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Gravitational Collapse and Expansion in the Newton Theory and General Relativity

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Abstract

The paper is devoted to the study of the gravitational collapse within the framework of the spherically symmetric problem in the Newton theory and general relativity on the basis of the pressure-free model of the continuum. In application to the Newton gravitation theory, the analysis consists of three stages. First, we assume that the gravitational force is determined by the initial sphere radius and constant density and does not change in the process of the sphere collapse. The obtained analytical solution allows us to find the collapse time in the first approximation. Second, we construct the step-by-step process in which the gravitational force at a given time moment depends on the current sphere radius and density. The obtained numerical solution specifies the collapse time depending on the number of steps. Third, we find the exact value of the collapse time which is the limit of the step-by-step solutions and study the collapse and the expansion processes in the Newton theory. In application to general relativity, we use the space model corresponding to the special four-dimensional space which is Euclidean with respect to space coordinates and Riemannian with respect to the time coordinate only. The obtained solution specifies two possible scenarios. First, sphere contraction results in the infinitely high density with the finite collapse time, which does not coincide with the conventional result corresponding to the Schwarzschild geometry. Second, sphere expansion with the velocity which increases with a distance from the sphere center and decreases with time.

Keywords

Gravitational Collapse, Newton Gravitation Theory, General Relativity

1. Introduction

Gravitational collapse is widely discussed in the literature concerning spherically symmetric problem of general relativity [1]-[4]. History and the state-of-the-art of the problem are presented in the review [5] which contains brief descriptions of the results obtained by 400 authors working in this field. The traditional model is a pressure-free continuum which is described by the Schwarzschild geometry and is referred to the so-called co-moving coordinates introduced by Tolman [6]. The first solution of the collapse problem was obtained in 1939 by Oppenheimer and Snyder [7]. After some assumptions, they arrived at the following asymptotic expressions for the metric coefficients of the Schwarzschild metric form for the sphere surface $r = R$:

$$g_{11}(R) = 1 + e^{tc/r_g}, \quad g_{44} = g_{11} e^{-2tc/r_g}\left(1 + e^{-tc/r_g}\right)$$

in which t is time and

$$r_g = 2mG/c^2 \tag{1}$$

is the so-called gravitational radius which is expressed in terms of the sphere mass m, the classical gravitation constant G and the velocity of light c. As can be seen, $g_{11}(t \to \infty) \to \infty$ and $g_{44}(t \to \infty) \to 0$. The Oppenheimer-Snyder solution is discussed by Burghardt [8].

An alternative solution of the collapse problem was presented by Weinberg [3] who arrived at the following expression for the collapse time (in the notations of the current paper):

$$t_c^{(W)} = \frac{\pi R_0}{2c}\sqrt{\frac{R_0}{r_g}} \tag{2}$$

in which $R_0 = R(t = 0)$. The Weinberg solution is also discussed by Burghardt [9]. More recent results are presented in [10].

In the present paper, the solution of the collapse problem is based on the special model of the four-dimensional space [11] which is Euclidean with respect to space coordinates and Riemannian with respect to time. Preliminary, the solution corresponding to the Newton gravitation theory is considered.

2. Gravitational Collapse and Expansion in the Newton Theory

2.1. Governing Equations

The behavior of continuum within the framework of the Newton gravitation theory is described by motion and compatibility equations which have the following form in spherical coordinates [12]:

$$\frac{\partial p}{\partial r} + \rho\frac{d\varphi}{dr} + \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = 0, \quad \frac{\partial}{\partial r}(\rho v) + \frac{2}{r}\rho v + \frac{\partial \rho}{\partial t} = 0 \tag{3}$$

in which p, ρ and v are pressure, density and velocity, whereas φ is the Newton gravitational potential that satisfies the Poisson equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 4\pi G\rho \tag{4}$$

In the general case, ρ depends on *r* and *t*. However, we further use the special model of the continuum which consists of a system of isolated and not non-interacting particles such that the pressure is zero. In this case, it is natural to assume that the density which is initially uniform, *i.e.* $\rho(t=0)=\rho_0$, does not depend on *r* in the process of gravitational collapse. Thus, we take $\rho=\rho(t)$. Using equations for the sphere mass for the current time and $t=0$

$$m=\frac{4}{3}\pi R^3(t)\rho(t),\quad m_0=\frac{4}{3}\pi R_0^3\rho_0 \tag{5}$$

and taking into account that $m=m_0$, we arrive at the following conditions:

$$R(t)=R_0\sqrt[3]{\rho_0/\rho(t)} \tag{6}$$

Now, we can integrate Equation (4). Applying Equations (1) and (5) to eliminate *G*, we can present the first integral of this equation as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r}=a^2r,\quad a^2=\frac{r_gc^2}{2R^3(t)} \tag{7}$$

and write the first equation of Equations (3) in the following form:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-a^2r \tag{8}$$

As follows from Equations (7), $a=a(t)$, *i.e.* in the process of collapse, the sphere radius decreases, the density increases and the gravitation force depends on time.

