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Hölder Derivative of Harmonic Functions on Sierpinski Gasket

Guangjun Yang[1], Ping Wang[2]

[1]College of Mathematics and Statistics, Yunnan University, Kunming, China
[2]Department of Mathematics, Penn State University, Schuylkill Haven, PA, USA
Email: yangzhuowei@outlook.com, pxw10@psu.edu

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Abstract

In the past years, we established analytic expressions of various fractals and discussed Hölder derivatives of the expressions. Based on our earlier results, we will study the properties of harmonic functions on a very important fractal, the Sierpinski gasket (SG). Our main result is that the harmonic function on SG satisfies a Hölder inequality of order $\alpha = \ln\frac{3}{5}\backslash\ln 2$.

Keywords

Holder Derivative, Harmonic Function, Sierpinski Gasket

1. Introduction

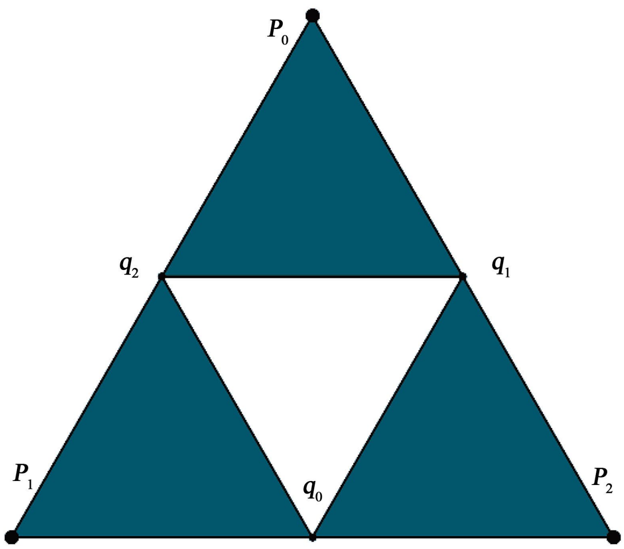
The concept of harmonic function on fractals was first introduced and studied by Kigami (see [1]-[3]) on Sierpinski gasket (SG). Since then, there have been various studies of the harmonic functions. Kigami proved the effective resistances for harmonic structures on p.c.f. semi-similar set (see [4]). Guariglia studied harmonic Sierpinski gasket and some of its applications (see [5]). Cao and Qiu considered boundary value problems for harmonic functions on domains in Sierpinski gaskets (see [6]). Very recently, Gopalakrishnan and Prasad investigated some harmonic functions on Vicsek fractal (see [7]). Our goal in this paper is to study the properties of harmonic functions on the Sierpinski gasket. By applying the methods we used in our recent work (see [8]), we will show that the harmonic function on Sierpinski gasket satisfies a Hölder inequality of order $\alpha = \ln\frac{3}{5}\backslash\ln 2$, namely, the harmonic function has a Hölder derivative of order $\alpha = \ln\frac{3}{5}\backslash\ln 2$. The order of this paper is organized as follows: In Section 2, based on the harmonic function introduced by Kigami (see [1]), we apply the method of [8] to establish a sequence of a priori estimates. In Section 3, we prove our main result that the harmonic function on SG satisfies a Hölder inequality of order $\alpha = \ln\frac{3}{5} \backslash \ln 2$.

2. Harmonic Functions on Sierpinski Gasket—Kagami's Definition

The initiator of Sierpinski gasket is an equilaternal triangle with sides of unit length, $\Delta P_0 P_1 P_2$, $P_0 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $P_1 = (0,0)$, $P_2 = (1,0)$. Take

$$F_0(p) = \frac{1}{2}P_0 + \frac{1}{2}p, F_1(p) = \frac{1}{2}P_1 + \frac{1}{2}p, F_2(p) = \frac{1}{2}P_2 + \frac{1}{2}p \tag{2.1}$$

After successive iterations, the Sierpinski gasket is obtained.



