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On the Strongly Hopfian Acts over Semigroups

Ksenya Anatolyevna Kolesnikova[1], Igor Borisovich Kozhukhov[1,2]

[1]National Research University MIET, Lomonosov Moscow State University, Moscow, Russia
[2]Russian Presidental Academy of National Economy and Public Administration, Moscow, Russia
Email: kozhuhov_i_b@mail.ru, ksenya.koless@gmail.com

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Abstract

We find the necessary and sufficient conditions on a coproduct of connected acts over a semigroup to be strongly hopfian. From this, we deduce the conditions of the strong hopfness for unitary acts over groups. Moreover, we prove that a finite coproduct of strongly hopfian acts over an arbitrary semigroup is strongly hopfian.

Keywords

Act Over Semigroup, Act Over Group, Strongly Hopfian Act

1. Introduction

An algebra A is called hopfian if it is not isomorphic to a proper homomorphic image of itself (an equivalent definition: every surjective endomorphism $\alpha: A \to A$ is injective (and therefore it is automorphism)). A conception of hopfness (and dual notion of co-hopfness) appeared in the group theory (see ([1], v. 2, Section 15), [2]-[4]). Abelian groups which are hopfian were also investigated in [5]. There are a lot of articles where hopfness and co-hopfness in modules over rings and other algebraic structures are considered.

The hopfness is a *finiteness condition*, since all the finite objects are hopfian. The hopfness is a weaker condition as the noetherness (maximal condition on congruences). Immediate position between hopfian and noetherian algebras occupy the strongly hopfian algebras. An algebra A is called *strongly hopfian* if, for every endomorphism $\varphi: A \to A$, a chain of kernels $\ker\varphi \subseteq \ker\varphi^2 \subseteq \cdots$ is stabilized, *i.e.* $\ker\varphi^{n+1} = \ker\varphi^n$ for some n.

Let S be a semigroup. A set X is called *an act over a semigroup* S if a mapping $X \times S \to X$ is given such that $x(st) = (xs)t$ for any $x \in X$ and $s,t \in S$ [6] [7]. The act is an algebraic model of *automaton* (there X is a set of states and S is a semigroup of input signals). Besides, an act is an *unary algebra*, *i.e.* an algebra with unary operations.

The hopfian acts over semigroups were considered in [8]-[11]. V. K. Kartashov proved ([12], Thm. 2) that every finitely generated commutative act is hopfian (it is some reformulation).

In the present work, we find the necessary and sufficient conditions to be strongly hopfian for a coproduct of the connected acts over arbitrary semigroup. Using this result we characterize strongly hopfian unitary acts over a group. As a consequence we obtain the fact that a coproduct of a finite number of acts is strongly hopfian if and only if each of them is strongly hopfian. Authors do not know whether this statement is true for the hopfness.

Non-defined terms from the theory of acts can be found in [6] [7], semigroup theory in [13], universal algebra in [14].

2. Preliminary Considerations

For a mapping $\varphi : A \to B$, a relation $\ker\varphi = \{(a,a') \mid \varphi(a) = \varphi(a')\}$ is called a *kernel* of φ. If φ is a homomorphism of algebras then $\ker\varphi$ is a congruence. For any set X we put $\Delta_X = \{(x,x) \mid [\in X\}$ (the equality relation on X).

An act X over a semigroup S is called *unitary* if S has a unity e and $xe = x$ for any $x \in X$.

A *coproduct* of universal algebras is a notion dual to the direct product. In case of acts, a coproduct $\coprod_{i\in I} X_i$ of acts X_i ($i \in I$) is a disjoint union of this acts (or isomorphic copies of them).

For an act X over a semigroup S, we can consider a graph where X is a set of vertices and $\{(x, xs) \mid x \in X, s \in S\}$ is a set of edges. An act X is called *connected* if the graph is connected (in the usual sense, *i.e.* as a non-oriented graph). It is clear that any graph is a coproduct of connected acts (its *connected components*).

Let G be a group and H be its subgroup, not necessarily normal. By G/H we will denote a set of right cosets Hg for $g \in G$. Remark that G/H is not necessarily a group but it is an act over the group G where a multiplication is defined by the rule $Hg \cdot g' = Hgg'$. The act G/H is unitary *cyclic* (even a *simple*) act over G. The following facts can be established straightforwardly.

