Applied Mathematics, 2025, 16(1), 143-168
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Finite Deformation, Finite Strain Nonlinear Micropolar NCCT for Thermoviscoelastic Solids with Rheology

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How to cite this paper: Surana, K.S. and Mathi, S.S.C. (2025) Finite Deformation, Finite Strain Nonlinear Micropolar NCCT for Thermoviscoelastic Solids with Rheology. *Applied Mathematics*, **16**, 143-168.
https://doi.org/10.4236/am.2025.161006

Received: December 11, 2024
Accepted: January 24, 2025
Published: January 27, 2025

Abstract

This paper presents a nonlinear micropolar nonclassical continuum theory (MPNCCT) for finite deformation, finite strain deformation physics of thermosviscoelastic solid medium with memory (polymeric micropolar solids) based on classical rotations ${}_c\boldsymbol{\Theta}$ and their rates. Contravariant second Piola-Kirchhoff stress and moment tensors, in conjunction with finite deformation measures derived by the authors in recent paper, are utilized in deriving the conservation and balance laws and the constitutive theories based on conjugate pairs in entropy inequality and the representation theorem. This nonlinear MPNCCT for TVES with rheology: 1) incorporates nonlinear ordered rate dissipation mechanism based on Green's strain rates up to order n; 2) also incorporates an additional ordered rate dissipation mechanism due to microconstituents, the viscosity of the medium and the rates of the symmetric part of the rotation gradient (of ${}_c\boldsymbol{\Theta}$) tensor up to order n, referred to as micropolar dissipation or micropolar viscous dissipation mechanism; 3) incorporates the primary mechanism of memory or rheology due to long chain molecules of the polymer and the viscosity of the medium by using the contravaraint second Piola-Kirchhoff stress tensor and its rates up to order m, resulting in a relaxation spectrum; 4) incorporates second mechanism of memory or rheology due to nonclassical physics, interaction of microconstituents with the viscous medium and long chain molecules by considering rates of the contravariant second Piola-Kirchhoff moment tensor up to order m, resulting in relaxation of second Piola-Kirchhoff moment tensor. This results in another relaxation spectrum for the second Piola-Kirchhoff moment tensor due to microconstituents, referred to as micropolar relaxation spectrum consisting of micropolar relaxation time constants of the material. This nonlinear MPNCCT for TVES with memory is thermodynamically and mathematically consistent, and the mathematical model consisting of conservation and balance laws and the constitutive theories has closure and naturally reduces to linear MPNCCT based on infinitesimal deformation assumption. BMM is the essential balance law for all MPNCCT and is used in the present work as well. In the absence of this balance law, a valid thermodynamically and mathematically consistent nonlinear MPNCCT is not possible. The nonlinear MPNCCT based on rotations (${}_c\boldsymbol{\Theta} + {}_\alpha\boldsymbol{\Theta}$) and ${}_\alpha\boldsymbol{\Theta}$ (ignoring ${}_c\boldsymbol{\Theta}$) is not considered due to the fact that even the linear MPNCCT based on these rotations is invalid and is thermodynamically and mathematically inconsistent MPNCCT.

Keywords

Nonclassical, Micropolar, Dissipation, Ordered Rate, Conservation and Balance Laws, Representation Theorem, Microviscous Dissipation, Microdissipation, Ordered Rate, Finite Deformation Theories, Finite Strain, Conservation and Balance Laws

1. Introduction

In a recent paper, the authors presented derivation of deformation measure valid for finite deformation, finite strain nonlinear 3M nonclassical continuum theories. Authors showed that the measures cannot be viewed as strain measures because the expression used in initiating their derivations clearly suggests against it. It is only after deriving the entropy inequality that the conjugate pairs in it confirm which measures are strain measures or can be made strain measures with minor modifications. It was shown that only the linear MPNCCT based on classical rotation ${}_c\boldsymbol{\Theta}$ is thermodynamically and mathematically consistent when BMM is used as a balance law in the mathematical model consisting of CBL, and the constitutive theories are derived using representation theorem. This mathematical model has closure. Authors showed all the other linear MPNCCT, such as those based on (${}_c\boldsymbol{\Theta} + {}_\alpha\boldsymbol{\Theta}$) or ${}_\alpha\boldsymbol{\Theta}$ (neglecting ${}_c\boldsymbol{\Theta}$), is not valid linear MPNCCT regardless of whether BMM is used as a balance law or not, and the mathematical models in these theories do not have closure.

In another recent paper, the author presented a finite deformation, finite strain nonlinear MPNCCT for compressible thermoelastic solid (TES) medium based on classical rotations ${}_c\boldsymbol{\Theta}$. BMM was used as an additional balance law. The contravariant second Piola-Kirchhoff stress and moment tensors, in conjunction with the nonlinear deformation measures and the conjugate pairs in the entropy inequality, were used to derive the final form of the conservation and balance laws and the constitutive theories. It was shown that the linear MPNCCT is a complete subset of this nonlinear MPNCCT. This nonlinear MPNCCT is thermodynamically and mathematically consistent, and the mathematical model has closure. The work was extended by the authors for finite deformation, finite strain nonlinear MPNCCT for compressible thermoviscoelastic solids without memory based on classical rotations ${}_c\boldsymbol{\Theta}$. In this work also, as in all MPNCCT presented by Surana *et al.* [1]-[23], BMM is an essential balance law; without this balance law, the resulting MPNCCT will be thermodynamically as well as mathematically inconsistent, and the mathematical model consisting of CBL and the constitutive theories will not have closure. The final form of CBL is derived using second Piola-Kirchhoff stress and moment tensors, and the constitutive theories are derived using conjugate pairs in entropy inequality, the representation theorem, and the nonlinear deformation measures derived. The nonlinear MPNCCT incorporates two nonlinear dissipation mechanisms: 1) The first one, the primary mechanism, is based on contravariant second Piola-Kirchhoff stress tensor and Green's strain rates up to order n, thus it is an ordered rate dissipation mechanism. 2) The second dissipation mechanism, the secondary dissipation mechanism, is due to microconstituents experiencing drag in the viscous medium; hence this mechanism is micropolar dissipation mechanism or micropolar viscous dissipation mechanism and is described by using contravariant second Piola-Kirchhoff moment tensor and rates of the symmetric part of rotation (${}_c\boldsymbol{\Theta}$) gradient tensor up to order n. Thus, this dissipation mechanism is also ordered rate dissipation mechanism. It was shown that corresponding linear MPNCCT for TVES is a complete subset of the nonlinear MPNCCT for TVES.

Surana *et al.* [1]-[23] have presented comprehensive literature review on linear MPNCCT. More recently in references [20] [21] the works of Eringen *et al* and others [24]-[43] have been discussed in detail for clear understanding of the validity of the published theories and methodologies used in them from the point of view of thermodynamic and mathematical consistency of the resulting MPNCCT and the closure of the mathematical model consisting of conservation and balance laws and the constitutive theories. It has been concluded in the works of Surana *et al.* [20] [21] that only linear MPNCCT based on classical rotations ${}_c\boldsymbol{\Theta}$ is thermodynamically and mathematically consistent when BMM is used as a balance law and when the constitutive theories are derived using entropy inequality, deformation measures and the representation theorem. Almost all published linear MPNCCT (to our knowledge) do not use BMM balance law, and the constitutive theories are rarely (if any) based on representation theorem; hence, these linear MPNCCT cannot be thermodynamically and mathematically consistent.

