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N-Fold Darboux Transformation and Various Solutions for the Coupled mKdV Equations

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Abstract

In this paper, we get the N-fold Darboux transformation with multi-parameters for the coupled mKdV equations with the help of a guage transformation of the spectral problem. As an application, some new multi-soliton solutions and complexiton solutions are obtained from choosing the appropriate seed solution. All obtained solutions and N-fold Darboux transformations are expressed using the Vandermonde-like determinants.

Keywords

The N-fold Darboux Transformation, The Coupled mKdV Equations, Multi-Soliton Solutions, Complexiton Solutions

1. Introduction

Soliton equations play an important role in the field of nonlinear science, whose have specific solutions to describe and explain the nonlinear phenomena, for example, super conductivity, plasma and elastic media, etc. [1]-[3]. In the past decades, a considerable number of methods [4]-[10] have been developed for obtaining explicit solutions of nonlinear evolution equations. Among them, the Darboux transformation is one of effective algorithmic procedures to generate explicit solution of some nonlinear evolution equations from the trivial seeds [11] [12]. The key step for constructing the Darboux transformation is to keep the corresponding spectral problems. The N-fold Darboux transformation can be regarded as the superposition of single Darboux transformation, thus allowing for the further acquisition of various solutions through symbolic computation [13]-[15].

In this paper, we construct the N-fold Darboux transformation for the coupled mKdV equations

$$u_t + \frac{1}{2}u_{xxx} - 3uu_x v = 0,$$
$$v_t + \frac{1}{2}v_{xxx} - 3vv_x u = 0, \tag{1}$$

which are still unknown to our knowledge. Equation (1) is an important member of the AKNS hierarchy and has various applications in mathematical and physical fields.

This paper is organized as follows: In Section 2, based on Neugebauer's idea, the N-fold DT of system (1) is constructed. In Section 3, the $2N$-soliton solutions and N-complexiton solutions of system (1) are obtained, which are also expressed as vandermonde-like determinants. In Section 4, some conclusion is given.

2. *N*-Fold Darboux Transformation

In order to construct a N-fold DT of the system (1), let us consider the following spectral problem

$$\phi_x = M\phi, \quad \phi = (\phi_1, \phi_2)^{\mathrm{T}}, \quad M = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix}, \tag{2}$$

and its auxiliary problem

$$\phi_t = N\phi, \quad N = \begin{pmatrix} -2i\lambda^3 - iuv\lambda + \frac{1}{2}(u_x v - uv_x) & 2u\lambda^2 + iu_x\lambda - \frac{1}{2}(u_{xx} - 2u^2 v) \\ 2v\lambda^2 - iv_x\lambda - \frac{1}{2}(v_{xx} - 2v^2 u) & 2i\lambda^3 + iuv\lambda - \frac{1}{2}(u_x v - uv_x) \end{pmatrix}, \tag{3}$$

the compatibility condition $\phi_{xt} = \phi_{tx}$ yields a zero curvature equation

$$M_t - N_x + [M, N] = 0,$$

which yields coupled mKdV system (1) by a direct computation.

Now, we consider a gauge transformation of the spectral problems (2) and (3)

$$\bar{\phi} = T\phi, \tag{4}$$

where T is defined by

$$(T_x + TM) = \bar{M}T, \tag{5}$$

$$(T_t + TN) = \bar{N}T. \tag{6}$$

By cross differentiating (5) and (6), we get

$$\bar{M}_t - \bar{N}_x + \bar{M}\bar{N} - \bar{N}\bar{M} = T(M_t - N_x + MN - NM)T^{-1}. \tag{7}$$

The lax pair (2) and (3) are transformed to

$$\bar{\phi}_x = \bar{M}\bar{\phi}, \bar{\phi}_t = \bar{N}\bar{\phi}. \tag{8}$$

(7) and (8) implie that $\bar{M}, \bar{N}$ has the same form as M, N expect replacing u and v with $\bar{u}$ and $\bar{v}$.

Suppose the Darboux matrix T in the forms of

$$T = T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{9}$$

where

$$A(\lambda)=A_N\left(\lambda^N+\sum_{k=0}^{N-1}A_k\lambda^k\right),\quad B(\lambda)=A_N\left(\sum_{k=0}^{N-1}B_k\lambda^k\right),$$

$$C(\lambda)=\frac{1}{A_N}\left(\sum_{k=0}^{N-1}C_k\lambda^k\right),\quad D(\lambda)=\frac{1}{A_N}\left(\lambda^N+\sum_{k=0}^{N-1}D_k\lambda^k\right),$$

$A_N, A_k, B_k, C_k, D_k\ (0\ge k\le N-1)$ are functions of x and t.

