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Least Product Relative Error Estimation for Partially Linear Multiplicative Model with Monotonic Constraint

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Abstract

We consider the partially linear multiplicative model with monotonic constraint for the analysis of positive response data. We propose a constrained least product relative error (LPRE) estimation procedure for the model by means of B-spline basis expansion. We have also established asymptotic properties of the proposed estimators under regularity conditions. We finally provide numerical simulations and a real data application to assess the finite sample performance of the developed methodology.

Keywords

Partially Linear Multiplicative Model, Monotonic Constraint, Least Product Relative Error, B-Spline, Asymptotic Property

1. Introduction

Linear multiplicative models are popular tools for analyzing data with positive responses. However, the linear structure of models is too restrictive on the regression relation, which may lead to a high risk of model misspecification when dealing with more complicated data. To compensate for this defect, scholars have proposed many general and powerful multiplicative models that allow nonparametric or semi-parametric modeling. Examples of these models include the partially linear multiplicative model, which takes the following structure:

$$Y = \exp\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta} + f(U)\right)\epsilon, \tag{1.1}$$

where Y is the response, $\boldsymbol{X} = \left(X_1, \cdots, X_p\right)^{\mathrm{T}}$ is a p-dimensional random covariates vector, $\boldsymbol{\beta} = \left(\beta_1, \cdots, \beta_p\right)^{\mathrm{T}}$ is a p-dimensional vector of unknown parameter, $f(\cdot)$ is an unknown smooth function, covariate U ranges over a non-degenerate compact interval, ϵ is the model error term. Both Y and ϵ defined in model (1.1) are positive. By applying logarithmic transformation to model (1.1), the above model becomes the usual partially linear model $Y^* = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta} + f(U) + \epsilon^*$ with $Y^* = \log(Y)$ and $\epsilon^* = \log(\epsilon)$. By allowing the response variable to depend linearly on some covariates and nonlinearly on remains through an unknown smooth function, the partially linear model enjoys both interpretation property of parametric modeling and flexibility of nonparametric modeling.

For logarithmic transformation of model (1.1), the absolute error based methods such as the least squares method and the least absolute deviation approach can be used to estimate $\boldsymbol{\beta}$ and $f(\cdot)$, however it may lose the intuitive explanatory sense for the transformed models. In many statistical applications, consideration of the relative error sometimes are more attractive than that of the absolute error for positive response data analysis. For example, under nonlinear regression framework, Khoshgoftaar *et al.* [1] studied the strong consistency of the estimators in the case of both squared relative error and absolute relative error. Park and Stefanski [2] derived the form of the best mean squared relative error prediction and adopted it into county-level gasoline usage prediction. Inspired by these, Chen *et al.* [3] proposed least absolute relative error (LARE) criterion for linear multiplicative models and they further applied proposed method to a study of stock returns in Hong Kong Stock Exchange. Xia *et al.* [4] discussed the interpretation of LARE criterion through a case study of stock price data from the views of buyers and sellers, they aimed to investigate the variable selection problem of linear multiplicative model with a diverging number of covariates. For model (1.1), Zhang and Wang [5] proposed the semi-parametric LARE criterion for estimating both parametric and nonparametric parts with help of kernel smoothing technique. However, the optimization of LARE criterion is non-smoothing and the computation is complicated. To this end, Chen *et al.* [6] developed the least product relative error (LPRE) criterion, the LPRE objective function is infinitely differentiable and strictly convex which makes the computation very convenient, while possess favourable statistical properties for resulting estimators. They further demonstrated the effectiveness of the LPRE estimation over some existing estimations under certain conditions. For model (1.1), Zhang *et al.* [7] proposed a profile LPRE estimation method for parametric components. This method has also been extended to other semi-parametric multiplicative models for estimation and inference, the recent literature include but is not restricted to Hu [8] for varying coefficient multiplicative models, Liu and Xia [9] for single index multiplicative models, Ming *et al.* [10] for multiplicative additive models, Zhang *et al.* [11] for varying coefficient single index multiplicative models. For more intuitive explanation of the advantages of the LPRE method, see the discussion of real monthly income for any two people in Ming *et al.* [10].