Fact 1. An act X over a group G is unitary cyclic if and only if X is isomorphic to an act G/H for some subgroup H of G.

Fact 2. An act X over a group G is unitary if and only if $X \cong \coprod_{i\in I} X_i$ where $X_i \cong G/H_i$ for some (not necessarily distinct) subgroups H_i of G.

Fact 3. There exists a homomorphism $G/H \to G/H'$ of acts over G if and only if $H \subseteq a^{-1}Ha$ for some $a \in G$ This homomorphism is always surjective (if it exists).

Let $X = \coprod_{i\in I} X_i$ be a coproduct of acts over a semigroup S. Define a binary relation $\preccurlyeq$ on the index set I as follows:

$$i \preccurlyeq j \Leftrightarrow \text{there exists a homomorphism } X_i \to X_j.$$

Let X be an act over a semigroup S and $\operatorname{End} X$ be its endomorphism semigroup. For $\varphi \in \operatorname{End} X$ we consider $\varphi^0 = \operatorname{id}_X$ (identical automorphism). If X is strongly hopfian then

$$\forall \varphi \in \operatorname{End} X \ \exists k \geq 0 \ \ker \varphi^k = \ker \varphi^{k+1}.$$

Therefore we can put $l(\varphi) = \inf\{k \geq 0 \mid \ker \varphi^k = \ker \varphi^{k+1}\}$. Further, we put $\operatorname{lh}(X) = \sup\{l(\varphi) \mid \varphi \in \operatorname{End} X\}$ (hopfian length of X). Note that the equality $\operatorname{lh}(X) = \infty$ is possible even if X is strongly hopfian.

3. Main Results

Let an act X be strongly hopfian. For every endomorphism $\varphi \in \operatorname{End} X$ we define a length $l(\varphi) = \min\{k \mid \ker \varphi^k = \ker \varphi^{k+1}\}$. We consider that $\varphi^0 = \operatorname{Id}_X$ is an identical automorphism. And a *hopfian length* of X we call $\operatorname{lh} X = \sup\{l(\varphi) \mid \varphi \in \operatorname{End} X\}$. If $\sup\{l(\varphi) \mid \varphi \in \operatorname{End} X\}$ does not exist then we say that $\operatorname{lh} X = \infty$.

Let $X = \coprod_{i \in I} X_i$ and $Y = \coprod_{j \in J} Y_j$ be acts over a semigroup S and X_i, Y_j be their connected components. Further, let $\varphi : X \to Y$ be a homomorphism. Consider any X_i. As X_i is connected, then $\varphi(X_i)$ is also connected, therefore $\varphi(X_i) \subseteq \varphi(Y_j)$ for some $j \in J$. So we have a mapping $\bar{\varphi} : I \to J$. Thus,

$$\bar{\varphi}(i) = j \Leftrightarrow \varphi(X_i) \subseteq Y(j).$$

For $i, j \in I$ we put $i \preceq j$ if there exists a homomorphism of acts $X_i \to X_j$. Clearly, the relation $\preceq$ is reflexive and transitive, *i.e.* it is a quasi-order.

Theorem 1. Let $X = \coprod_{i \in I} X_i$ be a coproduct of the connected acts over a semigroup S. Then X is strongly hopfian if and only if the following condition hold:

(i) X_i is strongly hopfian for any $i \in I$;

(ii) a set $J = \{i \mid \operatorname{lh}(X_i) = \infty\}$ is finite;

(iii) there exists a natural number L such that $\operatorname{lh}(X_i) \leq L$ for $i \in I \setminus J$;

(iv) there exists a natural number K such that $k \leq K$ for any chain $i_1 \preceq i_2 \preceq \cdots \preceq i_k$ of the distinct elements of I.

Proof. Necessity. Let X be strongly hopfian. Suppose that (i) does not hold. Then X_i is not strongly hopfian for some $i \in I$. Hence there is $\alpha \in \operatorname{End} X_i$ such that $\ker \alpha^k \neq \ker \alpha^{k+1}$ for all k. Put

$$\varphi(x) = \begin{cases} \alpha(x) & \text{if } x \in X_i, \\ x & \text{if } x \in X_j \text{ for some } j \neq i. \end{cases}$$

Obviously, $\ker \varphi^k \neq \ker \varphi^{k+1}$ for all k. As φ is an endomorphism, then X is not strongly hopfian which is false.