Denote the three vertices of the initial triangle as $V_0 = (x_1, y_1) = \{P_0, P_1, P_2\}$, the six vertices of three triangles as $V_1 = \{P_0, P_1, P_2, q_0, q_1, q_2\}$, where $P_0 = F_0(P_0)$, $P_1 = F_1(P_1)$, $P_2 = F_2(P_2)$, $q_0 = F_1(P_2) = F_2(P_1)$, $q_1 = F_0(P_2) = F_2(P_0)$, $q_2 = F_0(P_1) = F_1(P_0)$. In general, denote the set of vertices of 3^n triangles resulting from m iteration of V_0 as V_m,

$$V_m = \left\{P = F_{w_1} \circ \cdots \circ F_{w_m}(P_i)\right\}$$

Denote

$$F_{w_1} \circ \cdots \circ F_{w_{m-1}} \circ F_w(P_i) = F_{w_1,\cdots,w_m}(P_i)$$
$$F_{w_1} \circ \cdots \circ F_{w_{m-1}} \circ F_w(q_i) = F_{w_1,\cdots,w_m}(q_i)$$

$w_1, \cdots, w_m, \in \{0,1,2\}$. Triangles with vertices in V_m are called cells, more precisely cells in level-m. For example, $\Delta P_0 q_1 q_2$, $\Delta P_1 q_0 q_2$, $\Delta P_2 q_0 q_1$ are level-1 cells.

Now, consider difference equations on V_m. Assign a value $u(q)$ to each point in V_m, which can be considered as the discrete value of a function u. For each

$P\in V_m$, let V_{mP} denote all neighboring points q of P. Since SG is a graph consisting of points and line segments, functions defined on vertices of this graph are called graph functions.

Harmonic function on V_m is actually defined through the difference analog of Laplace operator

$$\frac{u(x+h)+u(x-h)-u(x)}{h^2}$$

on a lattice with points in V_m.

Without the denominator, Laplacian Δu corresponds to

$$\begin{aligned}&\sum_i\left[u(x_i+h)+u(x_i-h)-2u(x_i)\right]\\&=\sum_i\left[\left(u(x_i+h)-u(x_i)\right)+\left(u(x_i-h)-u(x_i)\right)\right]\end{aligned}$$

For $x_i=P\in V_m$, if V_{mP} denotes the set of all neighboring points, the difference set of u on V_{mP} can be denoted as

$$H_m(u)(P)_{P\in V_m}=\sum_{q\in V_{mP}}\left(u(q)-u(P)\right),\tag{2.2}$$

when $m=0$, $P\in V_0$ and

$$\begin{aligned}H_0u(P)&=\begin{pmatrix}(H_0u)(P_0)\\(H_0u)(P_1)\\(H_0u)(P_2)\end{pmatrix}=H_0\begin{pmatrix}u(P_0)\\u(P_1)\\u(P_2)\end{pmatrix}=\begin{pmatrix}u(P_1)-u(P_0)+u(P_2)-u(P_0)\\u(P_0)-u(P_1)+u(P_2)-u(P_1)\\u(P_0)-u(P_2)+u(P_1)-u(P_2)\end{pmatrix}\\&=\begin{pmatrix}u(P_1)+u(P_2)-2u(P_0)\\u(P_0)+u(P_2)-2u(P_1)\\u(P_0)+u(P_1)-2u(P_2)\end{pmatrix}=\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}u(P_0)\\u(P_1)\\u(P_2)\end{pmatrix}=H_0\begin{pmatrix}u(P_0)\\u(P_1)\\u(P_2)\end{pmatrix}.\end{aligned}\tag{2.3}$$

where $H_0=\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}$ and $(H_0u)(P)$ is the second order difference of lattice point V_0. $H_1u(P)$ contains six points in V_1: P_0,P_1,P_2,q_0,q_1,q_2.