2.2. First Approximation

To obtain the solution in the first approximation that can be useful for the further more rigorous analysis, assume that $R=R_0$ and $\rho=\rho_0$ which means that the gravitation force does not depend on time and corresponds to the initial parameters of the sphere. Then, Equation (8) takes the form

$$\frac{\partial v_0}{\partial t}+v_0\frac{\partial v_0}{\partial r}=-a_0^2r,\quad a_0^2=\frac{r_gc^2}{2R_0^3} \tag{9}$$

Assume that the solution of this equation is

$$v(r,t)=rf(t) \tag{10}$$

Then, Equation (9) yields

$$f''+f^2=-a_0^2,\quad (\cdot)'=\mathrm{d}(\cdot)/\mathrm{d}t$$

This is the Riccati equation [13] whose solution allows us to find the velocity

$$v(r,t)=-a_0r\tan(C_0a_0+a_0t) \tag{11}$$

Using the initial condition $v(r,t=0)=0$, we finally get

$$v(r,t)=-a_0r\tan(a_0t) \tag{12}$$

Now, the second equation in Equations (3) takes the form

$$\rho'-3\rho a_0\tan(a_0t)=0$$

and has the following solution:

$$\rho = \frac{B_0}{\cos^3(a_0 t)} \tag{13}$$

Using the initial condition $\rho(t=0)=\rho_0$, we arrive at

$$\rho = \frac{\rho_0}{\cos^3(a_0 t)} \tag{14}$$

Finally, Equation (6) allows us to determine the sphere radius

$$R(t) = R_0 \cos(a_0 t) \tag{15}$$

Collapse results in concentration of the sphere mass at a point and an infinitely high density. As follows from the foregoing solution, this happens if $t = t_c^{(0)} = \pi/2a_0$. Substituting a_0 from Equations (9), we get

$$t_c^{(0)} = \frac{\pi R_0}{c}\sqrt{\frac{R_0}{2r_g}} \tag{16}$$

Equations (12) and (15) allow us to obtain the following useful result:

$$v(r,t) = r\frac{R'(t)}{R_0} \tag{17}$$

As an example, consider the sphere with the parameters analogous to Sun, *i.e.*, $R_0 = 6.96\times10^8$ m, $r_g = 2960$ m. For this sphere, Equation (16) yields $t_c^{(0)} = 2500$ sec. Dependence $R(t)/R_0$ is shown in **Figure 1** (line 1).

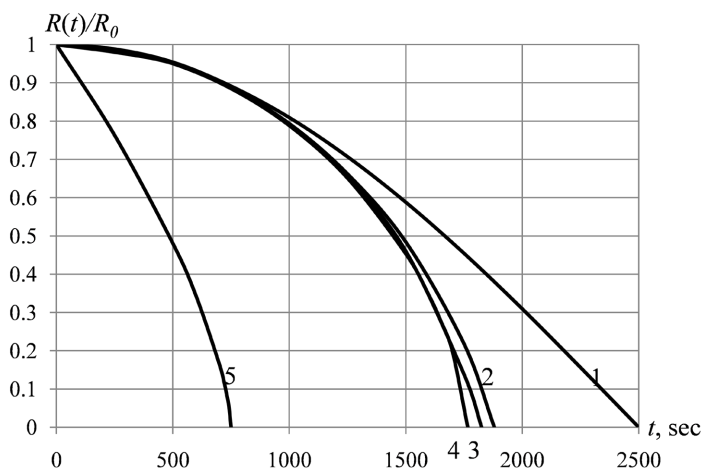

Figure 1. Dependences $R(t)/R_0$ corresponding to the first approximation (1), step-by step solutions with 20 (2) and 80 (3) steps, numerical solution for the Newton theory (4) and GR solution (5).