Let $\phi(\lambda_j)=(\phi_1(\lambda_j),\phi_2(\lambda_j))^{\mathrm{T}}$, $\psi(\lambda_j)=(\psi_1(\lambda_j),\psi_2(\lambda_j))^{\mathrm{T}}$ be two basic solutions of the spectral problems (2) and (3). From (5) and (6), there exist constants $r_j\ (1\ge j\le 2N)$, which satisfies

$$A(\lambda_j)\phi_1(\lambda_j)+B(\lambda_j)\phi_2(\lambda_j)-r_j\left(A(\lambda_j)\psi_1(\lambda_j)+B(\lambda_j)\psi_2(\lambda_j)\right)=0,$$

$$C(\lambda_j)\phi_1(\lambda_j)+D(\lambda_j)\phi_2(\lambda_j)-r_j\left(C(\lambda_j)\psi_1(\lambda_j)+D(\lambda_j)\psi_2(\lambda_j)\right)=0.$$

That is A_k, B_k, C_k and D_k are given by a linear algebraic system

$$\sum_{k=0}^{N-1}(A_k+\alpha_jB_k)\lambda_j^k=-\lambda_j^N,\quad \sum_{k=0}^{N-1}(C_k+\alpha_jD_k)\lambda_j^k=-\alpha_j\lambda_j^N \tag{10}$$

with

$$\delta_j=\frac{\phi_2(\lambda_j)-r_j\psi_2(\lambda_j)}{\phi_1(\lambda_j)-r_j\psi_1(\lambda_j)},\quad 0\ge j\le 2N, \tag{11}$$

where constants λ_j and r_j ($\lambda_k\ne\lambda_j$, $r_k\ne r_j$, as $k\ne j$) are chosen properly such that the determinant of coefficients for (10) are nonzero. So, A_k, B_k, C_k and $D_k\ (0\ge k\le N-1)$ are uniquely determined by (10).

From Equation (9), we have

$$\det T(\lambda_j)=A(\lambda_j)D(\lambda_j)-B(\lambda_j)C(\lambda_j). \tag{12}$$

On the other side, from system (10), we know that

$$A(\lambda_j)=-\alpha_jB(\lambda_j),\quad C(\lambda_j)=-\alpha_jD(\lambda_j). \tag{13}$$

Hence, it implies that

$$\det T(\lambda_j)=0,$$

which shows that $\lambda_j\ (1\ge j\le 2N)$ are $2N$ roots of $\det T(\lambda)$, in other words,

$$\det T(\lambda)=\prod_{j=1}^{2N}(\lambda-\lambda_j). \tag{14}$$

Based on the above facts, we will prove the following propositions.

Proposition 1 Assume that A_N satisfies

$$\partial_x\ln(A_N)=0,\quad A_N^2=1, \tag{15}$$

then the matrix $\bar{M}$ has the same form as M, respectively, where the transformation between the old potential u,v into new ones are defined as

$$\bar{u}=u+2iB_{N-1}, \tag{16}$$

$$\bar{v}=v-2iC_{N-1}, \tag{17}$$

and

$$\begin{aligned} A_{N-m,x} &= 2iB_{N-1}C_{N-m} + C_{N-m}u - B_{N-m}v, \\ B_{N-m,x} &= -2i\left(B_{N-M-1} - B_{N-1}D_{N-m}\right) - A_{N-m}u + D_{N-m}u, \\ C_{N-m,x} &= -2i\left(A_{N-M}C_{N-1} - C_{N-M-1}\right) + \left(A_{N-m} - D_{N-m}\right)v, \\ D_{N-m,x} &= -2iB_{N-m}C_{N-1} - C_{N-m}u - B_{N-m}v. \end{aligned} \tag{18}$$

Proof Let $T^{-1} = T^{*}/\det T$ and

$$\left(T_x + TM\right)T^{*} = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{19}$$

where $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are $(2N+3)$th-order polynomials in λ, $g_{12}(\lambda), g_{21}(\lambda)$ are $(2N+2)$th-order polynomials in λ, from Equation (2) and (11), we find a Riccati equation

$$\delta_{jx} = v + 2i\lambda\delta_j - u\delta_j^2. \tag{20}$$

From (14) and (20), we can get that $\lambda_j\left(1 \geq j \leq 2N\right)$ are roots of $g_{ki}(\lambda)(k,i=1,2)$, in this way, together with (14) and (19) gives

$$\left(T_x + TM\right)T^{*} = \left(\det T\right)p(\lambda), \tag{21}$$

with

$$p(\lambda) = \begin{pmatrix} p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix}, \tag{22}$$

where $p_{kj}^{(l)}\left(k,j=1,2;l=0,1\right)$ are undetermined functions independent of λ. And Equation (21) can be written as

$$\left(T_x + TM\right) = p(\lambda)T. \tag{23}$$

By compared with the coefficients of $\lambda^{N+1}, \lambda^{N}$ in Equation (23), we can find that