From references [39] [40] and the other published works of Eringen and Eringen *et al.* [24]-[38], we find that 3M nonlinear theories are included in the general form of the derivations of NCCT presented in them. However, there are many serious concerns regarding the approaches used and theories presented in these works: 1) We have demonstrated in numerous papers (for example [20] [21]) that a thermodynamically and mathematically consistent valid NCCT is not possible without using BMM as a balance law [13] [14] [44]. In references [25] [43] and other works of Eringen and Eringen *et al.*, the BMM balance law is never used as a balance law. 2) A nonsymmetric tensor cannot be used as a constitutive tensor as done in [39] [40] and others. This violates representation theorem [45]-[56]. A constitutive theory that cannot be supported by representation theorem is in violation of mathematical consistency. Thus, all constitutive theories [39] [40] and others using this approach are questionable constitutive theories. 3) Adding ${}_{\alpha}\boldsymbol{\Theta}$, rigid rotations of microconstituent to deformation gradient tensor that already contains classical rigid rotations ${}_{c}\boldsymbol{\Theta}$ and defining this as a strain measure that is work conjugate to nonsymmetric Cauchy stress tensor is not supported by any physics and mathematics. This approach only leads to erroneous constitutive theories. 4) The use of polynomials or potentials in the argument tensors of the constitutive tensor has no basis in any deformation physics. We point out that in deriving constitutive theories, the additive decompositions: $\boldsymbol{\sigma}^{(0)} = {}_{s}\boldsymbol{\sigma}^{(0)} + {}_{a}\boldsymbol{\sigma}^{(0)}$, ${}_{s}\boldsymbol{\sigma}^{(0)} = {}_{s}^{e}\boldsymbol{\sigma}^{(0)} + {}_{s}^{d}\boldsymbol{\sigma}^{(0)}$ are absolutely essential to address the deformation physics correctly. This is never done in any of the works (to our knowledge). Thus, our view is that a thermodynamically and mathematically consistent finite deformation, finite strain nonlinear MPNCCT for compressible TVES matter with rheology is not available in the published works.

Details of Scope of Work

In this paper, we present a nonlinear MPNCCT for finite deformation, finite strain deformation physics, addressing both volumetric and distortional deformation physics of compressible thermoviscoelastic solid matter with memory *i.e.*, polymeric solids. The NCCT presented in this paper consists of modified conservation and balance laws of CCM, additionally, it uses balance of moment of moments, a balance law shown to be essential in 3M NCCT [13] [14] [21] and the constitutive theories that are derived using entropy inequality and the representation theorem. Contravariant second Piola-Kirchhoff stress and moment tensors $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$, in conjunction with finite deformation conjugate strain measures in the entropy inequality that are supported by deformation measures derived for nonlinear 3M NCCT, are used to derive the conservation and balance laws. Balance of moment of moments balance law [13] [14] [44], shown to be an essential balance law, is also a part of the CBL contained in this paper. Since the details of the derivation of CBL have already been presented, only their final forms are summarized in this paper for the sake of brevity. We must consider the additive decomposition of $\boldsymbol{\sigma}^{[0]}$ to obtain suitable and valid constitutive tensors for describing volumetric and distortional deformation physics and to avoid the constitutive theory for ${}_{a}\boldsymbol{\sigma}^{[0]}$ as it is defined by BAM.

$$\boldsymbol{\sigma}^{[0]} = {}_{s}\boldsymbol{\sigma}^{[0]} + {}_{a}\boldsymbol{\sigma}^{[0]};\ {}_{s}\boldsymbol{\sigma}^{[0]} = {}_{s}^{e}\boldsymbol{\sigma}^{[0]} + {}_{s}^{d}\boldsymbol{\sigma}^{[0]} \tag{1}$$

The constitutive theory for equilibrium stress tensor ${}_{s}^{e}\boldsymbol{\sigma}^{[0]}$ describes volumetric deformation physics that remains the same as in references [57] [58]; hence, only the final form of the equations for ${}_{s}^{e}\boldsymbol{\sigma}^{[0]}$ is presented. Constitutive theories for ${}_{s}^{d}\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ address distortional physics as well as dissipation mechanism and rheology. The rate of work conjugate pair $\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]}$ in the entropy inequality suggests that elasticity must be due to $\boldsymbol{\varepsilon}_{[0]}$ and dissipation mechanism is due to $\dot{\boldsymbol{\varepsilon}}_{[0]} = \boldsymbol{\varepsilon}_{[1]}$ *i.e.*, Green's strain rate of order one. In the present work, we generalized the dissipation mechanism; hence we consider dissipation to be dependent on Green's strain rates of up to order n *i.e.*, $\boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \cdots, n$. Thus, at this stage:

$$ {}_s^d\boldsymbol{\sigma}^{[0]} = {}_s^d\boldsymbol{\sigma}^{[0]}\left(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]}, \theta\right); i = 1, 2, \cdots, n \tag{2} $$

is valid. Since the material has rheology, the constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ must be at least a first-order differential equation in time to describe rheology due to viscosity and long chain molecules. This requires that we must use ${}_s^d\boldsymbol{\sigma}^{[0]}$ and ${}_s^d\boldsymbol{\sigma}^{[1]}$, leading to ${}_s^d\boldsymbol{\sigma}^{[1]}$ as a constitutive tensor with ${}_s^d\boldsymbol{\sigma}^{[0]}$ as its argument tensor. Thus, now we have:

$$ {}_s^d\boldsymbol{\sigma}^{[1]} = {}_s^d\boldsymbol{\sigma}^{[1]}\left(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]}, {}_s^d\boldsymbol{\sigma}^{[0]}, \theta\right); i = 1, 2, \cdots, n \tag{3} $$

If we generalize and assume that rheology depends on stress rates of up to order m *i.e.*, ${}_s^d\boldsymbol{\sigma}^{[j]}$; $j = 1, 2, \cdots, m$, then ${}_s^d\boldsymbol{\sigma}^{[m]}$ is the constitutive tensor, and ${}_s^d\boldsymbol{\sigma}^{[j]}$; $j = 0, 1, \cdots, m-1$ are its argument tensors. Thus, we finally have:

$$ {}_s^d\boldsymbol{\sigma}^{[m]} = {}_s^d\boldsymbol{\sigma}^{[m]}\left(\boldsymbol{\varepsilon}_{[i]}, {}_s^d\boldsymbol{\sigma}^{[j]}, \theta\right); i = 0, 1, \cdots, n\,;\ j = 1, 2, \cdots, m-1 \tag{4} $$

We use (4) and representation theorem to derive constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[m]}$. The other conjugate pair in entropy inequality, $\boldsymbol{m}^{[0]}$ and ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$, suggests that $\boldsymbol{m}^{[0]}$ and ${}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}$ are work conjugate. Microconstituent rotations in a viscous medium result in secondary dissipation dependent on rotation rates of the microconstituents. If we generalize this to consider dissipation dependent on rotation rates up to order $\underset{\sim}{n}$, then we can write the following for the constitutive tensor $\boldsymbol{m}^{[0]}$.

$$ \boldsymbol{m}^{[0]} = \boldsymbol{m}^{[0]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}, {}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}, \theta\right); i = 1, 2, \cdots, n \tag{5} $$