Suppose (ii) is not fulfilled. Then the set J is infinite. Therefore, there exists a sequence $j_1, j_2, \cdots$ of distinct elements of J. For any j_t we take $\varphi_t \in \operatorname{End} X_{j_t}$ such that $\ker \varphi_t^{t+1} \supset \ker \varphi_t^t$. Define an endomorphism $\varphi \in \operatorname{End} X$ by the rule

$$\varphi(x) = \begin{cases} \varphi_t(x) & \text{if } x \in X_{j_t}, \\ x & \text{if } x \in X_i \quad \text{where } i \notin \{j_1, j_2, \cdots\}. \end{cases}$$

It is seen that $\ker\varphi^k \neq \ker\varphi^{k+1}$ for all $k \in \mathbb{N}$, and it contradicts with the fact that X is strongly hopfian.

Suppose (iii) is not fulfilled. Then there are $i_1, i_2, \cdots \in I$ such that $\mathrm{lh}\left(X_{i_t}\right) \geq t$ for $t = 1, 2, \cdots$. Therefore, there exist $\varphi_t \in \operatorname{End} X_{i_t}$ such that $\ker\varphi_t^{t-1} \subset \ker\varphi_t^t$. Put $Y = \coprod_{t=1}^{\infty} X_{i_t}$. Construct a mapping $\varphi: X \to X$ as follows:

$$\varphi(x) = \begin{cases} \varphi_t(x) & \text{if } x \in X_{i_t} \text{ for some } t, \\ x & \text{if } x \notin Y. \end{cases}$$

Obviously, φ is an endomorphism of the act X and we have the strong inclusions $\ker\varphi^{t-1} \subset \ker\varphi^t$ for all t. It contradicts to the assumption that X is strongly hopfian.

Suppose (iv) is not fulfilled. Construct a sequence of chains $\Gamma_1, \Gamma_2, \cdots$ in I such that $|\Gamma_t| = t$ for all t and $\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$. Take $\Gamma_1 = \{i\}$ where $i \in I$ is arbitrary. Let $\Gamma_1, \cdots, \Gamma_m$ be constructed. Take a chain Γ of k elements where $k \geq 1 + 2 + \cdots + (m+1)$. Put $\Gamma' = \Gamma \setminus (\Gamma_1 \cup \cdots \cup \Gamma_m)$. Clearly $|\Gamma'| \geq m+1$. Deleting (if it is necessary) from the chain Γ' some elements we obtain a chain Γ_{m+1} of $m+1$ elements, and $\Gamma_{m+1} \cap (\Gamma_1 \cup \cdots \cup \Gamma_m) = \varnothing$.

For every chain $\Gamma_t = \{i_1 \preceq i_2 \preceq \cdots \preceq i_t\}$ we define a subact $X^{(t)} = X_{i_1} \sqcup X_{i_2} \sqcup \cdots \sqcup X_{i_t}$. By definition of the relation $\preceq$, there exist the homomorphisms $\varphi_1^{(t)}: X_{i_1} \to X_{i_2}$, $\varphi_2^{(t)}: X_{i_2} \to X_{i_3}$, $\cdots$, $\varphi_{t-1}^{(t)}: X_{i_{t-1}} \to X_{i_t}$. Construct a homomorphism $\varphi^{(t)}$ putting

$$\varphi^{(t)}(x) = \begin{cases} \varphi_i^{(t)}(x) & \text{if } x \in X_i \text{ and } i < t, \\ x & \text{if } x \in X_t. \end{cases}$$