2.3. Step-by-Step Solution

Now recall that the foregoing solution does not take into account that parameter a in Equation (8) depends on t which means that the gravitational force does not change in the process of the sphere collapse. To allow for this change, we use the following step-by-step procedure. Divide the collapse time $t_c^{(0)}$ into a system of increments $t_1, t_2, \cdots, t_n$ and assume that the foregoing solution corresponding to $a = a_0$ is valid for $0 \le t \le t_1$. Then, for $t = t_1$, we have the expressions following from Equations (12), (14) and (15), *i.e.*,

$$v_0(r,t_1) = -a_0 r \tan(a_0 t_1),\ \rho^{(0)}(t_1) = \frac{\rho_0}{\cos^3(a_0 t_1)},\ R(t_1) = R_1 = R_0 \cos(a_0 t_1) \quad (18)$$

For $t_1 \le t \le t_2$, we change R_0 to R_1 and Equation (8) becomes

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial r} = -a_1^2 r, \quad a_1^2 = \frac{r_g c^2}{2R_1^3}$$

The solution is specified by Equation (11), *i.e.*,

$$v_1(r,t) = -a_1 r \tan(C_1 a_1 + a_1 t)$$

Constant C_1 is determined from the following condition: $v_0(r,t_1) = v_1(r,t_1)$ in which $v_0(r,t_1)$ is specified by Equations (18). The result is

$$C_1 = \frac{1}{a_1}\tan^{-1}\frac{a_0 \tan(a_0 t_1) - a_1 \tan(a_1 t_1)}{a_1 - a_0 \tan(a_0 t_1)\tan(a_1 t_1)}$$

The second equation in Equations (3) yields

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} - 3\rho_1 a_1 \tan(C_1 a_1 + a_1 t) = 0$$

The solution is similar to Equation (13), *i.e.*,

$$\rho_1 = \frac{B_1}{\cos^3(C_1 a_1 + a_1 t)}$$

Constant B_1 can be found from the condition $\rho^{(0)}(t_1) = \rho_1(t_1)$ in which $\rho^{(0)}(t_1)$ is specified by Equations (17). The result is

$$B_1 = \rho_0 \frac{\cos^3(C_1 a_1 + a_1 t_1)}{\cos^3(a_0 t_1)}$$

The sphere radius for $t = t_2$ can be found from Equation (6) which takes the form $R_0^3 \rho_0 = R_2^3 \rho_1(t_2)$. Finally, we get

$$R_2 = R_0 \frac{\cos(C_1 a_1 + a_1 t_2)}{\cos(C_1 a_1 + a_1 t_1)} \cos(a_0 t_1)$$

The described step-by-step process proceeds until R_n becomes zero which corresponds to the collapse time $t_c^{(n)}$. The results of calculation for the sphere with parameters of Sun are presented in Figure 1 for $n = 20$ (line 2) and $n = 80$ (line 3). The collapse times are $t_c^{(20)} = 1880\ \text{sec}$ and $t_c^{(80)} = 1825\ \text{sec}$.

2.4. Exact Solution

The foregoing results allow us to conclude that the sequence of collapse times t_c^n obtained by step-by-step calculation converges in the Newton gravitation theory to the collapse time t_c^N. To determine this time, return to Equation (8) and transform parameter a^2 using Equations (1) and (5) for r_g and m. The result is

$a^2 = (4/3)\pi G\rho(t)$ and Equation (8) reduces to

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -k_n^2 r\rho(t), \quad k_n^2 = \frac{4}{3}\pi G \tag{19}$$

Applying Equation (6), we get

$$\rho(t) = \frac{\rho_0}{\bar{R}^3(t)}, \quad \bar{R}(t) = \frac{R(t)}{R_0}. \tag{20}$$

Substituting this result in the second equation of Equations (3), we have

$$\frac{\partial v}{\partial r} + \frac{2}{r}v - 3\frac{\bar{R}'}{\bar{R}} = 0 \tag{21}$$

Now, assume that $v(r,t)$ is specified by equation similar to Equation (17), *i.e.*,

$$v(r,t) = r\frac{\bar{R}'(t)}{\bar{R}(t)} \tag{22}$$

Substituting Equation (22) in Equation (21), we can conclude that this equation is satisfied identically. Finally, substituting Equations (20) and (22) in Equation (19), we arrive at

$$\bar{R}'' + \frac{k_n^2\rho_0}{\bar{R}^2} = 0$$

This is the well-known equation which describes the free-fall motion and has the following first integral [11]:

$$\bar{R}' = \pm\sqrt{\left(C_1 + \frac{2k_n^2\rho_0}{\bar{R}}\right)} \tag{23}$$

2.5. Collapse Problem

For the collapse problem, we should take sign "−" in Equation (23). As follows from the foregoing analysis, in this case we have $v(t=0) = 0$ which means that $\bar{R}'(t=0) = 0$ and $\bar{R}(t=0) = 1$. Then, $C_1 = -2k^2\rho_0$ and

$$\bar{R}' = -k_n\sqrt{2\rho_0\left(\frac{1}{\bar{R}} - 1\right)} \tag{25}$$

Integration yields the following implicit expression for $\bar{R}(t)$:

$$\frac{1}{2}\sin^{-1}\left(1 - 2\bar{R}\right) + \sqrt{\bar{R}\left(1 - \bar{R}\right)} + \frac{\pi}{4} = k_n t\sqrt{2\rho_0} \tag{26}$$