The interaction of the microconstituents with long chain molecules in a viscous medium results in another mechanism of rheology. Upon cessation of an external stimulus, the presence of long chain molecules inhibits microconstituents from resuming their original state immediately. As long chain molecules relax, the microconstituents progressively approach their original undeformed position. This mechanism of rheology is referred to as micropolar rheology, with a corresponding micropolar relaxation time. This describes the relaxation phenomenon of the second Piola-Kirchhoff moment tensor. This mechanism requires that the constitutive theory for $\boldsymbol{m}^{[0]}$ be a differential equation in time, which necessitates that at the very minimum, $\boldsymbol{m}^{[0]}$ and $\boldsymbol{m}^{[1]}$ must be considered in deriving the constitutive theory for the moment tensor. We generalize and assume that micropolar rheology depends on rate of $\boldsymbol{m}^{[0]}$ up to order $\underset{\sim}{m}$ *i.e.*, $\boldsymbol{m}^{[k]}; k = 1, 2, \cdots, \underset{\sim}{m}$. Now, $\boldsymbol{m}^{[\underset{\sim}{m}]}$ can be considered constitutive tensor and $\boldsymbol{m}^{[k]}; k = 0, 1, \cdots, \underset{\sim}{m}-1$ as its argument tensor. Thus, finally we have:

$$ \boldsymbol{m}^{[\underset{\sim}{m}]} = \boldsymbol{m}^{[\underset{\sim}{m}]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}, \boldsymbol{m}^{[j]}, \theta\right); i = 0, 1, \cdots, \underset{\sim}{n}\,;\ j = 0, 1, \cdots, \underset{\sim}{m}-1 \tag{6} $$

We use (6) in conjunction with representation theorem to derive constitutive theory for $\boldsymbol{m}^{[\underset{\sim}{m}]}$.

The CBL and the constitutive theories for nonlinear MPNCCT expressed in terms of ${}^{d}_{s}\boldsymbol{\sigma}^{[j]};\ j=0,1,\cdots,m$, $\boldsymbol{m}^{[k]};\ k=0,1,\cdots,\underset{\sim}{m}$, $\boldsymbol{\varepsilon}_{[i]};\ i=0,1,\cdots,n$ and ${}^{c\Theta}\boldsymbol{\varepsilon}_{[j]};\ j=0,1,\cdots,\underset{\sim}{n}$ are a system of nonlinear partial differential equations in spatial coordinates and time that have closure. The linear MPNCCT for TVES with memory is a complete subset of the nonlinear MPNCCT for compressible TVES with memory presented in this paper.

2. Consideration of Various Measures

Following the notations used in references [20] [21] [57] [58] we have the following (see list of notations for description of variables)

$$\bar{\boldsymbol{P}}=\left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}}\,;\ \bar{\boldsymbol{P}}=\left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}} \tag{7}$$

$$\bar{\boldsymbol{M}}=\left(\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}}\,;\ \bar{\boldsymbol{M}}=\left(\bar{\boldsymbol{m}}_{(0)}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}} \tag{8}$$

Correspondence rule, contravariant first and second Piola-Kirchhoff stress and moment tensors are as follows (compressible matter).

$$\begin{aligned}
&\left\{d\bar{F}\right\}=\left\{dF\right\};\left\{d\bar{M}\right\}=\left\{dM\right\}\\
&\left[\sigma^{*}\right]^{\mathrm{T}}=|J|\left[\sigma^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1};\left[m^{*}\right]^{T}=|J|\left[m^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}\\
&\left\{d\bar{F}\right\}=\left[J\right]\left\{dF\right\};\left\{d\bar{M}\right\}=\left[J\right]\left\{dM\right\};\\
&\left[\sigma^{[0]}\right]^{T}=|J|\left[J\right]^{-1}\left[\sigma^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1};\left[m^{(0)}\right]^{\mathrm{T}}=|J|\left[J\right]^{-1}\left[m^{(0)}\right]^{\mathrm{T}}\left[J^{\mathrm{T}}\right]^{-1}
\end{aligned} \tag{9}$$

We remark that since $\boldsymbol{\sigma}^{(0)}$ is not symmetric, $\boldsymbol{\sigma}^{*}$ and $\boldsymbol{\sigma}^{[0]}$ are nonsymmetric as well. However, when BMM is used as a balance law, $\boldsymbol{m}^{(0)}$ is symmetric, hence $\boldsymbol{m}^{[0]}$ is symmetric but $\boldsymbol{m}^{*}$ is not symmetric.

Classical rotations ${}_{c}\boldsymbol{\Theta}$ and its gradients, deformation gradient tensor, etc. are given in the following:

$$\begin{aligned}
{}_{c}\boldsymbol{\Theta}&=\nabla\times\boldsymbol{u}=\boldsymbol{e}_i\times\boldsymbol{e}_j\frac{\partial u_i}{\partial x_j}=\epsilon_{ijk}\boldsymbol{e}_k\frac{\partial u_i}{\partial x_j}\\
&=\boldsymbol{e}_1\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)+\boldsymbol{e}_2\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+\boldsymbol{e}_3\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\\
&=\boldsymbol{e}_1\left({}_{c}\Theta_1\right)+\boldsymbol{e}_2\left({}_{c}\Theta_2\right)+\boldsymbol{e}_3\left({}_{c}\Theta_3\right)
\end{aligned} \tag{10}$$

$$\left[J\right]=\left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right]=\left[{}_{s}J\right]+\left[{}_{a}J\right] \tag{11}$$

$$\left[{}_{s}J\right]=\frac{1}{2}\left(\left[J\right]+\left[J\right]^{\mathrm{T}}\right);\left[{}_{a}J\right]=\frac{1}{2}\left(\left[J\right]-\left[J\right]^{\mathrm{T}}\right) \tag{12}$$

$$\left[{}^{c\Theta}J\right]=\frac{\partial\left\{{}_{c}\Theta\right\}}{\partial\{x\}}=\left[{}^{c\Theta}_{s}J\right]+\left[{}^{c\Theta}_{a}J\right] \tag{13}$$

$$\left[{}^{c\Theta}_{s}J\right]=\frac{1}{2}\left(\left[{}^{c\Theta}J\right]+\left[{}^{c\Theta}J\right]^{\mathrm{T}}\right);\left[{}^{c\Theta}_{a}J\right]=\frac{1}{2}\left(\left[{}^{c\Theta}J\right]-\left[{}^{c\Theta}J\right]^{\mathrm{T}}\right) \tag{14}$$

Equations (7)-(14) are basic measures, definitions and relations that are used in deriving the CBL and the constitutive theories.

3. Conservation and Balance Laws

Surana *et al.* have presented details of the derivation of conservation and balance laws for nonlinear MPNCCT for compressible solid matter using $\boldsymbol{\sigma}^*$ and $\boldsymbol{m}^*$ as stress and moment tensors. These CBL in their final form were expressed in terms of $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$. We begin with CBL (Equations (47)-(52)) expressed in terms of $\boldsymbol{\sigma}^*$ and $\boldsymbol{m}^*$, which include: Conservation of mass (CM), balance of linear momenta (BLM), balance of angular momenta (BAM), first law of thermodynamics (FLT) and the second law of thermodynamics (SLT).

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x},t) \tag{15}$$

$$\rho_0 \frac{Dv}{Dt} - \rho_0 \boldsymbol{F}^b - \nabla \cdot \boldsymbol{\sigma}^* = 0 \tag{16}$$

$$\nabla \cdot \boldsymbol{m}^* + \epsilon : \boldsymbol{\sigma}^* + \rho_0 \boldsymbol{m}^b = 0 \tag{17}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{18}$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} - {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b\right) = 0 \tag{19}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} - {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b\right) \le 0 \tag{20}$$

This mathematical model contains: $\boldsymbol{u}(3), \boldsymbol{\sigma}(9), \boldsymbol{m}(6), \boldsymbol{q}(3), \theta(1)$, a total of 22 dependent variables, but has only balance of linear momenta (3), balance of angular momenta (3), first law of thermodynamics (1), seven partial differential equations. Thus, additional 15 equations are needed for closure. These are provided by the constitutive theories for stress tensor (6), moment tensor (6) and heat vector (3).