$$p_{11}^{(1)} = p_{22}^{(1)} = -i, \tag{24}$$

$$p_{21}^{(0)} = \frac{v - 2iC_{N-1}}{A_N^2}, \quad p_{11}^{(0)} = -p_{22}^{(0)} = \partial_x \ln\left(A_N\right) \tag{25}$$

$$p_{12}^{(0)} = A_N^2\left(u + 2iB_{N-1}\right). \tag{26}$$

Substituting (15) and (16) into (24)-(26), we can obtain that

$$p_{11}^{(1)} = p_{22}^{(1)} = -i, \quad p_{11}^{(0)} = -p_{22}^{(0)} = 0,$$

$$p_{12}^{(0)} = u + 2iB_{N-1}, \quad p_{21}^{(0)} = v - 2iC_{N-1}.$$

From (5) and (23), it is easy to see that $\bar{M} = p(\lambda)$. Therefore, the proof of Proposition 1 is completed.

Next, let the solutions $\phi\left(\lambda_j\right)$ and $\psi\left(\lambda_j\right)$ also satisfy (3), we try to prove $\bar{N}$ in (6) has the same form as N under the transformation (4) and (17).

Proposition 2 Assume that A_N submits the differential equation with respect to the variable t

$$\partial_t ln\left(A_N\right) = 4uC_{N-1}\left(A_{N-1} - D_{N-1}\right). \tag{27}$$

Then, the matrix $\bar{N}$ has the same form as N, namely,

$$\bar{N}=\begin{pmatrix} -2i\lambda^3-i\bar{u}\bar{v}\lambda+\frac{1}{2}\left(\bar{u}_x\bar{v}-\bar{u}\bar{v}_x\right) & 2\bar{u}\lambda^2+i\bar{u}_x\lambda-\frac{1}{2}\left(\bar{u}_{xx}-2\bar{u}^2\bar{v}\right) \\ 2\bar{v}\lambda^2-i\bar{v}_x\lambda-\frac{1}{2}\left(\bar{v}_{xx}-2\bar{v}^2\bar{u}\right) & 2i\lambda^3+i\bar{u}\bar{v}\lambda-\frac{1}{2}\left(\bar{u}_x\bar{v}-\bar{u}\bar{v}_x\right) \end{pmatrix}, \tag{28}$$

the old potentials u,v are mapped into new ones $\bar{u},\bar{v}$ according to the same Darboux transformation (4), (16) and (17).

Proof Let $T^{-1}=T^{*}/\det T$ and

$$\left(T_t+TN\right)T^{*}=\begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \tag{29}$$

where $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $(2N+3)$th-order polynomials in λ, $f_{12}(\lambda), f_{21}(\lambda)$ are $(2N+2)$th-order polynomials in λ. On the basic of (3), (11) and (13), we can get a Riccati equation

$$\begin{aligned}\delta_{jt}&=2v\lambda_j^2-iv_x\lambda_j-\frac{1}{2}\left(v_{xx}-2v^2u\right)+\left(4i\lambda_j^3+2iuv\lambda-j-\left(u_xv-uv_x\right)\right)\delta_j\\ &\quad-\left(2u\lambda_j^2+iu_x\lambda_j\frac{1}{2}\left(u_{xx}-2u^2v\right)\right)\delta_j^2.\end{aligned} \tag{30}$$

Through a series of calculation, we can get that $\lambda_j\,(1\ge j\le 2N)$ are roots of $f_{ki}(\lambda)(k,i=1,2)$. From (14) and (29), we arrive at

$$\left(T_t+TN\right)T^{*}=\left(\det T\right)Q(\lambda), \tag{31}$$

with

$$Q(\lambda)=\begin{pmatrix} q_{11}^{(3)}\lambda^3+q_{11}^{(2)}\lambda^2+q_{11}^{(1)}\lambda+q_{11}^{(0)} & q_{12}^{(2)}\lambda^2+q_{12}^{(1)}\lambda+q_{12}^{(0)} \\ q_{21}^{(2)}\lambda^2+q_{21}^{(1)}\lambda+q_{21}^{(0)} & q_{22}^{(3)}\lambda^3+q_{22}^{(2)}\lambda^2+q_{22}^{(1)}\lambda+q_{22}^{(0)} \end{pmatrix},$$

where $q_{n,s}^{(l)}\,(n,s=1,2;l=0,1,2)$ are undetermined functions independent of λ. Equation (31) can be written as

$$\left(T_t+TN\right)=Q(\lambda)T. \tag{32}$$

According to compare the coefficients of $\lambda^{N+3},\lambda^{N+2},\lambda^{N+1}$ in Equation (32), we have