Determine the collapse time $t_c^{(N)}$ corresponding to the Newton theory. Taking into account that $\bar{R}\left(t = t_c^{(R)}\right) = 0$, using Equations (10) and (19) and applying Equation (26), we get

$$t_c^{(N)} = \frac{\pi}{2k_n\sqrt{2\rho_0}} = \frac{\pi R_0}{2c}\sqrt{\frac{R_0}{r_g}} \tag{27}$$

Equations (22) and (25) allow us to determine the velocity as

$$v = \frac{k_n r}{\bar{R}}\sqrt{2\rho_0\left(\frac{1}{\bar{R}} - 1\right)}$$

For $t=0$, we have $\bar{R}=1$ and $v=0$ which means that the static initial state exists for the collapse problem in the Newton theory.

As an example, consider as earlier the sphere with parameter of Sun. For this sphere, calculation yields $t_c^{(N)}=1766$ sec. Numerical solution of Equation (26) for $\bar{R}(t)$ is shown in **Figure 1** (line 4)

It should be noted that the forgoing formulation of the collapse problem in the absence of pressure is not realistic since pressure can prevent the collapse. This scenario is supported by the existence of the known static solution of Equations (3) for the constant pressure which is

$$p_0=\frac{2\pi}{3}GR^2\rho^2\left(1-\frac{r^2}{R^2}\right)$$

2.6. Expansion Problem

Now return to Equation (23) and take sign "+", *i.e.*,

$$\bar{R}'=\sqrt{C_1+\frac{2k_n^2\rho_0}{\bar{R}}} \tag{28}$$

This solution corresponds to the sphere expansion. The applied pressure-free model of continuum looks more realistic for the expansion problem than for the collapse problem considered above. For $t\to\infty$ we have $\bar{R}\to\infty$ and $\bar{R}'\to 0$. Then, $C_1=0$ in Equation (28) and

$$\bar{R}'\sqrt{\bar{R}}=k_n\sqrt{2\rho_0}$$

Integration yields

$$\frac{2}{3}\bar{R}^{3/2}=k_nt\sqrt{2\rho_0}+C$$

Taking into account that $\bar{R}(t=0)=1$, we get $C=2/3$ and

$$\bar{R}=\sqrt[3]{\left(1+\frac{3k_nt}{2}\sqrt{2\rho_0}\right)^2}=\sqrt[3]{\left(1+t\sqrt{6\pi G\rho_0}\right)^2} \tag{29}$$

In the second part of this equation coefficient k_n is substituted from Equation (19). Using Equations (20) and (26), we can find the density, *i.e.*,

$$\rho=\frac{\rho_0}{\left(1+t\sqrt{6\pi G\rho_0}\right)^2} \tag{30}$$

Finally, Equations (22) and (29) allow us to determine the velocity as

$$v=\frac{2r\sqrt{6\pi G\rho_0}}{1+t\sqrt{6\pi G\rho_0}} \tag{31}$$

Note that $v(t=0)\neq 0$, which means that the initial static state in the expansion problem does not exist. For $t\to\infty$ we have $\rho\to 0$, $v\to 0$ and $R\to\infty$ which corresponds to an infinite homogeneous static space.

3. Gravitational Collapse and Expansion in General Relativity

3.1. General Solution

The study is undertaken within the mechanical interpretation of general relativity as a phenomenological theory based on the traditional model of space as a homogeneous isotropic continuum whose physical microstructure is ignored. The collapse and the expansion problems are considered further for the special four-dimensional space which is Euclidean with respect to space spherical coordinates r,θ,φ and Riemannian with respect to the time coordinate only [11]. To introduce the special geometry, consider first the metric coefficients corresponding to the Newton gravitation theory [14], *i.e.*