4. Constitutive Theories

Conjugate pairs $\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta}, \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}}$ and $\boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}}$ in conjunction with axioms of constitutive theory [57] [58] suggest the choice of $\boldsymbol{q}, \boldsymbol{\sigma}^*$ and $\boldsymbol{m}^*$ as constitutive tensors with $\boldsymbol{g}, \boldsymbol{J}$ and ${}^{c\Theta}\boldsymbol{J}$ as their argument tensors in addition to temperature θ. $\boldsymbol{\sigma}^*, \boldsymbol{m}^*, \boldsymbol{J}$ and ${}^{c\Theta}\boldsymbol{J}$ are all nonsymmetric tensors and $\boldsymbol{J}$ is not objective. Thus, the following do not hold, *i.e.*:

$$\boldsymbol{\sigma}^* \ne \boldsymbol{\sigma}^*(\boldsymbol{J}, \theta) \tag{21}$$

$$\boldsymbol{m}^* \ne \boldsymbol{m}^*\left({}^{c\Theta}\boldsymbol{J}, \theta\right) \tag{22}$$

but

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{23}$$

is a valid choice. Intentionally, in the derivation of energy equation and entropy inequality, we have used $\boldsymbol{\sigma}^*$ and $\boldsymbol{m}^*$ for simplicity. At this stage, we transform these to $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$.

From BAM, we have:

$$\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b = -\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \tag{24}$$

Using (24), the last term in (20) can be written as:

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) = -{}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \right) \tag{25}$$

A simple calculation shows that

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \right) = {}_a\boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} = \boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \tag{26}$$

Therefore,

$${}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* + \rho_0 \boldsymbol{m}^b \right) = -\boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \tag{27}$$

Using (27), the entropy inequality (20) can be written as:

$$\rho_0 \left(\frac{D\theta}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} + \boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}} \le 0 \tag{28}$$

Consider $\boldsymbol{\sigma}^* : \dot{\boldsymbol{J}}$ term in (28)

$$\begin{aligned}
\boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} &= \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : \dot{\boldsymbol{J}}^{\mathrm{T}} = \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} \cdot \dot{\boldsymbol{J}} \\
&= \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : \left({}_s\dot{\boldsymbol{J}} + {}_a\dot{\boldsymbol{J}} \right) \\
&= \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : {}_s\dot{\boldsymbol{J}} + \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : {}_a\dot{\boldsymbol{J}} \\
&= \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : {}_s\dot{\boldsymbol{J}} + \boldsymbol{\sigma}^* : {}_a\dot{\boldsymbol{J}}
\end{aligned} \tag{29}$$

Consider $\left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : {}_s\dot{\boldsymbol{J}}$ term

$$\begin{aligned}
\left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} : {}_s\dot{\boldsymbol{J}} &= \left(\boldsymbol{\sigma}^* \right)^{\mathrm{T}} \cdot \frac{1}{2} \left(\bar{\boldsymbol{L}} \cdot \boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{L}}^{\mathrm{T}} \right) \\
&= \frac{1}{2} \boldsymbol{J} \cdot \left(\boldsymbol{\sigma}^{[0]} \right)^{\mathrm{T}} : \left(\left(\bar{\boldsymbol{D}} + \bar{\boldsymbol{W}} \right) \cdot \boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} \cdot \left(\bar{\boldsymbol{D}} - \bar{\boldsymbol{W}} \right) \right) \\
&= \frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} - {}_a\boldsymbol{\sigma}^{[0]} \right) : \left(\left(\bar{\boldsymbol{D}} + \bar{\boldsymbol{W}} \right) \cdot \boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} \cdot \left(\bar{\boldsymbol{D}} - \bar{\boldsymbol{W}} \right) \right) \\
&= \frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) : \bar{\boldsymbol{D}} \cdot \boldsymbol{J} + \frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) : \boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{D}} \\
&\quad + \frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) : \bar{\boldsymbol{W}} \cdot \boldsymbol{J} - \frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) : \boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{W}} \\
&\quad - \frac{1}{2} \boldsymbol{J} \cdot \left({}_a\boldsymbol{\sigma}^{[0]} \right) : \bar{\boldsymbol{D}} \cdot \boldsymbol{J} - \frac{1}{2} \boldsymbol{J} \cdot \left({}_a\boldsymbol{\sigma}^{[0]} \right) : \boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{D}} \\
&\quad - \frac{1}{2} \boldsymbol{J} \cdot \left({}_a\boldsymbol{\sigma}^{[0]} \right) : \bar{\boldsymbol{W}} \cdot \boldsymbol{J} + \frac{1}{2} \boldsymbol{J} \cdot \left({}_a\boldsymbol{\sigma}^{[0]} \right) : \boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{W}}
\end{aligned} \tag{30}$$

We consider each term in (30)

$$\frac{1}{2} \left(\boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) \right) : \left(\bar{\boldsymbol{D}} \cdot \boldsymbol{J} \right) = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{D}} \cdot \boldsymbol{J} \right) = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} \tag{31}$$

$$\frac{1}{2} \boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) : \boldsymbol{J} \cdot \bar{\boldsymbol{D}} = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{D}} \cdot \boldsymbol{J} \right) = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} \tag{32}$$

$$\frac{1}{2} \left(\boldsymbol{J} \cdot \left({}_s\boldsymbol{\sigma}^{[0]} \right) \right) : \bar{\boldsymbol{W}} \cdot \boldsymbol{J} = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} \cdot \boldsymbol{J} : \bar{\boldsymbol{W}} \cdot \boldsymbol{J} = \frac{1}{2}\, {}_s\boldsymbol{\sigma}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}} \cdot \bar{\boldsymbol{W}} \cdot \boldsymbol{J} \right) = 0 \tag{33}$$

$$\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_s\boldsymbol{\sigma}^{[0]}\right)\right):\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}=\frac{1}{2}\left({}_s\boldsymbol{\sigma}^{[0]}\right):\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=0 \tag{34}$$

$$-\frac{1}{2}\left(\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right)\right):\bar{\boldsymbol{D}}\cdot\boldsymbol{J}=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}:\bar{\boldsymbol{D}}\cdot\boldsymbol{J}=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=0 \tag{35}$$

$$-\frac{1}{2}\boldsymbol{J}\cdot{}_s\boldsymbol{\sigma}^{[0]}:\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}=\frac{1}{2}\left({}_s\boldsymbol{\sigma}^{[0]}\right):\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{D}}\cdot\boldsymbol{J}\right)=0 \tag{36}$$

$$-\frac{1}{2}\boldsymbol{J}\cdot\left({}_a\boldsymbol{\sigma}^{[0]}\right):\left(\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=-\frac{1}{2}\boldsymbol{J}\cdot{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\right)=-\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\cdot\boldsymbol{J}\right) \tag{37}$$

$$\begin{aligned}\frac{1}{2}\boldsymbol{J}\cdot{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\right)&=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}\cdot\boldsymbol{J}\right)^{\mathrm{T}}\\&=\frac{1}{2}\,{}_a\boldsymbol{\sigma}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\bar{\boldsymbol{W}}^{\mathrm{T}}\cdot\boldsymbol{J}\right)\end{aligned} \tag{38}$$