$$q_{11}^{(2)}=q_{22}^{(2)}=0, q_{11}^{(3)}=-q_{22}^{(3)}=-2i,$$

$$q_{21}^{(2)}=2\left(v-2iC_{N-1}\right), q_{12}^{(2)}=2\left(u+2iB_{N-1}\right),$$

$$q_{11}^{(1)}=-iuv+2vB_{N-1}-4iB_{N-1}C_{N-1}-2uC_{N-1},$$

$$q_{22}^{(1)}=2uC_{N-1}+iuv-2vB_{N-1}+4iC_{N-1}B_{N-1},$$

$$q_{12}^{(1)}=iu_x+2uA_{N-1}+4iB_{N-2}-4iB_{N-1}D_{N-1}-2uD_{N-1},$$

$$q_{21}^{(1)}=-iv_x+2vD_{N-1}-4iC_{N-2}-2vA_{N-1}+4iC_{N-1}A_{N-1},$$

$$\begin{aligned}q_{12}^{(0)}&=-\frac{1}{2}\left(u_{xx}-2u^2v\right)+2uA_{N-2}+iu_xA_{N-1}+4iB_{N-3}+2iuvB_{N-1}\\ &\quad-\left(2vB_{N-1}-4iB_{N-1}C_{N-1}-2uC_{N-1}\right)B_{N-1}-D_{N-2}\left(4iB_{N-1}+2u\right)\\ &\quad-D_{N-1}\left(iu_x+2uA_{N-1}+4iB_{N-2}-4iB_{N-1}D_{N-1}-2uD_{N-1}\right),\end{aligned}$$

$$q_{21}^{(0)} = -4iC_{N-3} - 2iuvC_{N-1} - \frac{1}{2}\left(v_{xx} - 2v^2u\right) + 2vD_{N-2}$$
$$-iv_xD_{N-1} - \left(2v - 4iC_{N-1}\right)A_{N-2}$$
$$-\left(-iv_x + 2vD_{N-1} - 4iC_{N-2} - 2vA_{N-1} + 4iC_{N-1}\right)A_{N-1}\Big)A_{N-1}$$
$$-\left(2uC_{N-1} - 2vB_{N-1} + 4iC_{N-1}B_{N-1}\right)C_{N-1},$$

$$q_{11}^{(0)} = \partial_t \ln\left(A_N\right) + \frac{1}{2}\left(u_xv - uv_x\right) + 2vB_{N-2} - iv_xB_{N-1}$$
$$-\left(2vB_{N-1} - 4iB_{N-1}\right)C_{N-1}\Big)A_{N-1} - \left(-4iB_{N-1} + 2u\right)C_{N-2}$$
$$-C_{N-1}\left(iu_x + 4iB_{N-2} - 4iB_{N-1}D_{N-1} - 2uD_{N-1}\right),$$

$$q_{22}^{(0)} = -\partial_t \ln\left(A_N\right) + 2uC_{N-2} + iu_xC_{N-1} - \frac{1}{2}\left(u_xv - uv_x\right)$$
$$-\left(2v - 4iC_{N-1}\right)B_{N-2} - \left(-iv_x - 4iC_{N-2} - 2vA_{N-1} + 4iC_{N-1}\right)A_{N-1}\Big)B_{N-1}$$
$$-\left(2uC_{N-1} + 4iC_{N-1}\right)B_{N-1}\Big)D_{N-1}.$$

Substituting (15)-(17) into above expressions, we can obtain

$$\begin{aligned}
&q_{11}^{(2)} = q_{22}^{(2)} = 0, \quad q_{11}^{(3)} = -q_{22}^{(3)} = -2i, \\
&q_{21}^{(2)} = 2\left(v - 2iC_{N-1}\right) = 2\overline{v}, \quad q_{12}^{(2)} = 2\left(u + 2iB_{N-1}\right) = 2\overline{u}, \\
&q_{11}^{(1)} = -i\overline{u}\,\overline{v}, \quad q_{22}^{(1)} = i\overline{u}\,\overline{v}, \quad q_{12}^{(1)} = i\overline{u}_x, \quad q_{21}^{(1)} = i\overline{v}_x, \\
&q_{12}^{(0)} = -\frac{1}{2}\left(\overline{u}_{xx} - 2\overline{u}^2\overline{v}\right), \quad q_{21}^{(0)} = -\frac{1}{2}\left(\overline{v}_{xx} - 2\overline{v}^2\overline{u}\right), \\
&q_{11}^{(0)} = \frac{1}{2}\left(\overline{u}_x\overline{v} - \overline{u}\overline{v}_x\right), \quad q_{22}^{(0)} = -\frac{1}{2}\left(\overline{u}_x\overline{v} - \overline{u}\overline{v}_x\right).
\end{aligned} \tag{33}$$

Therefore, we have $Q(\lambda) = \overline{N}$, that is, the proof is completed.