$$g_{11} = 1,\ g_{22} = r^2,\ g_{44} = 1 - \frac{r_g}{r}$$

As can be seen, the space coefficients g_{11} and g_{22} belong to the Euclidean space, whereas the Riemannian space appears only in the time coefficient g_{44} which is associated with gravitation. Indeed, in the absence of gravitation $r_g = 0$ and the space becomes Euclidean. Thus, the space is Euclidean with respect to the space coordinates and is Riemannian with respect to the time coordinate only. The idea is to extend this result to the general case. Undertake the following virtual experiment. Assume that we observe in a traditional three-dimensional Euclidean space a solid body loaded with self-balanced forces inducing a certain stressed state inside the body in the absence of gravitation. As known, the general relativity equations contain differential operators acting on a metric tensor of a Riemannian space in the left-hand sides and the so-called energy-momentum tensor in the right-hand sides. The energy-momentum tensor, in general, consists of the stress tensor and kinematic terms [15]. Thus, the general relativity equations describe the Riemannian space which is induced inside the body by the stresses only. However, such situation is not possible in Riemannian geometry, the dimension of the Euclidean space (n_E) in which the Riemannian space with n_R dimensions can exist is $n_E = n_R(n_R + 1)/2$ [16]. Taking $n_R = 3$, we get $n_E = 6$. However, the stresses solid is in a three-dimensional Euclidean space which means that in the absence of gravitation, the space is Euclidean, whereas gravitation induces the Riemannian space associated with time only. So, we arrive at the special model of the four-dimensional space in which the space is Euclidean with respect to space coordinates and Riemannian with respect to time. This model allows us to study the problems which cannot be solved within the traditional Riemannian space model. One of such problems is associated with gravitation in solids. For static problems in elastic solids, the system of general relativity equations includes three conservation equations which are actually the equilibrium equations. These equations contain, in general, six components of the stress tensor which cannot be found from three equations. In the theory of elasticity [17], the equilibrium equations are supplemented with compatibility equations which require the stressed space to be Euclidean and do not exist in the Riemannian space. Thus, a paradoxical situation occurs, the gravitational stresses in solids which are traditionally found in the classical solid mechanics cannot be determined in the theory of relativity. In the proposed space model, the space is Euclidean with respect to space coordinates, compatibility equations exist and the problem can be solved [18]. Moreover, this model allows us to overcome some principal problems of general relativity in the traditional Riemannian space. As known [2] [3] [14], the system of the field equations is not complete in this space and should be supplemented with additional coordinate conditions. In the proposed model the number of the unknown metric coefficients is less than in the general Riemannian space and no additional conditions are required. This property of equations is demonstrated further for a spherically symmetric problem.

The line element for the proposed four-dimensional space in spherical coordinates has the following form [11]:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + 2g_{14}c\mathrm{d}r\mathrm{d}t - g_{44}c^2\mathrm{d}t^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 \tag{32}$$

Here, $g_{14}\ g_{44}$ depends on r and t. In contrast to traditional coordinate frames, space is not "orthogonal" to time in the four-dimensional space corresponding to Equation (32). The metric form in Equation (32) was derived in 1921-22 by Gullstrand and Painleve [19] [20] from the Schwarzschild form

$$\mathrm{d}s^2 = g_{11}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 - g_{44}c^2\mathrm{d}t^2 \tag{33}$$

by coordinate transformation. Indeed, identical transformations in Equation (32) allow us to reduce it to

$$\mathrm{d}s^2 = \left(1 + \frac{g_{14}^2}{g_{44}}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 - g_{44}\left(c - \frac{g_{14}}{g_{44}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\mathrm{d}t^2 \tag{34}$$

In static problems, $\mathrm{d}r/\mathrm{d}t = 0$ and Equation (34) reduces to Equation (33) in which

$$g_{11} = 1 + \frac{g_{14}^2}{g_{44}}$$

In dynamic problems, the derivative $\mathrm{d}r/\mathrm{d}t$ is not known and direct reduction of Equation (32) to Equation (33) is not possible. Here, Equation (32) is not the results of coordinate transformation and follows from the proposed model of a four-dimensional space.

For the line element in Equation (32), the Einstein field equations are

$$E_1^1 = -\frac{1}{r^2g^2}\left[g\left(rg'_{44} - g_{14}^2\right) + 2rg_{44}\dot{g}_{14} - rg_{14}\dot{g}_{44}\right] \tag{35}$$

$$\begin{aligned} E_2^2 = \frac{1}{4rg^2}\big(&4g_{14}g_{44}g'_{14} - 4g_{14}^2g'_{44} - 2g_{44}g'_{44} - 2rg_{44}g''_{44} + rg'^{2}_{44} - 2rg_{14}^2g''_{44} \\ &+ 2rg_{14}g'_{14}g'_{44} - 4g_{44}\dot{g}_{14} + 2g_{14}\dot{g}_{44} + 4rg_{14}g'_{14}\dot{g}_{14} + 2rg'_{14}\dot{g}_{44} - 4rg\dot{g}'_{14}\big) \end{aligned} \tag{36}$$

$$E_4^4 = \frac{1}{r^2g^2}\left(rg_{14}g'_{44} - gg_{14} - 2rg_{44}g'_{14}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{rg_{14}^2}{g}\right) \tag{37}$$

$$E_1^4 = -\frac{g_{14}}{rg^2}\left(2g_{14}g'_{14} + g'_{44}\right) = \frac{g_{14}}{r}\frac{\partial}{\partial r}\left(\frac{1}{g}\right) \tag{38}$$

$$E_4^1 = -\frac{g_{14}}{rg^2}\left(2g_{44}\dot{g}_{14} - g_{14}\dot{g}_{44}\right) = \frac{g_{14}^2 g_{44}}{rg^2}\frac{\partial}{c\partial t}\ln\frac{g_{14}}{g_{44}^2}, \quad g = g_{44} + g_{14}^2 \tag{39}$$