Substituting (31)-(38) in (30), we obtain

$$\left(\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}}:{}_s\dot{\boldsymbol{J}}={}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]} \tag{39}$$

Substituting from (39) into (29) we obtain

$$\boldsymbol{\sigma}^{*}:\dot{\boldsymbol{J}}={}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}+\boldsymbol{\sigma}^{*}:{}_a\dot{\boldsymbol{J}} \tag{40}$$

Substituting from (40) into (28), and noting that $\boldsymbol{\sigma}^{*}:{}_a\dot{\boldsymbol{J}}$ term cancel, we obtain the following form of entropy inequality:

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{m}^{*}:{}^{c\Theta}\dot{\boldsymbol{J}}\le 0 \tag{41}$$

We substitute

$$\left(\boldsymbol{m}^{*}\right)^{\mathrm{T}}=\boldsymbol{J}\cdot\left(\boldsymbol{m}^{[0]}\right)^{\mathrm{T}} \tag{42}$$

in the last term of (41).

$$\boldsymbol{m}^{*}:{}^{c\Theta}\dot{\boldsymbol{J}}=\boldsymbol{m}^{[0]}\cdot\boldsymbol{J}^{\mathrm{T}}:{}^{c\Theta}\dot{\boldsymbol{J}}=\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right) \tag{43}$$

Substituting (43) in (41), we obtain the final form of entropy inequality

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_s\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right)\le 0 \tag{44}$$

Thus, $\dot{\boldsymbol{\varepsilon}}_{[0]}$ and $\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)$ are rate of work conjugate to ${}_s\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ (symmetric). We should confirm the validity of the rate terms in (44) by comparing them with the nonlinear deformation measures derived.

The nonlinear deformation measures for MPNCCT are:

$$[J]^{\mathrm{T}}[J]-[I];\left[{}_aJ^{(\alpha)}\right];2[J]^{\mathrm{T}}\left[\frac{\partial_a^d\left[J^{(\alpha)}\right]}{\partial x}\right] \tag{45}$$

or in the notations of MPNCCT based on classical rotations ${}_c\boldsymbol{\Theta}$, we have

$$[J]^{\mathrm{T}}[J]-[I];\left[{}_aJ\right];2[J]^{\mathrm{T}}\left[{}^{c\Theta}J\right] \tag{46}$$

Based on (46) and rates used in (41), the choice of $\boldsymbol{q}, \boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ as constitutive tensors is justified. Last term in (41) requires further considerations.

$$\boldsymbol{m}^{[0]} : \left(\boldsymbol{J}^{\mathrm{T}} \cdot \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right) \right) = \boldsymbol{m}^{[0]} : \frac{1}{2} \left(\boldsymbol{J}^{\mathrm{T}} \cdot \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right) + \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right)^{\mathrm{T}} \cdot \boldsymbol{J} \right) \tag{47}$$

Thus, now we can write entropy inequality (41) as:

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_{s}\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \frac{1}{2} \left(\boldsymbol{J}^{\mathrm{T}} \cdot \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right) + \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right)^{\mathrm{T}} \cdot \boldsymbol{J} \right) \le 0 \tag{48}$$

Similar changes can also be made in the energy equation.

If we use

$$\left(\boldsymbol{\sigma}^{*} \right)^{\mathrm{T}} = \boldsymbol{J} \cdot \left(\boldsymbol{\sigma}^{[0]} \right)^{\mathrm{T}} \tag{49}$$

and define

$${}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}_{c}^{\Theta}\boldsymbol{\varepsilon}_{[1]} = \frac{1}{2} \left(\boldsymbol{J}^{\mathrm{T}} \cdot \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right) + \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right)^{\mathrm{T}} \cdot \boldsymbol{J} \right) \tag{50}$$

Thus, the conservation and the balance laws (15)-(20) can be written as:

$$\rho_0(\boldsymbol{x}) = |J| \rho_0(\boldsymbol{x}, t) \tag{51}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^{b} - \nabla \cdot \left(\boldsymbol{\sigma}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) = 0 \tag{52}$$

$$\nabla \cdot \left(\boldsymbol{m}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) + \epsilon : \left(\boldsymbol{\sigma}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) + \rho_0 \boldsymbol{m}^{b} = 0 \tag{53}$$

$$\epsilon_{ijk} m_{ij}^{(0)} = 0 \tag{54}$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - {}_{s}\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : {}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = 0 \tag{55}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_{s}\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \left({}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} \right) \le 0 \tag{56}$$

in which

$${}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}_{c}^{\Theta}\boldsymbol{\varepsilon}_{[1]} = \frac{1}{2} \left(\boldsymbol{J}^{\mathrm{T}} \cdot \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right) + \left({}_{c}^{\Theta}\dot{\boldsymbol{J}} \right)^{\mathrm{T}} \cdot \boldsymbol{J} \right) \tag{57}$$

Remarks

(1) The linear MPNCCT for infinitesimal deformation is a complete subset of the nonlinear MPNCCT described by CBL (51)-(57). In this case:

$$\begin{gathered} \boldsymbol{J} \approx \boldsymbol{I} \,;\, \boldsymbol{\sigma}^{[0]} = \boldsymbol{\sigma}^{(0)};\, \boldsymbol{m}^{[0]} = \boldsymbol{m}^{(0)} \\ \dot{\boldsymbol{\varepsilon}}_{[0]} = \dot{\boldsymbol{\varepsilon}} \,;\, {}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = {}_{c}^{\Theta}\dot{\boldsymbol{\varepsilon}} \end{gathered} \tag{58}$$

in which $\boldsymbol{\varepsilon}$ is the linear strain measure and ${}_{c}^{\Theta}\boldsymbol{J}$ is the symmetric part of the classical rotation gradient tensor. Thus, (51)-(57) reduce to linear MPNCCT for infinitesimal deformation.

(2) When ${}_{c}\boldsymbol{\Theta}$ is not considered, the CBL (51)-(57) reduce to finite deformation, finite strain CCM physics.

(3) When ${}_{c}\boldsymbol{\Theta}$ is a free field (*i.e.*, ${}_{c}\boldsymbol{\Theta}$ is not described) and when the deformation is infinitesimal, then $[J] = [I]$ and $|J| = 1$, and the CBL presented here

automatically reduce to CBL for small deformation, small strain infinitesimal physics.

(4) All conjugate pairs in the entropy inequality define constitutive tensors and their argument tensors that are supported by the representation theorem, hence would yield thermodynamically and mathematically consistent constitutive theories.

Additive decomposition of ${}_s\boldsymbol{\sigma}^{[0]}$ into ${}_s^e\boldsymbol{\sigma}^{[0]}$ equilibrium and ${}_s^d\boldsymbol{\sigma}^{[0]}$, deviatoric tensors is necessary to address volumetric deformation (constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$) distortional deformation (constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$) physics that are mutually exclusive.

$$ {}_s\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]} + {}_s^d\boldsymbol{\sigma}^{[0]} \tag{59} $$

Substituting (59) into entropy inequality (56)

$$ \rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_s^e\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-{}_s^d\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{m}^{[0]}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)\le 0 \tag{60} $$

Based on (60), choice of $\boldsymbol{q}$, ${}_s^e\boldsymbol{\sigma}^{[0]}$, ${}_s^d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ as constitutive tensor is supported by the axioms of constitutive theory [31] [32].