$$\begin{aligned}(H_1u)(P)&=\begin{pmatrix}(H_1u)(P_0)\\(H_1u)(P_1)\\(H_1u)(P_2)\\(H_1u)(q_0)\\(H_1u)(q_1)\\(H_1u)(q_2)\end{pmatrix}=\begin{pmatrix}(H_1u)(P)_{P\in V_0}\\(H_1u)(q)_{q\in V_1\setminus V_0}\end{pmatrix}\\&=\begin{pmatrix}u(q_1)+u(q_2)-2u(P_0)\\u(q_0)+u(q_2)-2u(P_1)\\u(q_0)+u(q_1)-2u(P_2)\\u(P_1)+u(P_2)+u(q_1)+u(q_2)-4u(q_0)\\u(P_0)+u(P_2)+u(q_0)+u(q_2)-4u(q_1)\\u(P_0)+u(P_1)+u(q_0)+u(q_1)-4u(q_2)\end{pmatrix}\end{aligned}\tag{2.4}$$

$$=\begin{pmatrix} -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 \\ 1 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 1 & 1 & -4 \end{pmatrix}\begin{pmatrix} u(P_0) \\ u(P_1) \\ u(P_2) \\ u(q_0) \\ u(q_1) \\ u(q_2) \end{pmatrix}$$

$$=\begin{pmatrix} T & J \\ J & X \end{pmatrix}\begin{pmatrix} (H_1u)(P)_{P\in V_0} \\ (H_1u)(q)_{q\in V_1\cup V_0} \end{pmatrix}.$$

where

$$T=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix},\quad J=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},\quad X=\begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix}$$

Therefore,

$$H_1=\begin{pmatrix} T & J \\ J & X \end{pmatrix}.$$

and $(H_2f)(P)$ is the second difference on V_2.

In general, because of the self-similarity of SG, $(H_mu)(P)$ are similar on all small triangles. Therefore, the discussion on each triangle will be similar to that with H_1. That is, $(H_mu)(P)$ is the second difference on V_m and

$$(H_mu)(P)=\begin{cases} H_m(u)(P) & P\in V_0, \\ H_m(u)(P) & P\in V_m\setminus V_0. \end{cases}$$

For u to be a harmonic function, $u(q_0),u(q_1),u(q_2)$ must satisfy respectively the following average properties.

$$(H_1u)(q)_{q\in V_1\setminus V_0}=\begin{pmatrix} u(P_1)+u(P_2)+u(q_1)+u(q_2)-4u(q_0) \\ u(P_0)+u(P_2)+u(q_0)+u(q_2)-4u(q_1) \\ u(P_0)+u(P_1)+u(q_0)+u(q_1)-4u(q_2) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Due to $H_1(u)=0$, or equivalently,

$$\begin{aligned} &u(P_1)+u(P_2)+u(q_1)+u(q_2)=4u(q_0), \\ &u(P_0)+u(P_2)+u(q_0)+u(q_2)=4u(q_1), \\ &u(P_0)+u(P_1)+u(q_0)+u(q_1)=4u(q_2). \end{aligned} \tag{2.5}$$

That is, the sum of the second order difference of u on interior points is equal to zero.

Combining the equations in (2.5) yields

$$u(P_0)+u(P_1)+u(P_2)=u(q_0)+u(q_1)+u(q_2). \tag{2.6}$$

Adding $u(q_0),u(q_1),u(q_2)$ to the three equations in (2.5) respectively and applying (2.6), we have

$$\begin{aligned} &u(P_1)+u(P_2)+u(q_0)+u(q_1)+u(q_2) \\ &=u(P_1)+u(P_2)+u(P_0)+u(P_1)+u(P_2)=5u(q_0), \end{aligned}$$

$$
\begin{aligned}
&u(P_0)+u(P_2)+u(q_0)+u(q_1)+u(q_2) \\
&=u(P_1)+u(P_2)+u(P_0)+u(P_1)+u(P_2)=5u(q_1), \\
&u(P_0)+u(P_1)+u(q_0)+u(q_1)+u(q_2) \\
&=u(P_1)+u(P_2)+u(P_0)+u(P_1)+u(P_2)=5u(q_2).
\end{aligned}
$$