Remark that $\ker\varphi^{(t)} \cap (X_i \times X_t) = \varnothing$ for $i < t$, however $\ker\left(\varphi^{(t)}\right)^2 \cap (X_{t-1} \times X_t) \neq \varnothing$ but $\ker\left(\varphi^{(t)}\right)^2 \cap (X_i \times X_t) = \varnothing$ only for $i < t-1$ and so on. Thus, we have $\ker\varphi^{(t)} \subset \ker\left(\varphi^{(t)}\right)^2 \subset \cdots \subset \ker\left(\varphi^{(t)}\right)^t$. We can present the act X in the view $X = \coprod_{t=1}^{\infty} X^{(t)} \sqcup X'$ where $X' = X \setminus \bigcup_{t=1}^{\infty} X^{(t)}$ is a subact of X or the empty set. Define an endomorphism $\varphi: X \to X$ as follows:

$$\varphi(x) = \begin{cases} \varphi^{(t)}(x) & \text{if } x \in X^{(t)} \text{ for some } t, \\ x & \text{if } x \in X'. \end{cases}$$

Since the lengths of the chains is not bounded from above that $\ker\varphi^t$ are distinguish, which contradicts the strongly hopfness of X.

Sufficiency. Assume that the conditions (i)-(iv) hold and $\varphi \in \operatorname{End} X$. As X_i are connected then φ induces a mapping $\bar{\varphi}: I \to I$ such that $\bar{\varphi}(i) = j \Leftrightarrow \varphi(X_i) \subseteq X_j$. The set I with the unary operation $\bar{\varphi}$ is a *unar* (in another terminology: *monounary algebra*, see [15]).

Because of the condition (iv), $\bar{\varphi}^K(i)$ lies in a cycle for any $i \in I$. As the lengths of cycles are less or equal to K then for any cycle C and any $i \in C$ we have $\bar{\varphi}^{K!}(i) = (i)$. Because of (ii) there are finitely many members of infinite length, let they be $X_{j_1}, \cdots, X_{j_s}$. Put $\psi = \varphi^{K!}$. Select from $X_{j_1}, \cdots, X_{j_s}$ the members which are invariant with respect to ψ (*i.e.* $\psi(X_i) \subseteq X_i$). Without loss of

generality we may consider that $X_{j_1},\cdots,X_{j_t}$ are invariant but $X_{j_{t+1}},\cdots,X_{j_s}$ are not (here $0\le t\le s$). Put $\psi_i=\psi|_{X_{j_i}}$ for $i=1,\cdots,t$. By the condition (i) each endomorphism of X_i has a finite length. Therefore, we may put
$L_0=\max\{L,l(\psi_1),\cdots,l(\psi_t)\}$.

Let $(x,y)\in\ker\varphi^{K+2L_0K!}$. Then $(\varphi^K(x),\varphi^K(y))\in\ker\varphi^{2L_0K!}$. Put $x'=\varphi^K(x)$, $y'=\varphi^K(y)$. Obviously, $x'\in X_i$, $y'\in X_j$ where i,j belong to a cycle of the unar I Denote this cycle by C. Thus $i,j\in C$.

We have $\psi^{2L_0}(x')=\psi^{2L_0}(y')$. As the lengths of cycles are less or equal to K then $\bar{\psi}=\bar{\varphi}^{K!}$ is identical mapping on C, *i.e.* $\psi(X_i)\subseteq X_i$ and $\psi(X_j)\subseteq X_j$. As $\psi^{2L_0}(x')=\psi^{2L_0}(y')$ then $i=j$, therefore $x',y'\in X_i$.

If $\mathrm{lh}(X_i)<\infty$, then by (iii) $\mathrm{lh}(X_i)\le L\le L_0$. If $\mathrm{lh}(X_i)=\infty$, then $X_i=X_{j_u}$ for some $u\le t$, and therefore $l(\psi\,|\,X_i)=l(\psi_u)\le L_0$. In both cases
$\ker\left(\psi|_{X_i}\right)^{L_0}=\ker\left(\psi|_{X_i}\right)^{2L_0}$. It follows that $\psi^{L_0}(x')=\psi^{L_0}(y')$. It means that $\varphi^{L_0K!}(x')=\varphi^{L_0K!}(y')$. Therefore $\varphi^{K+L_0K!}(x)=\varphi^{K+L_0K!}(y)$.

Thus $(x,y)\in\ker\varphi^{K+L_0K!}$. We proved that $\ker\varphi^{K+L_0K!}\subseteq\ker\varphi^{K+2L_0K!}$. Put $M=K+L_0K!$. We have $\ker\varphi^{M}\subseteq\ker\varphi^{M+L_0K!}\subseteq\ker\varphi^{M+1}\subseteq\ker\varphi^{M}$. It implies $\ker\varphi^{M+1}=\ker\varphi^{M}$. Thus X is strongly hopfian. □

Corollary. A coproduct $X_1\sqcup\cdots\sqcup X_n$ of a finite number of acts over a semigroup is strongly hopfian if and only if each of X_i is strongly hopfian.