4.1. Constitutive Theory for Equilibrium Stress ${}_s^e\boldsymbol{\sigma}^{[0]}$

In our recent papers and also references [57] [58], details of constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ have been presented. We summarize it in this paper and give the final form of the constitutive equations. In compressible solids, density is deterministic in Lagrangian description from conservation of mass once J is known. Hence, density $\rho(\boldsymbol{x},t)$ cannot be a dependent variable in the mathematical model. Thus, in compressible solids, equation of state (a function of density and temperature) is a consequence of density and temperature but is not needed to define density. For deriving constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$, we must begin with CBL in Eulerian description for compressible matter in which density is a dependent variable. Using entropy inequality in Eulerian description and following references [57] [58], we can derive constitutive theory for equilibrium contravariant Cauchy stress tensor.

$$ {}_e\boldsymbol{\sigma}^{(0)} = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} = \bar{p}\left(\bar{\rho},\bar{\theta}\right)\boldsymbol{\delta}\,;\ \bar{p}\left(\bar{\rho},\bar{\theta}\right) = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\,;\ \text{Compressible} \tag{61} $$

$$ {}_e\boldsymbol{\sigma}^{(0)} = \bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta}\,;\ \text{Incompressible} \tag{62} $$

Using (61) and (62), we can obtain constitutive theories for ${}_s^e\boldsymbol{\sigma}^{[0]}$

$$ {}_s^e\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|\left(\boldsymbol{J}^{-1}\right)\cdot p(\rho,\theta)\cdot\left(\boldsymbol{J}^{-1}\right)^{\mathrm{T}} = |\boldsymbol{J}|\,p(\rho,\theta)\left(\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}^{-1}\right)(\text{compressible}) \tag{63} $$

$$ {}_s^e\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|\left(\boldsymbol{J}^{-1}\right)\cdot p(\theta)\left(\boldsymbol{J}^{-1}\right)^{\mathrm{T}} = |\boldsymbol{J}|\,p(\theta)\left(\boldsymbol{J}^{\mathrm{T}}\cdot\boldsymbol{J}\right)^{-1}\ (\text{incompressible}) \tag{64} $$

and the reduced form of entropy inequality in Lagrangian description is given by:

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_s^d\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{m}^{(0)}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)\leq 0 \tag{65}$$

Constitutive theory for ${}_s^d\boldsymbol{\sigma}$

The constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ must address: (i) distortional deformation physics, (ii) dissipation mechanism and (iii) rheology.

(1) From the rate of work conjugate pairs ${}_s^d\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}$ in the reduced form of the entropy inequality, we immediately conclude that ${}_s^d\boldsymbol{\sigma}^{[0]}:\boldsymbol{\varepsilon}^{[0]}$ is the work conjugate pair. Thus, for elasticity (distortional deformation physics), $\boldsymbol{\varepsilon}_{[0]}$ must be an argument tensor of ${}_s^d\boldsymbol{\sigma}^{[0]}$.

(2) An ordered rate dissipation mechanism dependent upon strain rates requires that $\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\cdots,n$ of up to order n to be argument tensors of ${}_s^d\boldsymbol{\sigma}^{[0]}$.

(3) Rheology, hence existence of relaxations or memory modulus requires that the constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[0]}$ be a differential in time, necessitating use of ${}_s^d\boldsymbol{\sigma}^{[0]}$ as well as ${}_s^d\boldsymbol{\sigma}^{[1]}$. Generalizing this concept and assuming that rheology depends upon rates of ${}_s^d\boldsymbol{\sigma}^{[0]}$ up to order m *i.e.*, ${}_s^d\boldsymbol{\sigma}^{[j]};\ j=0,1,\cdots,m$ must be considered in the constitutive theory. Choice of ${}_s^d\boldsymbol{\sigma}^{[m]}$ as constitutive tensor and ${}_s^d\boldsymbol{\sigma}^{[j]};\ j=0,1,\cdots,m-1$ as argument tensors of ${}_s^d\boldsymbol{\sigma}^{[m]}$ are valid. Now, we can write (including θ as an argument tensor ${}_s^d\boldsymbol{\sigma}^{[m]}$):

$${}_s^d\boldsymbol{\sigma}^{[m]}={}_s^d\boldsymbol{\sigma}^{[m]}\left(\boldsymbol{\varepsilon}_{[0]},\boldsymbol{\varepsilon}_{[i]},{}_s^d\boldsymbol{\sigma}^{[0]},{}_s^d\boldsymbol{\sigma}^{[j]},\theta\right);\ i=1,2,\cdots,n;\ j=1,2,\cdots,m-1 \tag{66}$$

The constitutive tensor and its argument tensors are all symmetric tensors of rank two except θ, a tensor of rank zero.

Now, we can derive constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[m]}$ using (66) and representation theorem [45]-[56]. Let ${}^\sigma\underset{\sim}{\boldsymbol{G}}^i;\ i=1,2,\cdots,{}^\sigma N$ be the combined generators of the argument tensors of ${}_s^d\boldsymbol{\sigma}^{[0]}$ in (66) that are symmetric tensors of rank two. Then, $\boldsymbol{I},{}^\sigma\underset{\sim}{\boldsymbol{G}}^i;\ i=1,2,\cdots,{}^\sigma N$ constitute the basis of the space of tensor ${}_s^d\boldsymbol{\sigma}^{[m]}$, referred to as integrity. Thus, we can write the following for ${}_s^d\boldsymbol{\sigma}^{[m]}$ using coefficients ${}^\sigma\alpha^i;\ i=0,1,\cdots,{}^\sigma N$.

$${}_s^d\boldsymbol{\sigma}^{[m]}={}^\sigma\alpha^0\boldsymbol{I}+\sum_{i=1}^{{}^\sigma N}{}^\sigma\alpha^i\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right);\ {}^\sigma\underset{\sim}{\alpha}^i={}^\sigma\underset{\sim}{\alpha}^i\left({}^\sigma\underset{\sim}{I}^j,\theta\right) \tag{67}$$

in which

$${}^\sigma\underset{\sim}{\alpha}^i={}^\sigma\underset{\sim}{\alpha}^i\left({}^\sigma\underset{\sim}{I}^j,\theta\right);\ j=1,2,\cdots,{}^\sigma M;\ i=1,2,\cdots,{}^\sigma N \tag{68}$$

The material coefficients in (68) are determined by expanding ${}^\sigma\underset{\sim}{\alpha}^i;\ i=0,1,\cdots,{}^\sigma N$ in Taylor series in the combined invariants ${}^\sigma\underset{\sim}{I}^j;\ j=1,2,\cdots,{}^\sigma M$ and the temperature θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^\sigma\underset{\sim}{I}^j;\ j=1,2,\cdots,{}^\sigma M$ and temperature θ (for simplicity of the resulting constitutive theory).

$${}^\sigma\underset{\sim}{\alpha}^i={}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}}+\sum_{j=1}^{{}^\sigma M}\frac{\partial^\sigma\underset{\sim}{\alpha}^i}{\partial^\sigma\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}}\left({}^\sigma\underset{\sim}{I}^j-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right)+\frac{\partial^\sigma\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right);\ i=0,1,\cdots,{}^\sigma N \tag{69}$$

Substituting for ${}^\sigma\underset{\sim}{\alpha}^0$ and ${}^\sigma\underset{\sim}{\alpha}^i;\ i=1,2,\cdots,{}^\sigma N$ from (69) into (67)