Therefore,

$$
\begin{pmatrix} u(q_0) \\ u(q_1) \\ u(q_2) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} u(P_0) \\ u(P_1) \\ u(P_2) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{2.7}
$$

where $\alpha = u(P_0), \beta = u(P_1), \gamma = u(P_2)$, boundary values. (2.7), showing internal values of u on q_0, q_1, q_2 determined by boundary values, is called "$\frac{1}{5}-\frac{2}{5}$" rule. It is important to note that if $u(q_0), u(q_1), u(q_2)$ are determined by "$\frac{1}{5}-\frac{2}{5}$" rule, (2.5) is automatically satisfied. That is, the harmonic condition is satisfied. The values of u at q_0, q_1, q_2 is the average value of u at four neighboring points, which is one important property of an ordinary harmonic function. This helps us better understand our new harmonic conditions.

Now, substituting $u(q_1)$, $u(q_2)$, $u(q_3)$ determined by (2.7) into $(H_0 u)(P)_{P\in V_0}$, we obtain

$$
\begin{aligned}
&\begin{pmatrix} u(q_1)+u(q_2)-2u(P_0) \\ u(q_0)+u(q_2)-2u(P_1) \\ u(q_0)+u(q_1)-2u(P_2) \end{pmatrix} = \frac{3}{5}\begin{pmatrix} u(P_1)+u(P_2)-2u(P_0) \\ u(P_0)+u(P_2)-2u(P_1) \\ u(P_0)+u(P_1)-2u(P_2) \end{pmatrix} \\
&= \frac{3}{5}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} u(P_0) \\ u(P_1) \\ u(P_2) \end{pmatrix} = \frac{3}{5}H_0\begin{pmatrix} u(P_0) \\ u(P_1) \\ u(P_2) \end{pmatrix}
\end{aligned} \tag{2.8}
$$

Generally, denote $F_{w_1,\cdots,w_{m-1}}$ as $\mathcal{F}_{m-1}$.

$$
\begin{aligned}
&H_m(u)(P) \\
&= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \Big\{\Big[u(\mathcal{F}_{m-1}(q_1))+u(\mathcal{F}_{m-1}(q_2))-2u(\mathcal{F}_{m-1}(P_0))\Big] \\
&\quad +\Big[u(\mathcal{F}_{m-1}(q_2))+u(\mathcal{F}_{m-1}(q_0))-2u(\mathcal{F}_{m-1}(P_1))\Big] \\
&\quad +\Big[u(\mathcal{F}_{m-1}(q_0))+u(\mathcal{F}_{m-1}(q_1))-2u(\mathcal{F}_{m-1}(P_2))\Big] \\
&\quad +\Big[u(\mathcal{F}_{m-1}(P_1))+u(\mathcal{F}_{m-1}(P_2))+u(\mathcal{F}_{m-1}(q_1))+u(\mathcal{F}_{m-1}(q_2))-4u(\mathcal{F}_{m-1}(q_0))\Big] \\
&\quad +\Big[u(\mathcal{F}_{m-1}(P_0))+u(\mathcal{F}_{m-1}(P_2))+u(\mathcal{F}_{m-1}(q_0))+u(\mathcal{F}_{m-1}(q_2))-4u(\mathcal{F}_{m-1}(q_1))\Big] \\
&\quad +\Big[u(\mathcal{F}_{m-1}(P_0))+u(\mathcal{F}_{m-1}(P_1))+u(\mathcal{F}_{m-1}(q_0))+u(\mathcal{F}_{m-1}(q_1))-4u(\mathcal{F}_{m-1}(q_2))\Big]\Big\}.
\end{aligned}
$$

It follows from the harmonic condition the following average value properties

$$
\begin{cases}
\Big[u(\mathcal{F}_{m-1}(P_1))+u(\mathcal{F}_{m-1}(P_2))+u(\mathcal{F}_{m-1}(q_1))+u(\mathcal{F}_{m-1}(q_2))-4u(\mathcal{F}_{m-1}(q_0))\Big]=0 \\
\Big[u(\mathcal{F}_{m-1}(P_0))+u(\mathcal{F}_{m-1}(P_2))+u(\mathcal{F}_{m-1}(q_0))+u(\mathcal{F}_{m-1}(q_2))-4u(\mathcal{F}_{m-1}(q_1))\Big]=0 \\
\Big[u(\mathcal{F}_{m-1}(P_0))+u(\mathcal{F}_{m-1}(P_1))+u(\mathcal{F}_{m-1}(q_0))+u(\mathcal{F}_{m-1}(q_1))-4u(\mathcal{F}_{m-1}(q_2))\Big]=0
\end{cases} \tag{2.9}
$$

which shows $H_m(u)(P)_{P\in V_m\setminus V_0}=0$ and the average value properties are equivalent. Therefore, the average value properties can also be used to define a harmonic function.