Proof. Decompose each X_i into a coproduct of connected components and apply Theorem 1. □

Remark. The authors do not know whether a similar statement is true for the hopfness.

Now let us move on to the unitary acts over the groups. Let G be a group and H be its subgroup.

Lemma 1 ([8], Lemma 1). A unitary cyclic act G/H is hopfian if and only if the following condition holds:

$$\forall a\in G\quad H\subseteq a^{-1}Ha\Leftrightarrow H=a^{-1}Ha. \qquad *$$

Lemma 2. A unitary cyclic act G/H over a group G is strongly hophian if and only if it is hopfian.

Proof. We need to proof only that every hopfian act G/H is strongly hopfian. Let G/H is hopfian and $\varphi\in\mathrm{End}(G/H)$. As any endoomorphism $\varphi:G/H\to G/H$ is subjective and G/H is hopfian, then φ is also injective. Then $\ker\varphi=\Delta_{G/H}$. Also $\ker\varphi^n=\Delta_{G/H}$ for all n. Therefore G/H is strongly hopfian. □

Remark. If G/H is hopfian then $\mathrm{lh}(G/H)=0$.

The authors proved in ([8], Thm. 1) the following statement.

Fact 4. A unitary act $X=\coprod_{i\in I}(G/H_i)$ over a group G is hopfian if and only if each of H_i satisfies ($*$) and there is no an infinity chain $i_1\preceq i_2\preceq\cdots$ of distinct elements of I.

A similar statement for the strong hopfness is so.

Theorem 2. A unitary act $X = \coprod_{i \in I} (G/H_i)$ over a group G is strongly hopfian if and only if the following conditions hold:

(v) $\forall i \in I \quad \forall a \in G \quad H_i \subseteq a^{-1} H_i a \Leftrightarrow H_i = a^{-1} H_i a$;

(vi) there exists a natural number K such that for any chain $i_1 \preceq i_2 \preceq \cdots \preceq i_k$ of distinct elements of I an inequality $k \le K$ is true.

Proof. The necessity follows from Theorem 1 in [8] (Fact 4) and Theorem 1. Prove the sufficiency. Let (v) and (vi) be satisfied. As (v) is fulfilled then by Lemma 2 the condition (i) of Theorem 1 holds. The condition (ii) holds since $J = \varnothing$. The condition (iii) holds since $\text{lh}(G/H_i) = 0$ (it follows from (v)). Finally the condition (iv) folds since it coincides with (vi). □

4. Examples

Here we give two examples: 1) a hopfian but not strongly hopfian act; 2) a strongly hopfian act of infinity hopfian length. Both acts are unitary acts over group.

Example 1. Let p be a prime number and $G = \mathbb{Z}_{p^\infty}$ be a quasi-cyclic group (a union of the ascending sequence of groups $H_n = \mathbb{Z}_{p^n}$: cyclic groups of order p^n). Then a coproduct $X = \coprod_{n \in \mathbb{N}} \mathrm{G/H_n}$ is hopfian but not strongly hopfian act over the group G.

Proof. Really, here $I = \mathbb{N}$ (set of natural number). We have an ascending sequence $1 \preceq 2 \preceq 3 \preceq \cdots$, therefore X is not strongly hopfian by Theorem 1. In the same time X is hopfian since ($*$) holds for any abelian group and its subgroup and there are no a descending sequence of distinct elements of I.

Remark. Although the quotient groups G/H_n in the Example 1 are isomorphic to one another (and they are isomorphic to the group G), G/H_n and G/H_m are not isomorphic for $m \ne n$ as the acts over the group G. It is known that $G/H \cong G/H'$ if and only if subgroups H and H' are conjugated.

Let us finish the article with another example. Note that any semigroup S is an act over itself. Denote this act by S_S. The subacts of this act are exactly the right ideals of the semigroup S, and the congruences are exactly the right congruences of the semigroup.