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[m]} = & \left({}^\sigma\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j - {}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) + \frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) \right) \boldsymbol{I} \\
& + \sum_{i=1}^{{}^\sigma N} \left({}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial^\sigma \underset{\sim}{\alpha}^i}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j - {}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) + \frac{\partial^\sigma \underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) \right) \left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i \right)
\end{aligned} \tag{70}
$$

Collecting coefficients of $\boldsymbol{I}$, ${}^\sigma\underset{\sim}{I}^j\boldsymbol{I}$, ${}^\sigma\underset{\sim}{\boldsymbol{G}}^i$, ${}^\sigma\underset{\sim}{I}^j\ {}^\sigma\underset{\sim}{\boldsymbol{G}}^i$, $\left(\theta-\theta|_{\underline{\Omega}}\right){}^\sigma\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}$ in (70)

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[m]} = & \left({}^\sigma\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) \right) \boldsymbol{I} + \sum_{j=1}^{{}^\sigma M} \frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} {}^\sigma\underset{\sim}{I}^j\boldsymbol{I} \\
& + \sum_{i=1}^{{}^\sigma N} \left({}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial^\sigma \underset{\sim}{\alpha}^i}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) \right) {}^\sigma\underset{\sim}{\boldsymbol{G}}^i \\
& + \sum_{i=1}^{{}^\sigma N}\sum_{j=1}^{{}^\sigma M} \left(\frac{\partial^\sigma \underset{\sim}{\alpha}^i}{\partial^\sigma \underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right) {}^\sigma\underset{\sim}{\boldsymbol{G}}^i \right) \\
& + \sum_{i=1}^{{}^\sigma N} \frac{\partial^\sigma \underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) {}^\sigma\underset{\sim}{\boldsymbol{G}}^i + \frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) \boldsymbol{I}
\end{aligned} \tag{71}
$$

If we define

$$
\begin{aligned}
& \boldsymbol{\sigma}^0\Big|_{\underline{\Omega}} = \left({}^\sigma\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^0\right)}{\partial\left(-{}^\sigma\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \right) \left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right); \quad {}^\sigma\underset{\sim}{a}_j = \frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^0\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \\
& {}^\sigma\underset{\sim}{b}_i = {}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^\sigma M} \frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^i\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}} \right); \quad {}^\sigma\underset{\sim}{c}_{ij} = \frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^i\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \\
& {}^\sigma\underset{\sim}{d}_i = -\frac{\partial^\sigma \alpha^i}{\partial\theta}\Bigg|_{\underline{\Omega}}; \quad {}^\sigma\alpha_{tm}\Big|_{\underline{\Omega}} = -\frac{\partial^\sigma \underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}
\end{aligned} \tag{72}
$$

then, using (72) in (71), we can write (71) in a more compact form.

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[m]} = & \ \sigma_0\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{i=1}^{{}^\sigma N} {}^\sigma\underset{\sim}{a}_i \left({}^\sigma\underset{\sim}{I}^j \right) \boldsymbol{I} + \sum_{j=1}^{{}^\sigma M} {}^\sigma\underset{\sim}{b}_j \left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i \right) + \sum_{j=1}^{{}^\sigma M}\sum_{i=1}^{{}^\sigma N} {}^\sigma\underset{\sim}{c}_{ij} \left({}^\sigma\underset{\sim}{I}^j \right)\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i \right) \\
& - \sum_{j=1}^{{}^\sigma N} {}^\sigma\underset{\sim}{d}^i \left(\theta - \theta|_{\underline{\Omega}} \right)\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i \right) - \left(\alpha_{tm}\right)_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right)\boldsymbol{I}
\end{aligned} \tag{73}
$$

Equations (72) defines material coefficients that can be functions of ${}^\sigma\underset{\sim}{I}^j;\ j=1,2,\cdots,{}^\sigma M$ and temperature θ in a known configuration $\underline{\Omega}$. This constitutive theory (73) is based on integrity *i.e.*, complete basis of the space of ${}_s^d\boldsymbol{\sigma}^{[m]}$ and has $\left(2\left({}^\sigma N\right)+\left({}^\sigma N\right)\left({}^\sigma M\right)+{}^\sigma M+1\right)$ material coefficients. Various desired linear or nonlinear constitutive theories can be derived using (73) by retaining desired generators and the invariants. The most simplified yet more general constitutive theory for ${}_s^d\boldsymbol{\sigma}^{[m]}$ is one in which ${}_s^d\boldsymbol{\sigma}^{[m]}$ is a linear function of the components of the argument tensors in (66). This constitutive theory will contain tensors $\boldsymbol{\varepsilon}_{[0]};\boldsymbol{\varepsilon}_{[1]};\ i=1,2,\cdots,n$; ${}_s^d\boldsymbol{\sigma}^{[m]};\ {}_s^d\boldsymbol{\sigma}^{[j]};\ j=1,2,\cdots,m-1$; $\operatorname{tr}\left(\boldsymbol{\varepsilon}_{[0]}\right);\ \operatorname{tr}\left(\boldsymbol{\varepsilon}_{[i]}\right);\ i=1,2,\cdots,n$; $\operatorname{tr}\left({}_s^d\boldsymbol{\sigma}^{[0]}\right);\ \operatorname{tr}\left({}_s^d\boldsymbol{\sigma}^{[j]}\right);\ j=1,2,\cdots,m$ and

$\left(\theta-\theta|_{\underline{\Omega}}\right)$.

If we define new material coefficients, then (73) can be reduced to ($\boldsymbol{\varepsilon}_{[0]}=\boldsymbol{\varepsilon}$)

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{[m]}+\sum_{k=1}^{m}\lambda_k\left({}_s^d\boldsymbol{\sigma}^{[k]}\right)=\sigma^0\big|_{\underline{\Omega}}\boldsymbol{I}+2\mu\boldsymbol{\varepsilon}+\lambda\operatorname{tr}\boldsymbol{\varepsilon}I+\sum_{i=1}^{n}2\eta_i\boldsymbol{\varepsilon}_{[i]}+\sum_{i=1}^{n}\kappa_i\left(\operatorname{tr}\boldsymbol{\varepsilon}_{[i]}\right)\boldsymbol{I}\\
+\sum_{k=0}^{m-1}\alpha_i\operatorname{tr}\left({}_s^d\boldsymbol{\sigma}^{[k]}\right)\boldsymbol{I}-\alpha_{tm}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned}
\tag{74}
$$

in which ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1=\boldsymbol{\varepsilon}_{[0]}$; ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i+1}=\boldsymbol{\varepsilon}_{[i]}$; $i=1,2,\cdots,n$. μ and λ are similar to Lame's coefficients in linear elasticity. η_i and κ_i are first and second viscosities corresponding to strain rates $\boldsymbol{\varepsilon}_{[i]}$; $i=1,2,\cdots,n$. α_i and α_{tm} are thermal coefficients. λ_k; $k=1,2,\cdots,m$ is a spectrum of relaxation times associated with stress rates ${}_s^d\boldsymbol{\sigma}^{[k]}$; $k=1,2,\cdots,m$ and β_k are material coefficients associated with $\operatorname{tr}\left({}_s^d\boldsymbol{\sigma}^{[k]}\right)$; $k=0,1,\cdots,m$. Thus, this constitutive theory has ordered rate mechanisms of dissipation as well as relaxation or rheology.