(2.9) implies the "$\frac{1}{5}-\frac{2}{5}$" rule

$$\begin{pmatrix} u(\mathcal{F}_{m-1}(q_0)) \\ u(\mathcal{F}_{m-1}(q_1)) \\ u(\mathcal{F}_{m-1}(q_2)) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} u(\mathcal{F}_{m-1}(P_0)) \\ u(\mathcal{F}_{m-1}(P_1)) \\ u(\mathcal{F}_{m-1}(P_2)) \end{pmatrix}. \tag{2.10}$$

From the "$\frac{1}{5}-\frac{2}{5}$" rule, we can obtain the extreme value principle

$$\min_{j=0,1,2} u\left(F_{w_1,\cdots,w_m}(P_j)\right) \le u\left(F_{w_1,\cdots,w_m}(q_i)\right) \le \max_{j=0,1,2} u\left(F_{w_1,\cdots,w_m}(P_j)\right), \quad i=0,1,2.$$

Notice that (2.9) and (2.10) hold for any integer $m>0$.

Now, the other terms of $H_m(u)(P)$,

$$\begin{aligned} &H_m(u)(P)_{V_0} \\ &= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \Big\{\left[u(\mathcal{F}_{m-1}(q_1)) + u(\mathcal{F}_{m-1}(q_2)) - 2u(\mathcal{F}_{m-1}(P_0))\right] \\ &\quad + \left[u(\mathcal{F}_{m-1}(q_2)) + u(\mathcal{F}_{m-1}(q_0)) - 2u(\mathcal{F}_{m-1}(P_1))\right] \\ &\quad + \left[u(\mathcal{F}_{m-1}(q_0)) + u(\mathcal{F}_{m-1}(q_1)) - 2u(\mathcal{F}_{m-1}(P_2))\right]\Big\} \end{aligned}$$

Applying (2.10), we have

$$\begin{aligned} &\left[u(\mathcal{F}_{m-1}(q_1)) + u(\mathcal{F}_{m-1}(q_2)) - 2u(\mathcal{F}_{m-1}(P_0))\right] \\ &= \frac{3}{5}\left[u(\mathcal{F}_{m-1}(P_1)) + u(\mathcal{F}_{m-1}(P_2)) - 2u(\mathcal{F}_{m-1}(P_0))\right] \\ &\left[u(\mathcal{F}_{m-1}(q_2)) + u(\mathcal{F}_{m-1}(q_0)) - 2u(\mathcal{F}_{m-1}(P_1))\right] \\ &= \frac{3}{5}\left[u(\mathcal{F}_{m-1}(P_2)) + u(\mathcal{F}_{m-1}(P_0)) - 2u(\mathcal{F}_{m-1}(P_1))\right] \\ &\left[u(\mathcal{F}_{m-1}(q_0)) + u(\mathcal{F}_{m-1}(q_1)) - 2u(\mathcal{F}_{m-1}(P_2))\right] \\ &= \frac{3}{5}\left[u(\mathcal{F}_{m-1}(P_0)) + u(\mathcal{F}_{m-1}(P_1)) - 2u(\mathcal{F}_{m-1}(P_2))\right] \end{aligned}$$

