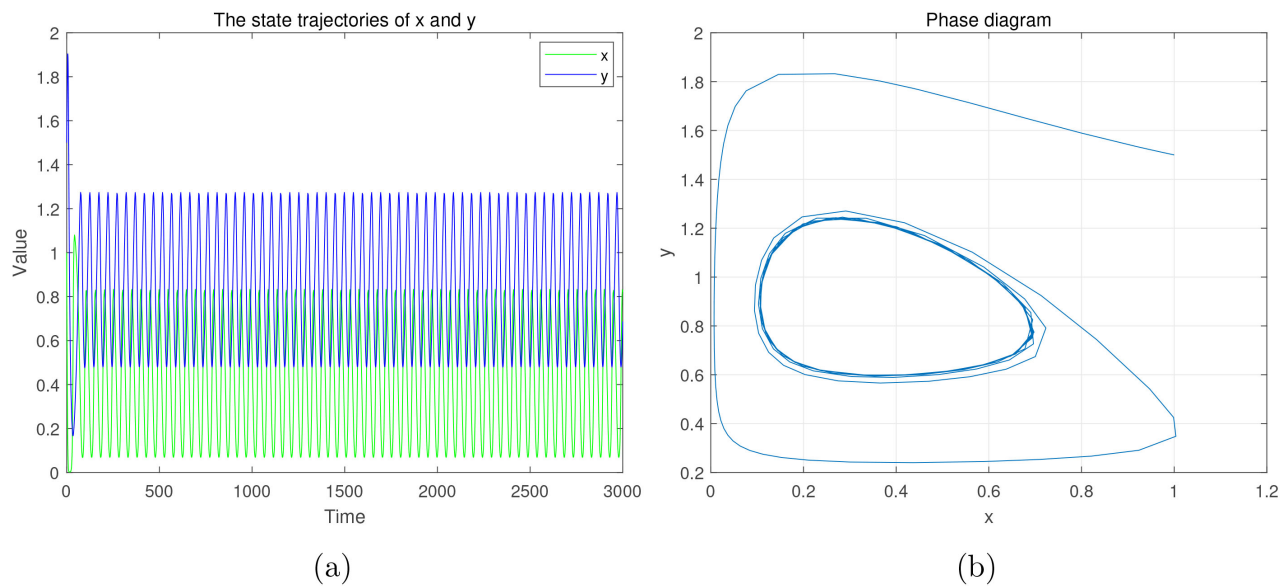


Figure 3. The system solution and phase diagrams of cannibalism rate in small-parameter predator individuals.

We can see from **Figure 3** that the cannibalism rate σ of the rate at which cannibalism occurs within predator individuals below a specific value destabilizes the coexistence equilibrium point and the system has periodic solution (see **Figure 3**).

2) Taking $n = 2.5 > n^*$, other parameters are the same as in (17), the numerical simulations are as follows.