It is worth noting that the above mentioned works on model (1.1) take unspecified form for nonparametric function $f(\cdot)$. In many applications the nonparametric component $f(\cdot)$ may be monotone constrained. For example, dose response curves in some clinical trials and growth curves in biomedical studies are known to be increasing. There are many references focused on monotone constrained partially linear model include but are not restricted to the following ones. Lu [12] used monotone B-spline to approximate the monotone nonparametric function and applied the generalized Rosen algorithm to compute the estimators jointly. Du *et al.* [13] studied the M-estimation of partially linear model under monotonic constraints. Sun *et al.* [14] investigated the isotonic partially linear error-in-variable model with randomly right censored response. Boente *et al.* [15] considered robust estimators for generalized partially linear regression model in which the nonparametric component is assumed to be a monotone function. Zhang and Wang [16] proposed a kernel based method for the monotone estimation of the nonparametric function component. There is a growing literature on monotone constrained partially linear model, but no such work exists for the model (1.1) with monotonic constraint up to now. It is, therefore, our impetus for solving this problem.

In this study, we extend the partially linear multiplicative model (1.1) to the situation in which the nonparametric component $f(\cdot)$ is specified with monotone constraint, and we assume that $f(\cdot)$ is a nondecreasing function without loss of generality. We investigate the LPRE-based estimators along with their theoretical results for the model (1.1) with monotonic constraint. Our work is also motivated by analyzing an environmental data set elaborated in Section 4, as we will see, there exists a monotonic relationship between concentration of NO_2 and traffic volume. Thus, our main goal is to examine the association between the levels of air pollutants and the number of cars per hour although the air pollutants are always influenced by potential confounding effects from other variables. It is not a unique model that fits the current data set, but our studies provide a useful perspective in exploring hidden structures for environmental data modeling.

The rest of this paper is organized as follows. In Section 2, we propose a constrained least product relative error estimation method for partially linear multiplicative model with monotonic constraint using spline approximation and constrained nonlinear programming, and then we provide the theoretical properties of the resulting estimators for both parametric and nonparametric components. In Section 3, we present some simulation studies to illustrate the merits of proposed method compared with existing ones. In Section 4, we apply our method to a real data application. We conclude the paper by mentioning some possible extensions in Section 5.

2. Estimation and Asymptotic Properties

This section focuses on the constrained LPRE estimation of model (1.1) and delves into the algorithm for computation. We also state the main asymptotic results of developed estimators for both the parametric and the nonparametric terms.

2.1. Estimation

Let $\{Y_i, \boldsymbol{X}_i, U_i\}$, $i = 1, \cdots, n$ be independent and identically distributed (i.i.d.) copies of $\{Y, \boldsymbol{X}, U\}$, the sample version of model (1.1) is given by

$$Y_i = \exp\left(\boldsymbol{X}_i^{\mathrm{T}} \boldsymbol{\beta} + f(U_i)\right)\epsilon_i. \tag{2.1}$$

To address the estimating issue of $\boldsymbol{\beta}$ and $f(\cdot)$ simultaneously, we adopt the B-spline basis functions approximation to covert the estimation problem of model (2.1) to the problem of regression coefficients estimating in the linear combinations framework. We first provide a brief review about the construction of these basis functions. Without loss of generality, we assume that the compact support set of U is $\mathcal{S}_U = [0,1]$. Let $\boldsymbol{B}(u) = \left(B_m(u) : 1 \le m \le J_n\right)^{\mathrm{T}}$ be the B-splines basis functions of order q, where $J_n = N_n + q$ and N_n is the number of interior knots for a knot sequence given as

$$\xi_1 = \cdots = 0 = \xi_q < \xi_{q+1} < \cdots < \xi_{N_n+q} < 1 = \xi_{N_n+q+1} = \cdots = \xi_{N_n+2q}.$$

For $q \le \iota \le J_n$, this knot sequence satisfies

$$\max_{q \le \iota \le J_n} \left|\xi_{\iota+1} - \xi_\iota\right| \Big/ \min_{q \le \iota \le J_n} \left|\xi_{\iota+1} - \xi_\iota\right| \le M,$$

for some constants $0 < M < \infty$. We refer the interested readers to Huang [17] for more details. Under some smoothness assumptions, we can approximate $f(\cdot)$ by

$$f(U_i) \approx \sum_{m=1}^{J_n} B_m(U_i)\gamma_m = \boldsymbol{B}^{\mathrm{T}}(U_i)\boldsymbol{\gamma}, \tag{2.2}$$

where $\boldsymbol{\gamma} = \left(\gamma_m : 1 \le m \le J_n\right)^{\mathrm{T}}$.