That is,

$$\begin{aligned} &\begin{pmatrix} u(\mathcal{F}_{m-1}(q_1)) + u(\mathcal{F}_{m-1}(q_2)) - 2u(\mathcal{F}_{m-1}(P_0)) \\ u(\mathcal{F}_{m-1}(q_2)) + u(\mathcal{F}_{m-1}(q_0)) - 2u(\mathcal{F}_{m-1}(P_1)) \\ u(\mathcal{F}_{m-1}(q_1)) + u(\mathcal{F}_{m-1}(q_0)) - 2u(\mathcal{F}_{m-1}(P_2)) \end{pmatrix} \\ &= \frac{3}{5}\begin{pmatrix} u(\mathcal{F}_{m-2}(q_2)) + u(\mathcal{F}_{m-2}(q_1)) - 2u(\mathcal{F}_{m-2}(P_0)) \\ u(\mathcal{F}_{m-2}(q_2)) + u(\mathcal{F}_{m-2}(q_0)) - 2u(\mathcal{F}_{m-2}(P_1)) \\ u(\mathcal{F}_{m-2}(q_0)) + u(\mathcal{F}_{m-2}(q_1)) - 2u(\mathcal{F}_{m-2}(P_2)) \end{pmatrix} \end{aligned}$$

Therefore,

$$H_m(u)(P) = \frac{3}{5}H_{m-1}(u)(P) = \cdots = \left(\frac{3}{5}\right)^m (H_0 u)(P)$$

The factor $\frac{3}{5}$ is called the renormalization factor, which plays an important role in our later discussions.

From

$$u(\mathcal{F}_m(q_0)) - u(\mathcal{F}_m(q_1)) = \frac{1}{5}\left[u(\mathcal{F}_m(P_1)) - u(\mathcal{F}_m(P_0))\right]$$

$$u(\mathcal{F}_m(q_0)) - u(\mathcal{F}_m(q_2)) = \frac{1}{5}\left[u(\mathcal{F}_m(P_2)) - u(\mathcal{F}_m(P_0))\right]$$

$$u(\mathcal{F}_m(q_1)) - u(\mathcal{F}_m(q_2)) = \frac{1}{5}\left[u(\mathcal{F}_m(P_2)) - u(\mathcal{F}_m(P_1))\right]$$

Thus,

$$\begin{aligned}
&H_m(u)(P)_{V_0} \\
&= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \frac{3}{5}\Big\{\left[u(\mathcal{F}_{m-1}(P_1)) + u(\mathcal{F}_{m-1}(P_2)) - 2u(\mathcal{F}_{m-1}(P_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-1}(P_2)) + u(\mathcal{F}_{m-1}(P_0)) - 2u(\mathcal{F}_{m-1}(P_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-1}(P_0)) + u(\mathcal{F}_{m-1}(P_1)) - 2u(\mathcal{F}_{m-1}(P_2))\right]\Big\} \\
&= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \frac{3}{5}\Big\{\Big[\left[u(\mathcal{F}_{m-2}F_0(P_1)) + u(\mathcal{F}_{m-2}F_0(P_2)) - 2u(\mathcal{F}_{m-2}F_0(P_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_1(P_1)) + u(\mathcal{F}_{m-2}F_1(P_2)) - 2u(\mathcal{F}_{m-2}F_1(P_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_2(P_1)) + u(\mathcal{F}_{m-2}F_2(P_2)) - 2u(\mathcal{F}_{m-2}F_2(P_0))\right]\Big] \\
&\quad + \Big[\left[u(\mathcal{F}_{m-2}F_0(P_0)) + u(\mathcal{F}_{m-2}F_0(P_2)) - 2u(\mathcal{F}_{m-2}F_0(P_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_1(P_0)) + u(\mathcal{F}_{m-2}F_1(P_2)) - 2u(\mathcal{F}_{m-2}F_1(P_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_2(P_0)) + u(\mathcal{F}_{m-2}F_2(P_2)) - 2u(\mathcal{F}_{m-2}F_2(P_1))\right]\Big] \\
&\quad + \Big[\left[u(\mathcal{F}_{m-2}F_0(P_0)) + u(\mathcal{F}_{m-2}F_0(P_1)) - 2u(\mathcal{F}_{m-2}F_0(P_2))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_1(P_0)) + u(\mathcal{F}_{m-2}F_1(P_1)) - 2u(\mathcal{F}_{m-2}F_1(P_2))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}F_2(P_0)) + u(\mathcal{F}_{m-2}F_2(P_1)) - 2u(\mathcal{F}_{m-2}F_2(P_2))\right]\Big]\Big\} \\
&= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \frac{3}{5}\Big\{\left[u(\mathcal{F}_{m-2}(q_2)) + u(\mathcal{F}_{m-2}(q_1)) - 2u(\mathcal{F}_{m-2}(P_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(P_1)) + u(\mathcal{F}_{m-2}(q_1)) - 2u(\mathcal{F}_{m-2}(q_2))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(q_0)) + u(\mathcal{F}_{m-2}(P_2)) - 2u(\mathcal{F}_{m-2}(q_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(P_0)) + u(\mathcal{F}_{m-2}(q_1)) - 2u(\mathcal{F}_{m-2}(q_2))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(q_2)) + u(\mathcal{F}_{m-2}(q_0)) - 2u(\mathcal{F}_{m-2}(P_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(q_1)) + u(\mathcal{F}_{m-2}(P_2)) - 2u(\mathcal{F}_{m-2}(q_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(P_0)) + u(\mathcal{F}_{m-2}(q_2)) - 2u(\mathcal{F}_{m-2}(q_1))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(q_2)) + u(\mathcal{F}_{m-2}(P_1)) - 2u(\mathcal{F}_{m-2}(q_0))\right] \\
&\quad + \left[u(\mathcal{F}_{m-2}(q_1)) + u(\mathcal{F}_{m-2}(q_0)) - 2u(\mathcal{F}_{m-2}(P_2))\right]\Big\}
\end{aligned}$$