Here, $(\cdot)' = \partial(\cdot)/\partial r$ and $\dot{(\cdot)} = \partial(\cdot)/c\partial t$. We use mixed tensor components because in spherical coordinates they coincide with physical components. Einstein tensor is proportional to the energy-momentum tensor, *i.e.*,

$$E_i^j = \chi T_i^j, \quad \chi = \frac{8\pi G}{c^4} \tag{40}$$

In the absence of pressure, the energy momentum tensor has the following form [16]:

$$T_1^1 = \rho v_1 v^1, \quad T_2^2 = T_3^3 = 0, \quad T_4^4 = \rho c^2, \quad T_1^4 = -\rho v_1 c, \quad T_4^1 = \rho v^1 c \tag{41}$$

Consider briefly the empty space surrounding the sphere [11] for which $T_i^j = 0$ and Equations (35)-(39) are homogeneous. Equation (39) in which $E_4^1 = 0$ allows us to conclude that g_{14} and g_{44} do not depend on time. This result is analogous to the Birkhoff theorem in the Schwarzschild solution [15]. Then, Equations (15) and (17) yield

$$rg'_{44} - g_{14}^2 = 0, \quad g = g_{44} + g_{14}^2 = C_1$$

The solution of these equations is

$$\left(g_{14}^e\right)^2 = \frac{C_1 C_2}{r}, \quad g_{44}^e = C_2 - \frac{C_1 C_2}{r}$$

Here, index "*e*" corresponds to the external space. Integration constants C_1 and C_2 can be found from the asymptotic condition according to which the obtained solution must reduce for $r \to \infty$ to the Newton theory, *i.e.*,

$$g_{14}^e = 0, \quad g_{44}^e = 1 - \frac{r_g}{r}$$

where r_g is given by Equation (2). The final solution becomes

$$g_{14}^e = \pm\sqrt{\frac{r_g}{r}}, \quad g_{44}^e = 1 - \frac{r_g}{r} \tag{42}$$

As can be seen, the obtained solution specifies two spaces corresponding to signs "+" and "−" in the first of these equations. It is important for the further analysis that

$$g_e = g_{44}^e + \left(g_{14}^e\right)^2 = 1 \tag{43}$$

Parameter g has a simple geometrical meaning, the determinant of the metric tensor in the four-dimensional space is $D = -r^4 g \sin^2\theta$.

As can be proved [11], the escape velocity for the sphere with radius R is specified by the following equation:

$$v_e = c\sqrt{\frac{r_g}{R}} \tag{44}$$

which means that for the sphere with radius $R = r_g$ the escape velocity $v_e = c$ and the sphere becomes invisible. Such object is analogous to a black hole, but the Schwarzschild singularity does not appear and the space inside the sphere is Euclidian with respect to space spherical coordinates.

Consider the internal space for the sphere with radius *R*. Equations (37), (40) and (41) yield

$$\frac{\partial}{\partial r}\left(\frac{r g_{14}^2}{g}\right) = \chi \rho c^2 \tag{45}$$

As in the Newton theory, we assume that in the absence of pressure $\rho = \rho(t)$. The solution of Equation (45) is

$$g_{14}^2 = g\left(k_r \rho c^2 r^2 + \frac{C}{r}\right), \quad k_r = \frac{1}{3}\chi c^2 \tag{46}$$

The integration constant $C = 0$. Otherwise, g_{14} becomes singular at the center of a sphere with any dimensions. Taking into account that $g = g_{44} + g_{14}^2$, we finally get

$$g_{14}^2 = kg\rho r^2, \quad g_{44} = g\left(1 - k\rho r^2\right) \tag{47}$$

Consider Equations (39), (40) and (41) which yield

$$\frac{g_{14}^2 g_{44}}{r g^2} \frac{\partial}{c\partial t} \ln \frac{g_{44}}{g_{14}^2} = \chi \rho c v^1$$

Substituting Equations (47) and taking into account Equation (29) for *k*, we arrive at

$$v^1 = -\frac{r}{3\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \tag{48}$$

Applying Equations (38), (40) and (21), we have the following equation

$$\frac{g_{14}}{r}\frac{\partial}{\partial r}\left(\frac{1}{g}\right) = -\rho v_1 c$$

which yields

$$v_1 = \pm \frac{g'}{cg}\sqrt{\frac{k_r}{\rho g}} \tag{49}$$

Thus, Equations (47), (48) and (49) allow us to express g_{14}, g_{44}, v^1 and v_1 in terms of two functions $\rho(t)$ and $g(r,t)$.