Follows Theorem 5.9 of Schumaker [18], the spline $f(\cdot)$ is monotonically non-decreasing on $\mathcal{S}_U$ if non-decreasing constraints are imposed on the coefficients $\boldsymbol{\gamma} = \left(\gamma_m : 1 \le m \le J_n\right)^{\mathrm{T}}$, *i.e.*,

$$\gamma_1 \le \gamma_2 \le \cdots \le \gamma_{J_n},$$

this ensures the nondecreasing property of nonparametric function $f(\cdot)$.

Based on the above analysis, we can rewrite model (2.1) as

$$Y_i \approx \exp\left(\boldsymbol{X}_i^{\mathrm{T}} \boldsymbol{\beta} + \boldsymbol{B}^{\mathrm{T}}(U_i)\boldsymbol{\gamma}\right)\epsilon_i, \tag{2.3}$$

which transforms the model (2.1) into an almost equivalent linear multiplicative model. Let $\boldsymbol{\Pi}_i = \left(\boldsymbol{X}_i^{\mathrm{T}}, \boldsymbol{B}^{\mathrm{T}}(U_i)\right)^{\mathrm{T}}$, then following Chen *et al.* [6], we estimate the related coefficients $\boldsymbol{\vartheta} = \left(\boldsymbol{\beta}^{\mathrm{T}}, \boldsymbol{\gamma}^{\mathrm{T}}\right)^{\mathrm{T}}$ by minimizing

$$\sum_{i=1}^{n} \left\{ \left| \frac{Y_i - \exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}}\boldsymbol{\vartheta}\right)}{Y_i} \right| \times \left| \frac{Y_i - \exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}}\boldsymbol{\vartheta}\right)}{\exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}}\boldsymbol{\vartheta}\right)} \right| \right\}, \tag{2.4}$$

subject to the constraint $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_{J_n}$, which is equivalent to solve the following

$$\mathcal{L}_n(\boldsymbol{\vartheta}) \equiv \sum_{i=1}^{n} \left\{ Y_i \exp\left(-\boldsymbol{\Pi}_i^{\mathrm{T}}\boldsymbol{\vartheta}\right) + Y_i^{-1} \exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}}\boldsymbol{\vartheta}\right) \right\}, \tag{2.5}$$

with constraint $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_{J_n}$.

Now we delve into the algorithm for solving (2.5), since minimization problem (2.5) requires constrained nonlinear programming, we use the constrOptim package in R software to ensure restrictive condition $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_{J_n}$ and achieve this optimization. To use the constrOptim algorithm, we need to specify the gradient of the objective function $\mathcal{L}_n(\boldsymbol{\vartheta})$ in (2.5) given as,

$$\frac{\partial \mathcal{L}_n(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}} = \sum_{i=1}^{n} \boldsymbol{\Pi}_i^{\mathrm{T}} \left\{ Y_i^{-1} \exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}} \boldsymbol{\vartheta}\right) - Y_i \exp\left(-\boldsymbol{\Pi}_i^{\mathrm{T}} \boldsymbol{\vartheta}\right) \right\}. \tag{2.6}$$

Denote the final estimator in (2.5) as $\hat{\boldsymbol{\vartheta}}$, we call it monotone constrained LPRE estimator, then the estimator for $f(\cdot)$ is given by $\hat{f}(\cdot) = \boldsymbol{B}^{\mathrm{T}}(U_i)\hat{\boldsymbol{\gamma}}$.

2.2. Asymptotic Properties

The asymptotic properties of the proposed estimators are studied in this subsection. Let $\boldsymbol{\beta}_0$ and $f_0(\cdot)$ be the true values of $\boldsymbol{\beta}$ and $f(\cdot)$ in model (1.1). We use $\|\cdot\|$ to denote the L^2 norm for functions. For some positive series a_n and b_n, $a_n \asymp b_n$ means $\lim_{n\to\infty} a_n/b_n = c$ for some nonzero constant $c > 0$. The following regularity conditions are required.

(C1). The covariate U_i has a continuous density $g_{U_i}(u)$ which is bounded away from 0 and infinity on $\mathcal{S}_U$ for every $i = 1, \cdots, n$.

(C2). $f(\cdot) \in C^{(r)}(\mathcal{S}_U)$ for some integer $r \ge 2$, and the spline order satisfies $q \ge r$. $C^{(r)}(\mathcal{S}_U) = \left\{\varphi \mid \varphi^{(r)} \in C(\mathcal{S}_U)\right\}$ denotes the space of r-th order smooth function.