The two propositions indicate that both of the lax pairs (2), (3) and (8) lead to the same system (1). The transformation (4), (16) and (17), $(\phi, u, v) \to (\overline{\phi}, \overline{u}, \overline{v})$ are called a DT of system (1).

Theorem The *N*-fold DT (16), (17) in terms of Vandermonde-like determinants can be written as

$$\begin{aligned}
\overline{u}[N] &= u + 2i\frac{\Delta B_{N-1}}{\Delta_0}, \\
\overline{v}[N] &= v - 2i\frac{\Delta C_{N-1}}{\Delta_0},
\end{aligned} \tag{34}$$

where Δ is determinant of the coefficients for the linear algebraic system (10), that is

$$\Delta_0 = \begin{vmatrix}
1 & \delta_1 & \lambda_1 & \delta_1\lambda_1 & \cdots & \lambda_1^{N-1} & \delta_1\lambda_1^{N-1} \\
1 & \delta_2 & \lambda_2 & \delta_2\lambda_2 & \cdots & \lambda_2^{N-1} & \delta_2\lambda_2^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \delta_{2N-1} & \lambda_{2N-1} & \delta_{2N-1}\lambda_{2N-1} & \cdots & \lambda_{2N-1}^{N-1} & \delta_{2N-1}\lambda_{2N-1}^{N-1} \\
1 & \delta_{2N} & \lambda_{2N} & \delta_{2N}\lambda_{2N} & \cdots & \lambda_{2N}^{N-1} & \delta_{2N}\lambda_{2N}^{N-1}
\end{vmatrix}$$

with ΔB_{N-1} is produced from Δ by replacing its $2N$-th column with

$\left(-\lambda_1^N,-\lambda_2^N,\cdots,-\lambda_{2N}^N\right)^{\mathrm{T}}$, ΔC_{N-1} is produced from Δ by replacing its $(2N-1)$-th column with $\left(-\delta_1\lambda_1^N,-\delta_2\lambda_2^N,\cdots,-\delta_{2N}\lambda_{2N}^N\right)^{\mathrm{T}}$, and $\delta_j\left(j=1,2,\cdots,2N\right)$ are given by (11).

3. The (2*N*)-Soliton Solutions

In this section, we shall apply the *N*-fold DT to obtain multi-soliton solutions of system (1). We start from the trivial solution $u=\alpha+\beta i$, $v=-\alpha+\beta i$ (as α and β are constants) as our seed solutions, then we choose two basic solutions of (2) and (3)

$$\phi\left(\lambda_j\right)=\begin{pmatrix}\cosh\mu_j\\ \frac{i\lambda_j}{u}\cosh\mu_j+\frac{c_j}{u}\sinh\mu_j\end{pmatrix},\quad \psi\left(\lambda_j\right)=\begin{pmatrix}\sinh\mu_j\\ \frac{i\lambda_j}{u}\sinh\mu_j+\frac{c_j}{u}\cosh\mu_j\end{pmatrix},$$

with

$$\mu_j=c_j\left(x+\left(uv+2\lambda_j^2\right)t\right),\quad c_j=\sqrt{uv+\lambda_j^2}\quad\left(1\geq j\leq 2N\right).$$

thus, δ_j can be written as

$$\delta_j=\frac{i\lambda_j}{u}+\frac{c_j}{u}\frac{\tanh_{\mu_j}-r_j}{1-r_j\tanh_{\mu_j}},\quad\left(1\geq j\leq 2N\right).\tag{35}$$

In what follows, we discuss soliton solutions of system (1) for the case of $N=1$, and $N=2$.

Let $N=1,\lambda=\lambda_j\left(j=1,2;j\neq k\right)$, solving the linear algebraic system (10), we have

$$A_0=\frac{\Delta A_0}{\Delta_0},\quad B_0=\frac{\Delta B_0}{\Delta_0},\quad C_0=\frac{\Delta C_0}{\Delta_0},\quad D_0=\frac{\Delta D_0}{\Delta_0},\tag{36}$$

where

$$\Delta_0=\begin{vmatrix}1&\delta_1\\1&\delta_2\end{vmatrix},\quad \Delta B_0=\begin{vmatrix}1&-\lambda_1\\1&-\lambda_2\end{vmatrix},\quad \Delta C_0=\begin{vmatrix}-\delta_1\lambda_1&\delta_1\\-\delta_2\lambda_2&\delta_2\end{vmatrix},$$

Resorting to expressions (16), (17), a multi-soliton solution of system (1) can be obtained as follows,

$$u[1]=\alpha+i\beta+\frac{2i\left(\lambda_1-\lambda_2\right)}{\delta_2-\delta_1},\quad v[1]=-\alpha+i\beta-\frac{2i\delta_1\delta_2\left(\lambda_2-\lambda_1\right)}{\delta_2-\delta_1},\tag{37}$$

where

$$\delta_j=\frac{i\lambda_j}{u}+\frac{c_j}{u}\frac{\tanh_{\mu_j}-r_j}{1-r_j\tanh_{\mu_j}},\quad \mu_j=c_j\left(x+\left(uv+2\lambda_j^2\right)t\right),\quad c_j=\sqrt{uv+\lambda_j^2},$$

and r_j,λ_j are arbitrary parameters.