Now, introduce the basic assumption of this analysis. For the sphere with radius *R*, the metric coefficients must be continuous and satisfy the following boundary conditions on the sphere surface:

$$g_{14}(R) = g_{14}^e(R), \quad g_{44}(R) = g_{44}^e(R) \tag{50}$$

However, in the collapse problem, *R* depends on *t*. So, the right-hand parts of Equations (32) should be written as [3]

$$g_{14}^e(R) = \pm\sqrt{\frac{r_g}{R(t)}}, \quad g_{44}^e(R) = 1 - \frac{r_g}{R(t)} \tag{51}$$

As follows from Equation (43) and (51), $g=1$ for the external space and for the boundary of the internal space. Thus, it is natural to assume that $g=1$ for the whole internal space. Taking $g=1$ in Equations (47), (48) and (49), we get

$$g_{14}^2 = k_r \rho r^2, \quad g_{44} = 1 - k_r \rho r^2, \quad v^1 = -\frac{r}{3\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}, \quad v_1 = 0 \tag{52}$$

As can be seen, at the sphere center $g_{14}=0$, $g_{44}=1$, the space is Euclidean and $v^1 = v_1 = 0$.

Now, return to the field equations, Equations (35)-(39). To obtain the solution in Equations (52), we used Equations (37), (38) and (39). For one unknown function $\rho(t)$, we have two remaining equations, Equations (35) and (36). Substituting Equations (52) in Equations (35) and (36), we arrive at one and the same equation for density

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \mp 3\rho c\sqrt{k\rho} = 0 \tag{53}$$

Thus, the system of the field equation turns out to be complete and reduces to Equation (53). The solution of this equation which satisfies the initial condition $\rho(t=0)=\rho_0$ is

$$\rho(t) = \frac{\rho_0}{\left(1 \mp \frac{3}{2}ct\sqrt{k_r \rho_0}\right)^2}$$

Taking into account Equation (46) for k_r and Equations (1), (5), (40) for r_g, m_0, χ, we find

$$k_r = (1/3)\chi c^2 = 8\pi G/3c^2 = r_g/\rho_0 R_0^3$$

and finally arrive at

$$\rho(t) = \frac{\rho_0}{\left(1 \mp t\sqrt{6\pi G\rho_0}\right)^2} = \frac{\rho_0}{\left(1 \mp \frac{3ct}{2R_0}\sqrt{\frac{r_g}{R_0}}\right)^2} \tag{54}$$

The sphere radius follows from Equations (6) and (56) which yield

$$R(t) = R_0\sqrt[3]{\left(1 \mp t\sqrt{6\pi G\rho_0}\right)^2} = R_0\sqrt[3]{\left(1 \mp \frac{3ct}{2R_0}\sqrt{\frac{r_g}{R_0}}\right)^2} \tag{55}$$

Now check the boundary conditions. Taking $r=R$ in Equations (54) and using Equations (56), (57), we arrive at Equations (53) for the external space which means that the boundary conditions on the sphere surface are satisfied. The velocity follows from Equations (34) and (56), *i.e.*,

$$v^1(r,t) = -\frac{2r\sqrt{6\pi G\rho_0}}{3\left(1 \mp t\sqrt{6\pi G\rho_0}\right)} = -\frac{cr\sqrt{r_g/R_0}}{1 \mp \frac{3ct}{2R_0}\sqrt{\frac{r_g}{R_0}}} \tag{56}$$

As follows from Equations (55) and (56), sign "−" corresponds to the sphere collapse under gravitation forces, whereas the solution with sign "+" describes the sphere expansion which can start from the collapsed sphere under the action of internal forces the nature of which is not discussed here.

3.2. Collapse Problem

To study the collapse problem, we should take sign "−" in the forgoing solution. Then

$$\rho(t)=\frac{\rho_0}{\left(1-t/t_c^{(R)}\right)^2},\ R(t)=R_0\sqrt[3]{\left(1-t/t_c^{(R)}\right)^2},\ v^1(r,t)=-\frac{2r}{t_c^{(R)}\left(1-t/t_c^{(R)}\right)} \tag{59}$$

in which

$$t_c^{(R)}=\frac{1}{\sqrt{6\pi G\rho_0}}=\frac{2R_0}{3c}\sqrt{\frac{R_0}{r_g}} \tag{60}$$

is the collapse time. It is interesting to note that $v^1(t=0)=2r/t_c^{(R)}$, *i.e.*, that the initial velocity is not zero which means that, as in the Newton theory, a static initial solution does not exist for collapse problem in General Relativity.