(C3). The matrix $E\left(\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\right)$ is nonsingular and its eigenvalues are uniformly bounded away from 0 and infinity.

(C4). The error ϵ satisfies $E\left(\epsilon + \epsilon^{-1} \mid \boldsymbol{X}, U\right) < \infty$.

(C5). The error ϵ satisfies $E\left(\epsilon \mid \boldsymbol{X}, U\right) = E\left(\epsilon^{-1} \mid \boldsymbol{X}, U\right)$.

Conditions (C1)-(C3) are standard for the nonparametric estimation in spline smoothing literature, which are essentially same as those in Shen *et al.* [19] and Guo *et al.* [20] Condition (C4) ensures the asymptotic normality of the estimator for parametric term, see Liu and Xia [9] and Hu [8]. Condition (C5) is needed for model identification and asymptotic variance of estimates, see Ming *et al.* [10] and Hu [8].

Theorem 1 *Under conditions (C1)-(C5), if* $N_n \asymp n^{1/(2r+1)}$, *then we have*

$$\left\|\hat{f}(u) - f_0(u)\right\|^2 = O_p\left(n^{-2r/(2r+1)}\right).$$

Theorem 1 indicates that the nonparametric estimates obtained by our proposed method achieve the optimal convergence rates. The following Theorem 2 states that the estimator $\hat{\boldsymbol{\beta}}$ is asymptotically normal.

Theorem 2 *Under the same assumptions of Theorem 1,* $\hat{\boldsymbol{\beta}}$ *converges in probability to the true value* $\boldsymbol{\beta}_0$, *i.e.*

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\right) \xrightarrow{d} N\left(0, \Sigma^{-1}\Lambda\Sigma^{-1}\right),$$

where $\xrightarrow{d}$ denotes convergence in distribution, $\Sigma = E\left\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\left(\epsilon + \epsilon^{-1}\right)\right\}$ and

$\Lambda = E\left\{ \boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \left(\epsilon - \epsilon^{-1} \right)^2 \right\}.$

Since the proofs of Theorems 1-2 follow along the same ideas as the proofs of Theorems in Ming *et al.* [10] although part of details differs, we omit the proofs in this paper.

3. Numerical Simulations

In this section, we carry out numerical simulations to investigate the finite sample performance of the proposed method. We generate the random samples from the following model:

$$Y_i = \exp\left(X_{i1}\beta_1 + X_{i2}\beta_2 + f\left(U_i\right) \right)\epsilon_i, \quad i = 1, \cdots, n,$$

where $n = 100, 200, 500$, $\boldsymbol{X}_i = \left(X_{i1}, X_{i2} \right)^{\mathrm{T}}$, X_{i1} and X_{i2} follows a bivariate normal distribution with mean 0, variance 1, and covariance 0.5, the true parameters $\beta_1 = 1$ and $\beta_2 = 3.5$, we set $f\left(U_i\right) = \left(U_i - 1\right)^3$ and the variable U_i's are sampled uniformly on $\left[0, 2\right]$. The following three cases for the random error ϵ are considered:

Case I. The log normal distribution $\log\left(\epsilon\right) \sim N\left(0,1\right)$.

Case II. The log uniform distribution on $\left(-2, 2\right)$, $\log\left(\epsilon\right) \sim U\left(-2, 2\right)$.

Case III. $\epsilon \sim U\left(0.5, \kappa\right)$ with κ being chosen such that $E\left(\epsilon\right) = E\left(1/\epsilon\right)$.

To implement the developed method, we need to choose the number of interior knots appropriately. We fix the spline order as cubic, as this is the most commonly used choice in the spline literature. As recommended by Ming *et al.* [10], N_n was set as $\lfloor n^{1/(2q+3)} \rceil$, this choice is small enough to avoid over-fitting with suitable sample size not too small and big enough to approximate smooth functions, and the results are similar for N_n varying on a set of candidate values. In this article, we propose a data-driven approach to select it. We use the Bayesian information criterion (BIC) to choose the optimal number of interior knots N_n by minimizing the following BIC function

$$\mathrm{BIC}\left(N_n\right) = \log\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[Y_i \exp\left(-\boldsymbol{\Pi}_i^{\mathrm{T}} \hat{\boldsymbol{\vartheta}} \right) + Y_i^{-1} \exp\left(\boldsymbol{\Pi}_i^{\mathrm{T}} \hat{\boldsymbol{\vartheta}} \right) \right] \right\} + \frac{\left(N_n + q\right)\log\left(n\right)}{n},$$

on the range $\lfloor n^{1/9} \rceil \le N_n \le \lfloor 8n^{1/9} \rceil + 1$, where $\lfloor a \rceil$ denotes the closest integer to a. Then the optimal N_n can be derived as $N_n^{opt} = \arg\min_{N_n} \mathrm{BIC}\left(N_n\right)$.