For the case of $N=2,\lambda=\lambda_j\left(j=1,2,3,4\right)$, solving the linear algebraic system (10), we have

$$A_1=\frac{\Delta A_1}{\Delta_1},\quad B_1=\frac{\Delta B_1}{\Delta_1},\quad C_1=\frac{\Delta C_1}{\Delta_1},\quad D_1=\frac{\Delta D_1}{\Delta_1},\tag{38}$$

where

$$\Delta_1=\begin{vmatrix}1&\delta_1&\lambda_1&\lambda_1\delta_1\\1&\delta_2&\lambda_2&\lambda_2\delta_2\\1&\delta_3&\lambda_3&\lambda_3\delta_3\\1&\delta_4&\lambda_4&\lambda_4\delta_4\end{vmatrix},\quad \Delta B_1=\begin{vmatrix}1&\delta_1&\lambda_1&-\lambda_1^2\\1&\delta_2&\lambda_2&-\lambda_2^2\\1&\delta_3&\lambda_3&-\lambda_3^2\\1&\delta_4&\lambda_4&-\lambda_4^2\end{vmatrix},$$

$$\Delta C_1=\begin{vmatrix}1&\delta_1&-\delta_1\lambda_1^2&\lambda_1\delta_1\\1&\delta_2&-\delta_2\lambda_2^2&\lambda_2\delta_2\\1&\delta_3&-\delta_3\lambda_3^2&\lambda_3\delta_3\\1&\delta_4&-\delta_4\lambda_4^2&\lambda_4\delta_4\end{vmatrix}.$$

Resorting to expressions (16) and (17), a multi-soliton solution of system (1) can be obtained as follows,

$$u[2]=\alpha+i\beta+\frac{2i\Delta B_1}{\Delta_1},\quad v[2]=-\alpha+i\beta-\frac{2i\Delta C_1}{\Delta_1}. \tag{39}$$

4. The Complexiton Solutions

If we take the trivial solution $u=v=0$, and choose conjugated spectral parameters as follows,

$$\begin{aligned}&\lambda_{4j-3}=\alpha_j+i\beta_j=\lambda_j^{(1)},\quad \lambda_{4j-2}=-\alpha_j-i\beta_j=\lambda_j^{(2)},\\&\lambda_{4j-1}=\alpha_j-i\beta_j=\bar{\lambda}_j^{(1)},\quad \lambda_{4j}=-\alpha_j+i\beta_j=\bar{\lambda}_j^{(2)}.\end{aligned} \tag{40}$$

where $j=1,2,\cdots$. then the corresponding compatible solution of the lax pairs (2) and (3) can be choosed as

$$\begin{aligned}\phi(\lambda_{4j-3})&=\begin{pmatrix}\phi_1(\lambda_{4j-3})\\\phi_2(\lambda_{4j-3})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^-)(\cos(\xi_j^-)+i\sin(\xi_j^-))\\\exp(-\eta_j^-)(\cos(\xi_j^-)-i\sin(\xi_j^-))\end{pmatrix}\\\phi(\lambda_{4j-2})&=\begin{pmatrix}\phi_1(\lambda_{4j-2})\\\phi_2(\lambda_{4j-2})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^+)(\cos(\xi_j^+)-i\sin(\xi_j^+))\\\exp(-\eta_j^+)(\cos(\xi_j^+)+i\sin(\xi_j^+))\end{pmatrix}\\\phi(\lambda_{4j-1})&=\begin{pmatrix}\phi_1(\lambda_{4j-1})\\\phi_2(\lambda_{4j-1})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^-)(\cos(\xi_j^-)-i\sin(\xi_j^-))\\\exp(-\eta_j^-)(\cos(\xi_j^-)+i\sin(\xi_j^-))\end{pmatrix}\\\phi(\lambda_{4j})&=\begin{pmatrix}\phi_1(\lambda_{4j})\\\phi_2(\lambda_{4j})\end{pmatrix}=\begin{pmatrix}\exp(\eta_j^+)(\cos(\xi_j^+)+i\sin(\xi_j^+))\\\exp(-\eta_j^+)(\cos(\xi_j^+)-i\sin(\xi_j^+))\end{pmatrix}\end{aligned} \tag{41}$$

where

$$\eta_j^-=-i\alpha_jx+(6i\alpha_j\beta_j^2-2i\alpha_j^3)t,\eta_j^+=i\alpha_jx-(6i\alpha_j\beta_j^2-2i\alpha_j^3)t,$$

$$\xi_j^-=-i\beta_jx+(6i\alpha_j^2\beta_j+2i\beta_j^3)t,\xi_j^+=-i\beta_jx+(6i\alpha_j^2\beta_j+2i\beta_j^3)t,$$

and α_j,β_j are arbitrary real constants.