As earlier, consider the sphere with the parameters of Sun for which Equation (60) yields $t_c^{(R)}=750\ \text{sec}$. This time is about one half of the result corresponding to the Newton theory ($t_c^{(N)}=1766\ \text{sec}$). The obtained result does not coincide with Equation (2) presented in the literature [2] [3] which gives the same collapse time that in the Newton theory, *i.e.* $t_c^{(W)}=t_c^{(N)}$. The difference is most probably associated with different metric forms used for analysis, the time $t_c^{(W)}$ is found from equations following from the Schwarzschild metric form presented in the co-moving coordinates in which $g_{44}=1$.

The dependence $R(t)/R_0$ corresponding to Equations (59) is shown in **Figure 1** (line 5). It should be emphasized that the scenario of the collapse with zero pressure, as it has been already noted, is not realistic. Pressure resists gravitational contraction and can prevent it. This scenario is supported by the existence of the static solution for the general relativity equations incorporating pressure [21] [22]. However, even in the absence of pressure, there is the effect that stops collapse. Indeed, as follows from Equation (46), the sphere radius cannot be smaller that r_g. Taking $R(t)=r_g$ in the second equation of Equations (59), we can find the lower limit of time for which the foregoing solution exists

$$t^*=\left(1-\sqrt{\frac{r_g^3}{R_0^3}}\right)t_c^{(R)}$$

For the sphere with the parameters of Sun, t^* is rather close to $t_c^{(R)}$, *i.e.*,

$$t^*=\left(1-8.69\times10^{-9}\right)t_c^{(R)}$$

3.3. Expansion Problem

Now consider the second case corresponding to sign "+" in the forgoing equations which yield

$$\rho(t)=\frac{\rho_0}{\left(1+t\sqrt{6\pi G\rho_0}\right)^2},\ R(t)=R_0\sqrt[3]{\left(1+t\sqrt{6\pi G\rho_0}\right)^2},\ v^1=\frac{2r\sqrt{6\pi G\rho_0}}{3\left(1+t\sqrt{6\pi G\rho_0}\right)} \quad (61)$$

This solution describes the sphere expansion. It looks more realistic than the collapse solution considered above because the absence of pressure is a reasonable assumption for the expansion problem. It is interesting to note Equations (61) coincide with Equations (29), (30) and (31) obtained in the previous Section for the Newton theory.

Suppose that the initial density is infinitely high and $R_0=0$. Taking $\rho_0\to\infty$ in the first and the last equations of Equations (61), we arrive at

$$\rho(t)=\frac{1}{6\pi Gt^2},\ v^1=\frac{2r}{3t} \quad (62)$$

The limit version of the second equation in Equations (61) can be obtained if we use Equation (5) for the sphere mass which does not change in the process of expansion. The result is

$$R^3(t)=\frac{9}{2}mGt^2$$

As an example, consider the universe with the age $t_u=13.787\times10^9$ years [23]. Taking into account that $G=6.67\times10^{-11}$ m^3/kg sec^2 and using the first equation in Equations (62), we get $\rho(t_u)=0.421\times10^{-26}$ kg/m^3. This result is in fair agreement with existing evaluations which have rather high scatter. The second equation in Equations (62) shows that the velocity is proportional to the radial coordinate which also corresponds to the existing observations. This equation describes a specific motion with zero coordinate acceleration, *i.e.*,

$$\frac{\mathrm{d}v^1}{\mathrm{d}t}=\frac{\partial v^1}{\partial t}+v^1\frac{\partial v^1}{\partial r}=0$$

For $t\to\infty$, we have $\rho\to0$, $v^1\to0$, $R\to\infty$ and $g_{14}\to0$, $g_{44}\to1$ which corresponds to an infinite static Euclidean space.

4. Conclusion

Gravitational collapse and expansion of a pressure-free sphere are analyzed within the framework of the Newton gravitation theory and general relativity. For the Newton theory, a step-by-step numerical solution on any step of which the gravitation force does not depend on time is constructed. The sequence of collapse times corresponding to solutions with various numbers of steps converges to the exact solution of the field equations which is determined analytically. For general relativity, the solution is found for the special model of four-dimensional space which is Euclidean with respect to space coordinates and Riemannian with respect to the time coordinate. This solution specifies two spaces: one of which describes the sphere collapse and the second corresponds to the sphere expansion. The first solution allows us to determine the collapse time which does not coincide with the result presented in the literature. The second solution specifies the expansion velocity which is proportional to the radial coordinate and decreases with time. At an infinite time, the space degenerates into the static Euclidean space with zero density and velocity.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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