In our simulation experiments, 500 repetitions are carried out for each error configuration. We compute the means of absolute biases (ABISE$_\ell$) for each estimated parametric coefficient $\hat{\beta}_\ell$, $\ell = 1, 2$ and mean squared error (MSE) for the estimated $\hat{\boldsymbol{\beta}}$. To assess the performance of estimator $\hat{f}\left(\cdot\right)$ for the monotone function $f\left(\cdot\right)$, we apply the square root of average square error (RASE) of $\hat{f}\left(\cdot\right)$, which is defined as

$$\mathrm{RASE}\left(\hat{f}\right) = \left\{ \frac{1}{n_{\mathrm{grid}}} \sum_{i=1}^{n_{\mathrm{grid}}} \left(\hat{f}\left(u_i\right) - f\left(u_i\right) \right)^2 \right\}^{1/2},$$

where $\left\{ u_i, i = 1, 2, \cdots, n_{\mathrm{grid}} \right\}$ are the grid points at which the function $f\left(\cdot\right)$ is evaluated, and we simply set n_{grid} equals to the sample sizes in each simulation.

Meanwhile, we compare our proposed monotone constrained LPRE estimator (M-LPRE) with three conventional competitors, including: 1) transformation least squares estimator using data $\{\log(Y_i), \boldsymbol{X}_i, U_i\}$ and regular B-spline approximation without monotone constraint (TLS); 2) transformation least squares estimator using monotone B-spline approximation (M-TLS); 3) classical LPRE estimator without monotone constraint (see Ming *et al.* [10]). **Tables 1-3** list the means and standard deviations (in parentheses) of ABISE$_\ell$, $\ell = 1, 2$, MSE and RASE for the estimators with different sample sizes and cases.

Table 1. The simulation results ($\times 10^{-2}$) of ABISEs, MSEs and RASEs for Case I.

n	Methods	ABIAS$_1$	ABIAS$_2$	MSE	RASE
100	TLS	9.7389 (7.2943)	9.6925 (7.3219)	2.9539 (3.5008)	21.738 (6.7734)
	M-TLS	9.7247 (7.2827)	9.6772 (7.2483)	2.9358 (3.4886)	17.877 (6.1711)
	LPRE	10.153 (7.7417)	10.073 (7.9802)	3.2795 (4.0361)	22.906 (7.1748)
	M-LPRE	10.085 (7.6489)	10.086 (7.8772)	3.2375 (4.0039)	18.662 (6.4405)
200	TLS	6.7492 (5.4977)	6.6565 (4.7724)	1.4275 (1.6476)	15.429 (5.0271)
	M-TLS	6.7855 (5.4965)	6.6675 (4.7479)	1.4315 (1.6459)	13.096 (4.8426)
	LPRE	7.0469 (5.6828)	7.0415 (5.0957)	1.5738 (1.8380)	16.335 (5.3119)
	M-LPRE	7.0928 (5.7057)	7.0122 (5.0885)	1.5781 (1.8394)	13.790 (4.9359)
500	TLS	4.2335 (3.1922)	4.4513 (3.2366)	0.5836 (0.6642)	9.7130 (3.2102)
	M-TLS	4.2227 (3.1807)	4.4396 (3.2398)	0.5811 (0.6617)	8.6214 (3.0531)
	LPRE	4.5267 (3.5264)	4.7574 (3.5601)	0.6818 (0.8095)	10.262 (3.3762)
	M-LPRE	4.5190 (3.5183)	4.7561 (3.5641)	0.6807 (0.8054)	9.0773 (3.2266)

Table 2. The simulation results ($\times 10^{-2}$) of ABISEs, MSEs and RASEs for Case II.