For the case of $r_j=0,1,(1\le j\le 4)$, according to (11), we have

$$
\begin{aligned}
\delta_{4j-3} &= \frac{\phi_2\left(\lambda_{4j-3}\right)}{\phi_1\left(\lambda_{4j-3}\right)} = \exp\left(-2\eta_j^-\right)\left(\cos 2\left(\xi_j^-\right) - i\sin 2\left(\xi_j^-\right)\right) = \delta_j^{(1)}, \\
\delta_{4j-2} &= \frac{\phi_2\left(\lambda_{4j-2}\right)}{\phi_1\left(\lambda_{4j-2}\right)} = \exp\left(-2\eta_j^+\right)\left(\cos 2\left(\xi_j^+\right) + i\sin 2\left(\xi_j^+\right)\right) = \delta_j^{(2)}, \\
\delta_{4j-1} &= \frac{\phi_2\left(\lambda_{4j-1}\right)}{\phi_1\left(\lambda_{4j-1}\right)} = \exp\left(-2\eta_j^-\right)\left(\cos 2\left(\xi_j^-\right) + i\sin 2\left(\xi_j^-\right)\right) = \overline{\delta}_j^{(1)}, \\
\delta_{4j} &= \frac{\phi_2\left(\lambda_{4j}\right)}{\phi_1\left(\lambda_{4j}\right)} = \exp\left(-2\eta_j^+\right)\left(\cos 2\left(\xi_j^+\right) - i\sin 2\left(\xi_j^+\right)\right) = \overline{\delta}_j^{(2)}.
\end{aligned} \tag{42}
$$

Resorting to expressions (40), (42) and DT (16) and (17), we can obtain the complexiton solutions of system (1),

$$
\begin{aligned}
\overline{u}\left[2N\right] &= u + 2i\frac{\Delta B_{2N-1}}{\Delta_{2N-1}}, \\
\overline{v}\left[2N\right] &= v - 2i\frac{\Delta C_{2N-1}}{\Delta_{2N-1}},
\end{aligned} \tag{43}
$$

where

$$
\Delta_{2N-1} = \det\left[\sigma_1^{(1)}, \sigma_1^{(2)}, \overline{\sigma_1^{(1)}}, \overline{\sigma_1^{(2)}}, \cdots, \sigma_N^{(1)}, \sigma_N^{(2)}, \overline{\sigma_N^{(1)}}, \overline{\sigma_N^{(2)}}\right]^{\mathrm{T}},
$$

$$
\Delta B_{2N-1} = \det\left[b_1^{(1)}, b_1^{(2)}, \overline{b_1^{(1)}}, \overline{b_1^{(2)}}, \cdots, b_N^{(1)}, b_N^{(2)}, \overline{b_N^{(1)}}, \overline{b_N^{(2)}}\right]^{\mathrm{T}},
$$

$$
\Delta C_{2N-1} = \det\left[c_1^{(1)}, c_1^{(2)}, \overline{c_1^{(1)}}, \overline{c_1^{(2)}}, \cdots, c_N^{(1)}, c_N^{(2)}, \overline{c_N^{(1)}}, \overline{c_N^{(2)}}\right]^{\mathrm{T}},
$$

with

$$
\begin{aligned}
\sigma_j^{(l)} &= \left(1, \delta_j^{(l)}, \lambda_j^l, \delta_j^{(l)}\lambda_j^{(l)}, \cdots, \lambda_j^{(l)^{2N-2}}, \delta_j^{(l)}\lambda_j^{(l)^{2N-2}}, \lambda_j^{(l)^{2N-1}}, \delta_j^{(l)}\lambda_j^{(l)^{2N-1}}\right), \\
b_j^{(l)} &= \left(1, \delta_1^{(l)}, \lambda_1^{(l)}, -\lambda_1^{(l)^2}, \cdots, \lambda_j^{(l)^{2N-2}}, \delta_j^{(l)}\lambda_j^{(l)^{2N-2}}, \lambda_j^{(l)^{2N-1}}, -\lambda_j^{(l)^{2N}}\right), \\
c_j^{(l)} &= \left(1, \delta_1^{(l)}, -\delta_1^{(l)}\lambda_1^{(l)^2}, \delta_1^{(l)}\lambda_1^{(l)}, \cdots, \lambda_j^{(l)^{2N-2}}, \delta_j^{(l)}\lambda_j^{(l)^{2N-2}}, -\delta_j^{(l)}\lambda_j^{(l)^{2N}}, \delta_j^{(l)}\lambda_j^{(l)^{2N-1}}\right),
\end{aligned} \tag{44}
$$

and $\overline{\sigma_j^{(1)}}, \overline{b_j^{(1)}}, \overline{c_j^{(1)}}$ are conjugated functions of $\sigma_j^{(l)}, b_j^{(l)}, c_j^{(l)}, \left(l = 1, 2\right)$.