$$= \sum_{w_1,\cdots,w_{m-1}\in\{0,1,2\}} \frac{3}{5}\Big\{\Big[u\big(\mathcal{F}_{m-2}(q_2)\big)+u\big(\mathcal{F}_{m-2}(q_1)\big)-2u\big(\mathcal{F}_{m-2}(P_0)\big)\Big]$$
$$+\Big[u\big(\mathcal{F}_{m-2}(q_2)\big)+u\big(\mathcal{F}_{m-2}(q_0)\big)-2u\big(\mathcal{F}_{m-2}(P_1)\big)\Big]$$
$$+\Big[u\big(\mathcal{F}_{m-2}(q_0)\big)+u\big(\mathcal{F}_{m-2}(q_1)\big)-2u\big(\mathcal{F}_{m-2}(P_2)\big)\Big]\Big\}$$

For $i,j\in\{0,1,2\}$, we have

$$\left|u\big(\mathcal{F}_m(q_i)\big)-u\big(\mathcal{F}_m(q_j)\big)\right|$$
$$\le \frac{1}{5}\Big\{\left|u\big(\mathcal{F}_m(P_1)\big)-u\big(\mathcal{F}_m(P_0)\big)\right|+\left|u\big(\mathcal{F}_m(P_2)\big)-u\big(\mathcal{F}_m(P_0)\big)\right|$$
$$+\left|u\big(\mathcal{F}_m(P_2)\big)-u\big(\mathcal{F}_m(P_1)\big)\right|\Big\}$$

Thus,

$$\max_{i,j\in\{0,1,2\}}\left|u\big(\mathcal{F}_m(q_i)\big)-u\big(\mathcal{F}_m(q_j)\big)\right| \le \frac{3}{5}\max_{i,j\in\{0,1,2\}}\left|u\big(\mathcal{F}_m(P_i)\big)-u\big(\mathcal{F}_m(P_j)\big)\right|$$

and successfully

$$\max_{i,j\in\{0,1,2\}}\left|u\big(\mathcal{F}_m(q_i)\big)-u\big(\mathcal{F}_m(q_j)\big)\right| \le \left(\frac{3}{5}\right)^m \max_{i,j\in\{0,1,2\}}\left|u(P_i)-u(P_j)\right| \le \left(\frac{3}{5}\right)^m C \quad (2.11)$$

where constant C is determined by boundary values. It is now possible to prove that harmonic function on SG satisfies Hölder inequality with $\alpha = \ln\frac{3}{5}\backslash\ln 2$.