n	Methods	ABISE$_1$	ABISE$_2$	MSE	RASE
100	TLS	11.440 (8.1737)	10.838 (8.3276)	3.8423 (4.0262)	24.858 (7.9266)
	M-TLS	11.304 (8.1259)	10.723 (8.2529)	3.7665 (3.9514)	20.267 (7.0257)
	LPRE	9.8071 (7.1019)	9.3712 (7.2280)	2.8647 (3.0107)	21.389 (7.1301)
	M-LPRE	9.6380 (6.9635)	9.1854 (7.1033)	2.7601 (2.9055)	17.667 (6.2289)
200	TLS	8.0140 (5.8346)	7.7360 (5.9461)	1.9333 (2.0871)	17.735 (5.4551)
	M-TLS	7.9488 (5.8484)	7.6923 (5.8878)	1.9109 (2.0591)	15.221 (5.2409)
	LPRE	6.6634 (4.9784)	6.4300 (4.9792)	1.3522 (1.4655)	14.836 (4.7409)
	M-LPRE	6.6228 (4.9507)	6.3848 (4.9110)	1.3315 (1.4410)	12.852 (4.5762)
500	TLS	4.9159 (3.9014)	4.6336 (3.3834)	0.7225 (0.8453)	11.117 (3.5822)
	M-TLS	4.9090 (3.8962)	4.6229 (3.3747)	0.7198 (0.8477)	9.8016 (3.4976)
	LPRE	4.0294 (3.2317)	3.8389 (2.7915)	0.4917 (0.5751)	9.1923 (2.9412)
	M-LPRE	4.0156 (3.2192)	3.8274 (2.7830)	0.4884 (0.5754)	8.1851 (2.8816)

Table 3. The simulation results ($\times 10^{-2}$) of ABISEs, MSEs and RASEs for Case III.

n	Methods	$ABISE_1$	$ABISE_2$	MSE	RASE
100	TLS	3.2162 (2.3359)	3.0322 (2.3445)	0.3047 (0.3208)	7.1122 (2.1965)
	M-TLS	3.2146 (2.3095)	3.0107 (2.3420)	0.3019 (0.3191)	6.4357 (2.0982)
	LPRE	3.1725 (2.3017)	2.9921 (2.3203)	0.2967 (0.3123)	7.0157 (2.1718)
	M-LPRE	3.1705 (2.2772)	2.9699 (2.3135)	0.2939 (0.3102)	6.3625 (2.0805)
200	TLS	2.2765 (1.6694)	2.1814 (1.6741)	0.1552 (0.1693)	5.1104 (1.5069)
	M-TLS	2.2662 (1.6723)	2.1767 (1.6696)	0.1544 (0.1685)	4.7502 (1.4204)
	LPRE	2.2435 (1.6464)	2.1490 (1.6455)	0.1505 (0.1639)	5.0379 (1.4909)
	M-LPRE	2.2334 (1.6478)	2.1434 (1.6398)	0.1497 (0.1630)	4.6889 (1.4076)
500	TLS	1.3782 (1.1084)	1.3298 (0.9610)	0.0581 (0.0688)	3.2795 (0.9612)
	M-TLS	1.3750 (1.1066)	1.3278 (0.9607)	0.0579 (0.0687)	3.1095 (0.9321)
	LPRE	1.3562 (1.0906)	1.3097 (0.9474)	0.0563 (0.0668)	3.2249 (0.9389)
	M-LPRE	1.3530 (1.0902)	1.3086 (0.9462)	0.0562 (0.0668)	3.0596 (0.9112)

We can make the following observations: 1) M-TLS and M-LPRE estimators perform better than the corresponding TLS and LPRE estimators, respectively. 2) For Case I, as is anticipated, M-TLS estimators perform best, for Cases II-III, M-LPRE generally outperform the others, this is because the error term of logarithmic partially linear multiplicative model is the standard norm distribution, makes least squares based estimator better, while Cases II-III make LPRE based estimators efficient. 3) For a given error distribution case, it is obvious that the mean and standard deviation of $ABISE_\ell$, $\ell = 1, 2$, MSE and RASE for all estimators decrease as the sample size increases, this result confirms the asymptotic consistency of the proposed estimation. Moreover, we present the estimated nonparametric curves and boxplots of MSEs and RASEs for the parameters and coefficient functions in Figure 1 under the Case II when the sample size $n = 200$. All these findings reflect the satisfactory performance of our proposed method under the considered settings.