According to properties of determinant, *N*-complexiton solutions of system (1) can be obtained as follows.

$$
\begin{aligned}
\overline{u} &= u + 2i\frac{\Lambda B_{2N-1}}{\Lambda_{2N-1}}, \\
\overline{v} &= v - 2i\frac{\Lambda C_{2N-1}}{\Lambda_{2N-1}},
\end{aligned} \tag{45}
$$

where

$$
\Lambda_{2N-1} = \det\left[\operatorname{Re}\sigma_1^{(1)}, \operatorname{Re}\sigma_1^{(2)}, \operatorname{Im}\sigma_1^{(1)}, \operatorname{Im}\sigma_1^{(2)}, \cdots, \operatorname{Re}\sigma_N^{(1)}, \operatorname{Re}\sigma_N^{(2)}, \operatorname{Im}\sigma_N^{(1)}, \operatorname{Im}\sigma_N^{(2)}\right]^{\mathrm{T}},
$$

$$
\Lambda_{B_{2N-1}} = \det\left[\operatorname{Re}b_1^{(1)}, \operatorname{Re}b_1^{(2)}, \operatorname{Im}b_1^{(1)}, \operatorname{Im}b_1^{(2)}, \cdots, \operatorname{Re}b_N^{(1)}, \operatorname{Re}b_N^{(2)}, \operatorname{Im}b_N^{(1)}, \operatorname{Im}b_N^{(2)}\right]^{\mathrm{T}},
$$

$$\Lambda_{C_{2N-1}} = \det\left[\operatorname{Re} c_1^{(1)}, \operatorname{Re} c_1^{(2)}, \operatorname{Im} c_1^{(1)}, \operatorname{Im} c_1^{(2)}, \cdots, \operatorname{Re} c_N^{(1)}, \operatorname{Re} c_N^{(2)}, \operatorname{Im} c_N^{(1)}, \operatorname{Im} c_N^{(2)}\right]^{\mathrm{T}},$$

with $\sigma_j^{(l)}, b_j^{(l)}, c_j^{(l)}, (l = 1, 2)$ are given by (44).

For simplicity, we shall discuss complexiton solutions of system (1) with a special case $j = 1$, which we called 1-complexiton solutions of system (1).

$$\begin{aligned} \bar{u} &= u + 2i\frac{\Lambda_{B_1}}{\Lambda_1}, \\ \bar{v} &= v - 2i\frac{\Lambda_{C_1}}{\Lambda_1}, \end{aligned} \tag{46}$$

where

$$\Lambda_1 = \det\left[\operatorname{Re}\sigma_1^{(1)}, \operatorname{Re}\sigma_1^{(2)}, \operatorname{Im}\sigma_1^{(1)}, \operatorname{Im}\sigma_1^{(2)}\right]^{\mathrm{T}},$$

$$\Lambda_{B_1} = \det\left[\operatorname{Re} b_1^{(1)}, \operatorname{Re} b_1^{(2)}, \operatorname{Im} b_1^{(1)}, \operatorname{Im} b_1^{(2)}\right]^{\mathrm{T}},$$

$$\Lambda_{C_1} = \det\left[\operatorname{Re} c_1^{(1)}, \operatorname{Re} c_1^{(2)}, \operatorname{Im} c_1^{(1)}, \operatorname{Im} c_1^{(2)}\right]^{\mathrm{T}},$$

with

$$\sigma_1^{(l)} = \left(1, \delta_1^{(l)}, \lambda_1^{(l)}, \delta_1^{(l)}\lambda_1^{(l)}\right),$$

$$b_1^{(l)} = \left(1, \delta_1^{(l)}, \lambda_1^{(l)}, -\lambda_1^{(l)^2}\right),$$

$$c_1^{(l)} = \left(1, \delta_1^{(l)}, -\delta_1^{(l)}\lambda_1^{(l)^2}, \delta_1^{(l)}\lambda_1^{(l)}\right),$$

$(l = 1, 2), \delta_1^{(l)}$ are given by (42), and $\lambda_1^{(l)}$ are given by (40).

5. Conclusion and Suggestions

In this paper, an explicit *N*-fold Darboux transformation is constructed for the coupled mKdV equation, and through these transformations, the determinant forms of the multi-soliton and complexiton solutions of system (1) are obtained.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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