Example 2. Let $T = (0,1)$ be a semigroup with the usual multiplication. Clearly, $I = (0,1/2]$ is an ideal of T. Let $S = T/I$ be a Rees quotient semigroup. We may think that $S = \{0\} \cup (1/2;1)$ with a multiplication

$$x * y = \begin{cases} xy & \text{if } xy > 1/2, \\ 0 & \text{if } xy \le 1/2. \end{cases}$$

Then the act S_S is strongly hopfian with infinite hopfian length. We will provide a scheme of the proof:

(a) to prove that the endomorphisms of the act S_S are exactly the mappings of view $\varphi_a(x) = ax$ for $a \le 1$;

(b) to note that $\varphi_1(x)$ is identical automorphism and hence $\ker \varphi_1^k = \Delta_X$ for every k;

(c) to note that $\varphi_a(x)$ is nilpotent for $a < 1$, *i.e.* $\varphi_a^k = 0$ for some k (namely,

$\varphi_a^k = 0 \Leftrightarrow a^k \le 1/2$); therefore $\ker \varphi_a^k = S \times S$ for $k \ge -\ln 2/\ln a$;

(d) it follows from (c) that $l(\varphi_a) = [-\ln 2/\ln a]$;

(e) to note that $\lim_{a \to 1} l(\varphi_a) = +\infty$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kurosh, A.G. (2003) The Theory of Groups, V. 1: Chelsea Publ. Co., N.Y., 1956, 1960, 272 p.; V. 2. AMS, 308 p.

[2] Varadarajan, K. (1992) Hopfian and Co-Hopfian Objects. *Publicacions Matemàtiques*, **36**, 293-317. https://doi.org/10.5565/publmat_36192_21

[3] Deo, S. and Varadarajan, K. (1997) Hopfian and Co-Hopfian Groups. *Bulletin of the Australian Mathematical Society*, **56**, 17-24. https://doi.org/10.1017/s0004972700030690

[4] Hirshon, R. (1970) On Hopfian Groups. *Pacific Journal of Mathematics*, **32**, 753-766. https://doi.org/10.2140/pjm.1970.32.753

[5] Kaygorodov, E.V. (2014) Hopfian Abelian Groups. Gorno-Altaisk State Univ., 128 p. (In Russian)

[6] Kilp, M., Knauer, U. and Mikhalev, A.V. (2000) Monoids, Acts and Categories. W. de Gruyter, xvii + 529 p.

[7] Kozhukhov, I.B. and Mikhalev, A.V. (2023) Acts over Semigroups. *Journal of Mathematical Sciences*, **269**, 362-401. https://doi.org/10.1007/s10958-023-06287-3

[8] Kozhukhov, I.B. and Kolesnikova, K.A. (2023) On Hopfianity and Co-Hopfianity of Acts over Groups. *Journal of Mathematical Sciences*, **269**, 356-361. https://doi.org/10.1007/s10958-023-06286-4

[9] Kozhukhov, I.B. and Kolesnikova, K.A. (2023) Coproduct of Co-Hopfian Acts, Proc. VIII Int. Conf. "Donetsk Readings 2023". Vol. 1, Donetsk State Univ., 167-169. (In Russian)

[10] Roueentan, M. and Khosravi, R. (2022) On Hopfian (Co-Hopfian) and Fitting S-Acts (I). 13 p.

[11] Roueentan, M. and Khosravi, R. (2024) A Note on Hopfian and Co-Hopfian S-Acts. *Iranian Journal of Science*. https://doi.org/10.1007/s40995-024-01753-2

[12] Kartashov, V.K. (2008) Independent Systems of Generators and the Hopf Property for Unary Algebras. *Discrete Mathematics and Applications*, **18**, 625-630. https://doi.org/10.1515/dma.2008.047

[13] Clifford, A.H. and Preston, G.B. (1961, 1967) The Algebraic Theory of Semigroups, V. I, II, Mathematical Surveys, Number 7. Amer. Math. Soc., Providence, xvi + 244 p. and xv + 350 p.

[14] Cohn, P.M. (1965) Universal Algebra. Harper & Row, xv + 333 p.

[15] Jakubíková-Studenovská, D. and Pócs, J. (2009) Monounary Algebras. UPJS, 302 p.