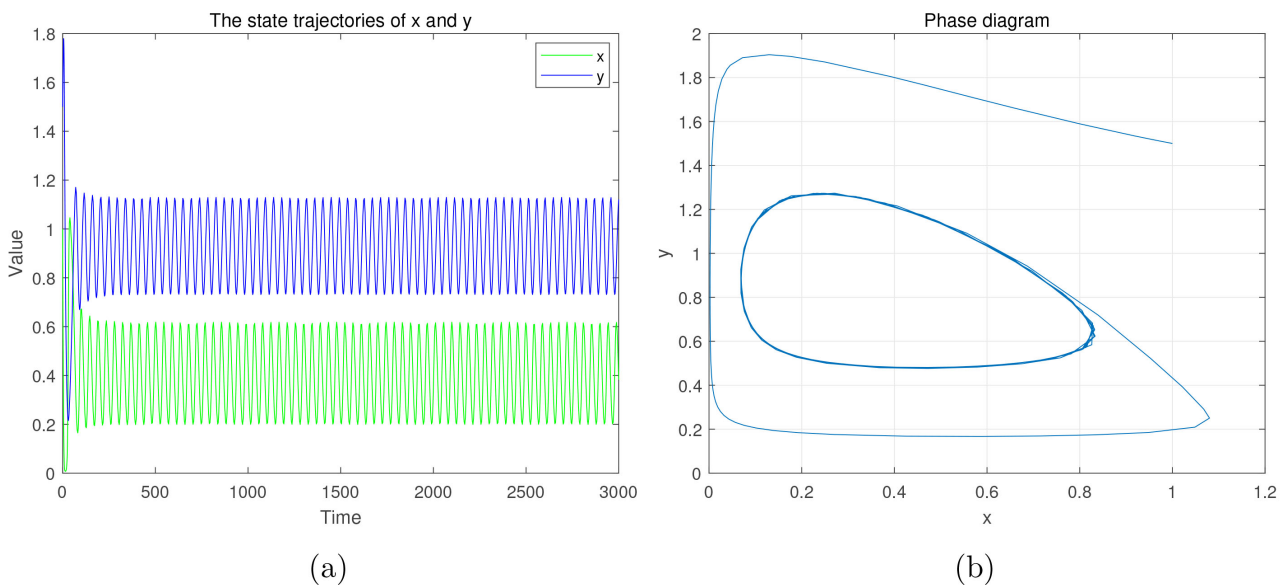


Figure 4. The system solution and phase diagrams of cannibalism rate in big-parameter predator individuals.

From **Figure 4**, we know that increase the conversion rate causes destabilizing the coexistence equilibrium point and the system has a stable limit cycle (see **Figure 4**).

Finally, we analyze the impact of the time delay τ on the model, fixed parameter

in (17), we get the following results:

1) Taking $\tau = 1.39 \in [0, \tau_0]$, the numerical simulations are as follows.

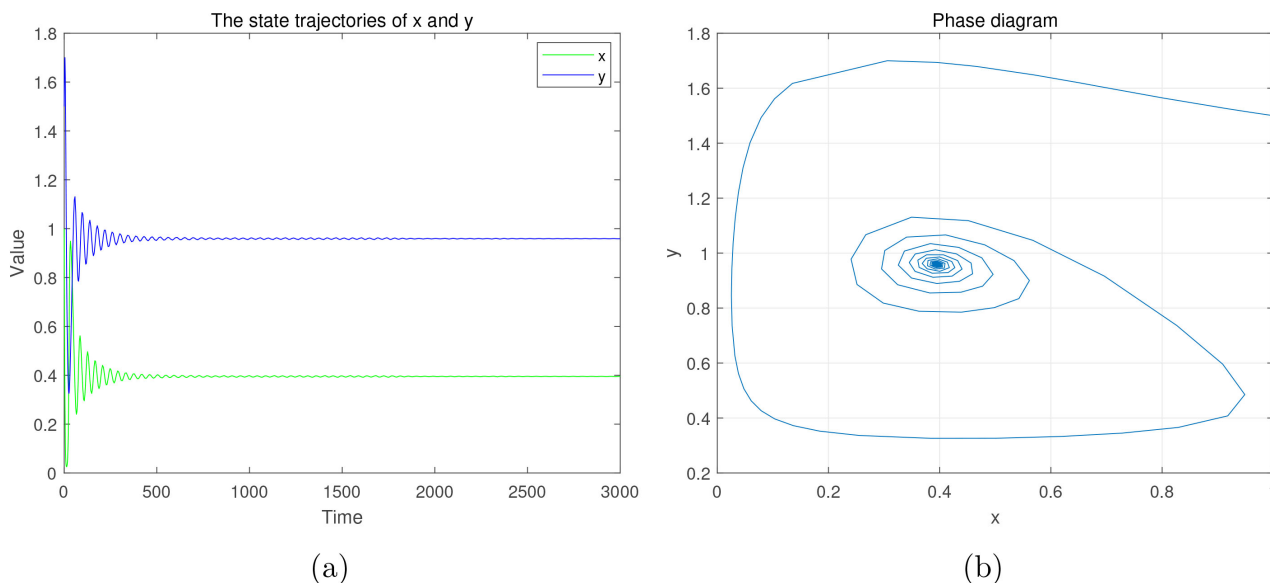


Figure 5. Solution and phase diagram of model when $\tau = 1.39$.