4. Real Data Application

In this section, we illustrate the proposed approach by analysing the air pollution data set. This data set was collected by the Norwegian Public Roads Administration which is available at http://lib.stat.cmu.edu/datasets/NO2.dat. There are 500 observations measured at Alnabru in Oslo, Norway, between October 2001 and August 2003. The purpose is to research how the concentration of the air pollution NO_2 depends on the traffic volume and three meteorological elements, thus, it is appropriate to treat hourly values of the logarithm of the concentration of NO_2 (particles) as the response variable (Y). According to the suggestion proposed by Du *et al.* [13], we take the logarithm of the number of cars per hour as predictor

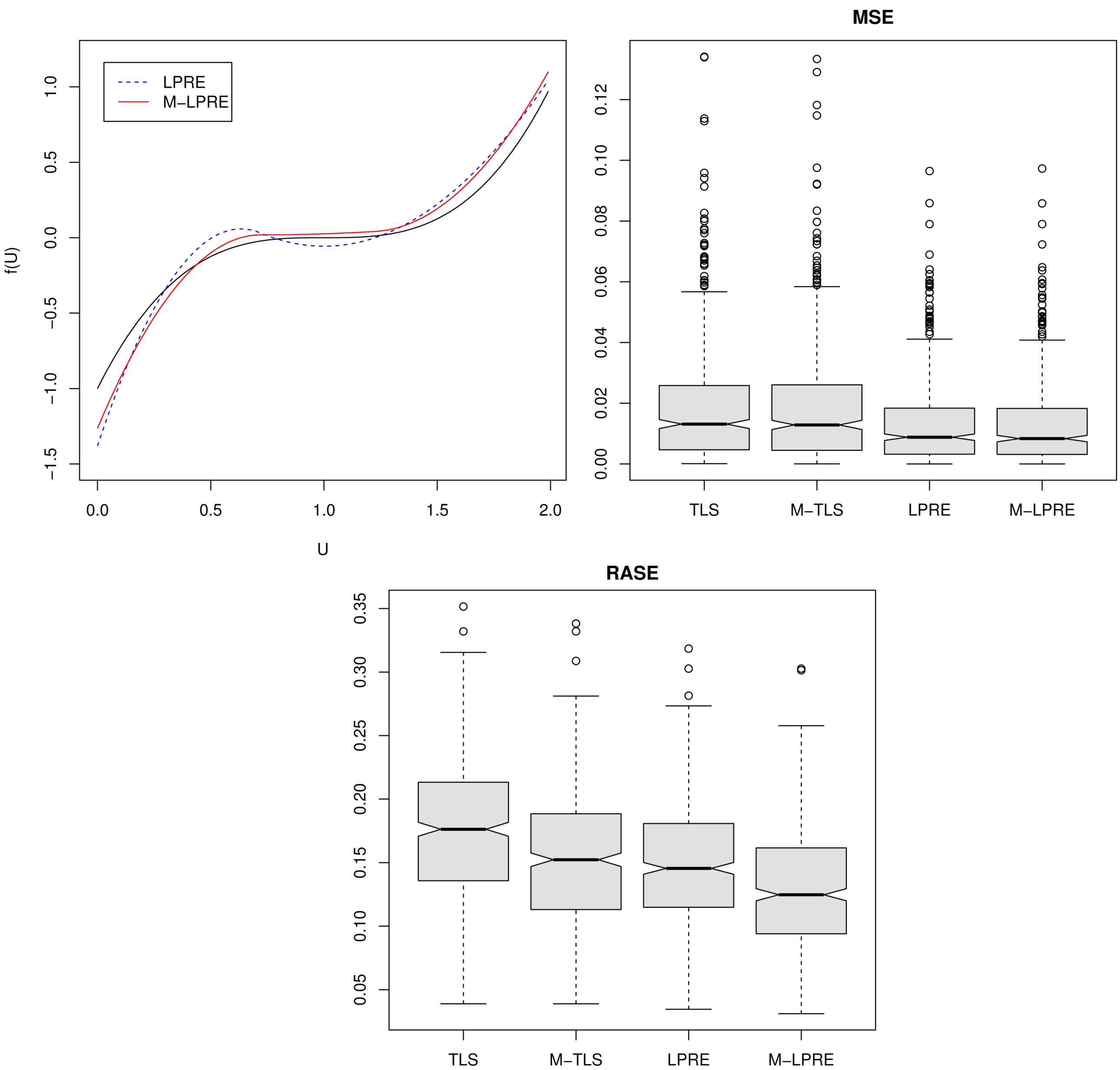

Figure 1. Estimated nonparametric curves and boxplots of MSEs and RASEs for Case II when sample size $n = 200$.