3. Main Result

Theorem 3.1 *Suppose* p,q *on SG satisfy* $|p-q| \le \left(\frac{1}{2}\right)^m$. *Then, harmonic function* u *satisfy the following inequality*

$$\left|u(p)-u(q)\right| \le C_1 |p-q|^\alpha, \quad (3.1)$$

where C_1 is a const and $\alpha = \ln\frac{3}{5}/\ln 2$.

Proof. Suppose $|p-q| > \left(\frac{1}{2}\right)^{m-1}$, we proceed in two cases:

1) p,q belong to some small cells at level- m.

2) p,q belong to different cells with a common vertex $F_{w_1,\cdots,w_m}(P_i)$.

Since all points in SG are iterations from three boundary points $\{P_0,P_1,P_2\}$, in case 1),

$$p = F_{w_1,\cdots,w_m,\cdots}(P_i) = F_{w_1,\cdots,w_m}(x)$$
$$q = F_{w_1,\cdots,w_m,\cdots}(P_j) = F_{w_1,\cdots,w_m}(y)$$

with the same iteration mapping up to level m. Therefore,

$$\left|u(p)-u(q)\right| = \left|u\big(F_{w_1,\cdots,w_m}(x)\big)-u\big(F_{w_1,\cdots,w_m}(y)\big)\right|$$
$$\le \left(\frac{3}{5}\right)^m C = \frac{3}{5}\left(\frac{3}{5}\right)^{m-1} C = \frac{3}{5}\left(\frac{1}{2}\right)^{(m-1)\alpha} C$$
$$\le |p-q|^\alpha \cdot \frac{3}{5} C.$$

As for case 2), since

$$\begin{aligned}\left|u(p)-u(q)\right| &= \left|u\left(F_{w_1,\cdots,w_m}(x)\right)-u\left(F_{w_1,\cdots,w_m}(P_i)\right)\right| \\ &\quad+\left|u\left(F_{w_1,\cdots,w_m}(P_i)\right)-u\left(F_{w_1,\cdots,w_m}(y)\right)\right| \\ &\le 2\left(\frac{3}{5}\right)^m C=\frac{6}{5}\left(\frac{3}{5}\right)^{m-1} C=\frac{6}{5}\left(\frac{3}{5}\right)^{(m-1)\alpha} C \\ &\le \left|p-q\right|^{\alpha}\cdot\frac{6}{5}C.\end{aligned}$$

We proved (3.1).

Kigami and others successfully established Laplace equation without the denominator in the differential on SG, obtaining important average value property and extreme value principle. But for other equations such as the Poisson equation $\Delta u=\phi$, the denominator in the differentials can not be ignored. It is difficult to discuss the Laplace operator or the Green's formula. In fact, function $u(x)$ on SG and Cantor function are "internal function", just like Koch curve. Since Hölder inequalities are satisfied, their derivatives should be α -order Hölder derivatives.

4. Conclusion

Based on the harmonic function introduced by Kigami, we apply the method we developed earlier to establish a sequence of a priori estimates with which we proved our main result that the harmonic function on SG satisfies a Hölder inequality of order $\alpha=\ln\frac{3}{5}\setminus\ln 2$. The significance of the derivative of a harmonic function is that it directly relates to Laplace's equation. It signifies any distribution with no local "peaks" or "valleys". This property is crucial in various fields like physics, where it models phenomena like electric potential or steady-state temperature distribution.

5. Future Work

It is reasonable to consider related differential equations as those involving α -order Hölder derivatives. The method used in this work involves heavily the analytic expression of SD fractal. Since we have also obtained the analytic expressions of other fractals, we believe in our future study, we could investigate Hölder derivative of harmonic functions on other fractals, such as Koch curve, Cantor set, etc. Of course, the future work would present different challenges as the structures and expressions of other fractals are entirely different. How we establish necessary a priori estimates for different fractals will not be the same and what obstacles ahead are yet to be seen.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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