2) Taking $\tau = 2.5 > \tau_0$, the numerical simulations are as follows.

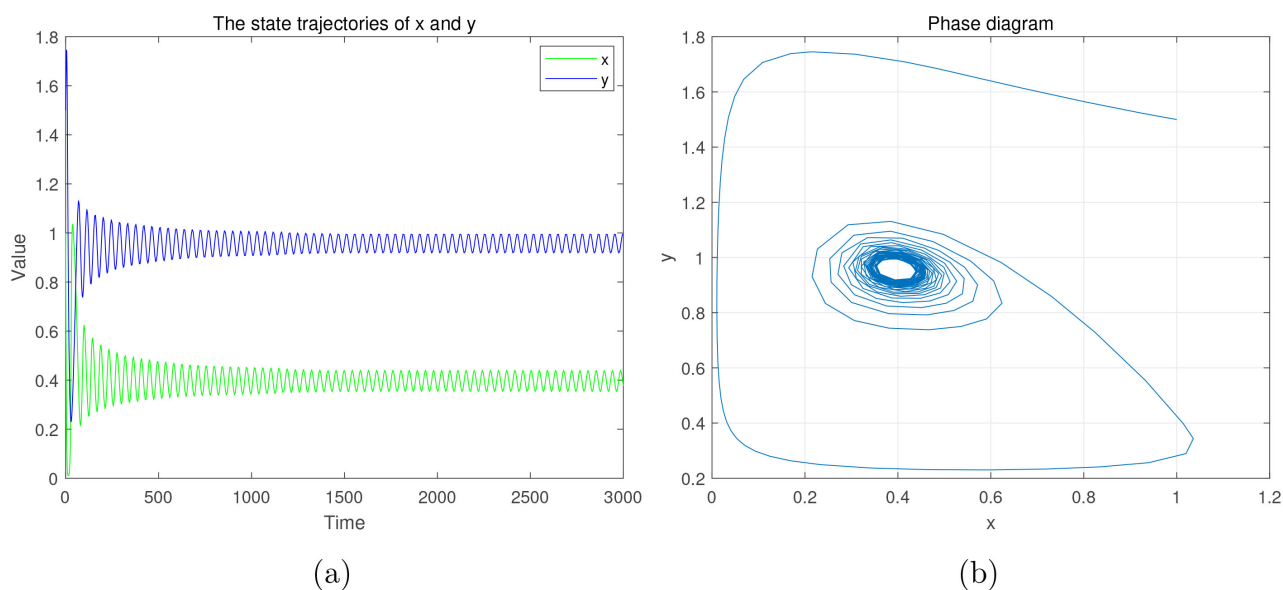


Figure 6. Solution and phase diagram of model when $\tau = 2.5$.

We can see from **Figure 6** that when the pregnancy delay parameter is greater than $\tau_0 = 1.402$, the system has a Hopf bifurcation (see **Figure 5** and **Figure 6**).

6. Conclusions

This paper proposes a predator-prey model that considers the existence of the fear

effect and the gestation delay, as well as cannibalism. Firstly, we prove the positivity and boundedness of the system (2). Secondly, it is obtained that the system (2) has four possible non-negative equilibrium points. The local stability of them is studied for $\tau \geq 0$, and the stability conditions are determined. The existence of Hopf bifurcation as a function of τ is proved. The stability and direction of the bifurcated periodic dynamics are investigated using the center manifold and normal form theory. Finally, numerical simulation results present the influence of different model parameters on the dynamics of system.

The research results indicate that cannibalism and delay pregnancy in predator populations have significant impacts on population dynamics. The final state and stability of a population mainly depend on predation behavior and parameter configuration. The research findings provide a new perspective for understanding population interactions in ecosystems, emphasizing the crucial role of cannibalism and delayed pregnancy of predator populations in ecosystem stability and population dynamics. Future research can further explore the dynamic characteristics of population models under different predation relationships, which provide insight into species interactions and stability in ecosystems.

Acknowledgements

The authors are grateful to Dr. Danfeng Pang for her valuable suggestion leading to a substantial improvement of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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