variable U, and the three covariate variables are temperature two meters above ground (X_1, °C), wind speed (X_2, m/s) and the temperature difference between 25 and 2 m above ground (X_3, °C). In the subsequent analysis, we use exponential Y as response variable for model (1.1), and then construct the model as follows:

$$Y = \exp\left(X_1\beta_1 + X_2\beta_2 + X_3\beta_3 + f(U)\right)\epsilon.$$

We use the first 420 samples from the data as the training data for modeling fitting and the remainder as test data for evaluating the prediction performance of the methods. We calculate the median of additive relative prediction error defined as $\text{MAPE} = \text{Median}\left\{\left|Y_i - \hat{Y}_i\right| / Y_i + \left|Y_i - \hat{Y}_i\right| / \hat{Y}_i, i = 1, \cdots, 80\right\}$ to measure the predictability, where $\hat{Y}_i$ is the fitted value of Y_i. The obtained MAPEs along with parameter estimator given in brackets are 0.6778 ($\hat{\beta}_1 = -0.0211$, $\hat{\beta}_2 = -0.1432$, $\hat{\beta}_3 = 0.1357$) and 0.6565 ($\hat{\beta}_1 = -0.0202$, $\hat{\beta}_2 = -0.1407$, $\hat{\beta}_3 = 0.1373$) for LPRE and M-LPRE methods, respectively. The estimated curves are displayed in **Figure 2**. As expected, the more cars would result in higher concentration of NO_2, therefore it is reasonable to assume the monotonicity of $f(\cdot)$, and our proposed method coincides with this empirical evidence.

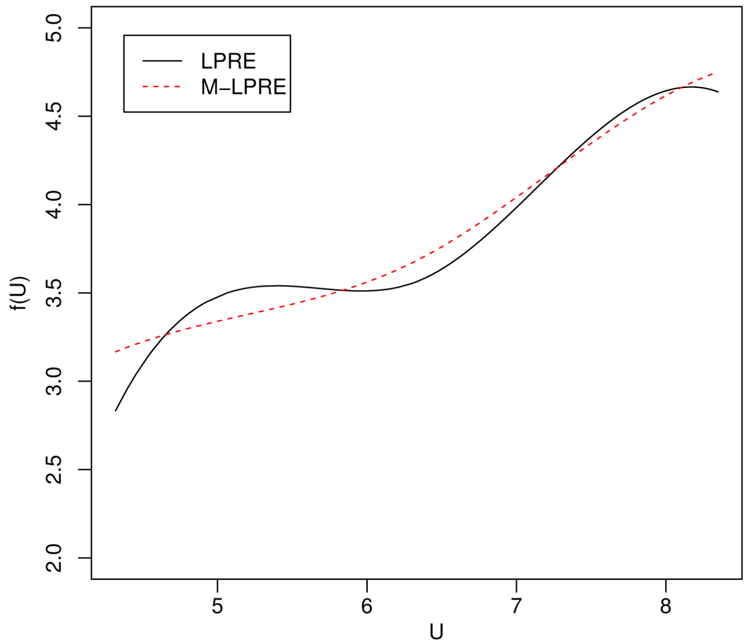

Figure 2. Estimated nonparametric curves in the air pollution study data set.

5. Conclusions

In this paper, we propose a novel partially linear multiplicative model in which the nonparametric component is assumed to be a monotone function. We use monotone B-spline basis expansion to estimate the nonparametric function based on the constrained least product relative error criterion. We then provide a uniform convergence rate and asymptotic normality of the proposed spline estimators. Numerical results suggest that the proposed estimation is promising over its competitors.

The proposed method has some useful extensions. First, we can add a penalty term to achieve sparsity when irrelevant variables exist in the model. Second, we can set an intercept term in the model and allow it to vary for different subgroups from a heterogeneous population, we then study subgroup analysis problem using popular concave pairwise penalized approach. Third, it would be of interest to extend the proposed method to monotone partially linear single index multiplicative model and investigate its theoretical property. Fourth, as one of the reviewers pointed out whether the optimization process is computationally feasible for large-scale data. We need pay efforts to constructing adaptive distributed algorithm or use subsampling method to solve the constraint problem imposed by large-scale data. For the relatively strong condition of monotonicity, pursuing constraint test on this aspect should be meaningful and interesting. We will pursue these detailed investigation issues in our future research.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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