4.2. Constitutive Theory for $\boldsymbol{m}^{[0]}$

In this section, we present details of the constitutive theory for Contravariant second Piola Kirchhoff moment tensor. Consider the rate of work conjugate pair $\boldsymbol{m}^{[0]}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)$ in the entropy inequality. We simplify ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$:

$$
\begin{aligned}
{}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}&=\frac{1}{2}\left(\boldsymbol{J}^{\mathrm{T}}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)+\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{\mathrm{T}}\cdot\boldsymbol{J}\right)\\
&=\frac{1}{2}\left(\left({}_s\boldsymbol{J}+{}_a\boldsymbol{J}^{\mathrm{T}}\right)\cdot\left({}_s^{c\Theta}\dot{\boldsymbol{J}}+{}_a^{c\Theta}\dot{\boldsymbol{J}}\right)+\left({}_s^{c\Theta}\dot{\boldsymbol{J}}-{}_a^{c\Theta}\dot{\boldsymbol{J}}\right)\cdot\left({}_s\boldsymbol{J}+{}_a\boldsymbol{J}\right)\right)\\
&=\frac{1}{2}\left({}_s\boldsymbol{J}\cdot\left({}_s^{c\Theta}\dot{\boldsymbol{J}}\right)+\left({}_s^{c\Theta}\dot{\boldsymbol{J}}\right)\cdot{}_s\boldsymbol{J}-\left({}_a\boldsymbol{J}\cdot\left({}_a^{c\Theta}\dot{\boldsymbol{J}}\right)+{}_a^{c\Theta}\dot{\boldsymbol{J}}\cdot{}_a\boldsymbol{J}\right)\right)\\
&={}_s\boldsymbol{J}\cdot{}_s^{c\Theta}\dot{\boldsymbol{J}}-{}_a\boldsymbol{J}\cdot{}_a^{c\Theta}\dot{\boldsymbol{J}}
\end{aligned}
\tag{75}
$$

Thus, we can write

$$
{}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}={}^{c\Theta}\boldsymbol{\varepsilon}_{[1]}={}_s\boldsymbol{J}\cdot\left({}_s^{\Theta}\boldsymbol{J}_{[1]}\right)-{}_a\boldsymbol{J}\cdot\left({}_a^{\Theta}\boldsymbol{J}_{[1]}\right)
\tag{76}
$$

Generalizing (76) to include rates of ${}_s^{\Theta}\boldsymbol{J}$ and ${}_a^{\Theta}\boldsymbol{J}$ up to orders n, we can write:

$$
{}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}={}_s\boldsymbol{J}\cdot\left({}_s^{\Theta}\boldsymbol{J}_{[i]}\right)-{}_a\boldsymbol{J}\cdot\left({}_a^{\Theta}\boldsymbol{J}_{[i]}\right);\ i=1,2,\cdots,\underset{\sim}{n}
\tag{77}
$$

$$
\text{and}\quad {}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}={}_s\boldsymbol{J}\cdot{}_s^{c\Theta}\boldsymbol{J}-{}_a\boldsymbol{J}\cdot{}_a^{c\Theta}\boldsymbol{J}
\tag{78}
$$

In deriving the constitutive theory for second Piola-Kirchhoff moment tensor, we consider:

(1) Based on conjugate pair $\boldsymbol{m}^{[0]}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)$, ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$ can be argument tensor of the constitutive tensor $\boldsymbol{m}^{[0]}$.

(2) We consider dissipation mechanism between the microconstituent and the viscous medium surrounding it. Rotation of the microconstituent must overcome the viscous drag due to the surrounding medium. These viscous drag forces are considered to be proportional to the rotation rates up to order $\underset{\sim}{n}$, thus the rates

${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[k]};\ k=1,2,\cdots,\underset{\sim}{n}$ are argument tensors of the moment constitutive tensor. This viscous mechanism is micropolar dissipation or micropolar viscous dissipation mechanism.

(3) Since the microconstituents are present in a viscous medium with long chain molecules, upon cessation of external stimulus, if the relaxation process of long chain molecules interferes with the microconstituents, then microconstituents do not immediately return to their original state, but require some time in doing so. This introduces relaxation phenomenon for the moment tensor that may be dependent on the rates of the moment tensor up to order $\underset{\sim}{m}$. This requires that we consider $\boldsymbol{m}^{[0]}$ as well as its rates up to order $\underset{\sim}{m}$ *i.e.*, $\boldsymbol{m}^{[k]};\ k=1,2,\cdots,\underset{\sim}{m}$. Now, instead of $\boldsymbol{m}^{[0]}$, we must choose $\boldsymbol{m}^{[\underset{\sim}{m}]}$ as constitutive tensor with $\boldsymbol{m}^{[k]};\ k=0,1,\cdots,\underset{\sim}{m}-1$ as its argument tensors.

For the constitutive tensor $\boldsymbol{m}^{[m]}$, its argument tensors are now defined and we can write:

$$\boldsymbol{m}^{[\underset{\sim}{m}]}=\boldsymbol{m}^{[\underset{\sim}{m}]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]},{}^{c\Theta}\boldsymbol{\varepsilon}_{[i]},\boldsymbol{m}^{[0]},\boldsymbol{m}^{[j]},\theta\right);\ i=1,2,\cdots,\underset{\sim}{n};\ j=1,2,\cdots,\underset{\sim}{m}-1 \tag{79}$$

Equation (79) only contains symmetric tensor of rank two except for θ, a tensor of rank zero. We can now use (79) and representation theorem to derive constitutive theory for $\boldsymbol{m}^{[\underset{\sim}{m}]}$. Let ${}^{m}\underset{\sim}{\boldsymbol{G}}^{i};\ i=1,2,\cdots,{}^{m}\underset{\sim}{N}$ be the combined generators of the argument tensors of $\boldsymbol{m}^{[\underset{\sim}{m}]}$ in (79) that are symmetric tensors of rank two, then $\boldsymbol{I},\ {}^{m}\underset{\sim}{\boldsymbol{G}}^{i};\ i=1,2,\cdots,{}^{m}\underset{\sim}{N}$ constitute integrity or complete basis of the space of tensor $\boldsymbol{m}^{[\underset{\sim}{m}]}$, hence we can express $\boldsymbol{m}^{[\underset{\sim}{m}]}$ as a linear combination of the basis using coefficients ${}^{m}\underset{\sim}{\alpha}^{i};\ i=0,1,\cdots,{}^{m}\underset{\sim}{N}$ in the current configuration.

$$\boldsymbol{m}^{[\underset{\sim}{m}]}={}^{m}\underset{\sim}{\alpha}^{0}\boldsymbol{I}+\sum_{i=1}^{{}^{m}N}{}^{m}\underset{\sim}{\alpha}^{i}\left({}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right) \tag{80}$$

in which

$${}^{m}\underset{\sim}{\alpha}^{i}={}^{m}\underset{\sim}{\alpha}^{i}\left({}^{m}\underset{\sim}{I}^{j},\theta\right);\ j=1,2,\cdots,{}^{m}\underset{\sim}{M};\ i=0,1,\cdots,{}^{m}\underset{\sim}{N} \tag{81}$$

where ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,{}^{m}\underset{\sim}{M}$ are the combined invariants of the argument tensors of $\boldsymbol{m}^{[\underset{\sim}{m}]}$ in (80). Material coefficients for the constitutive theory (80) are determined by considering Taylor series expansion of ${}^{m}\underset{\sim}{\alpha}^{i};\ i=0,1,\cdots,{}^{m}\underset{\sim}{N}$ in ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,{}^{m}\underset{\sim}{M}$ and θ about a known configuration $\underset{\sim}{\Omega}$ (axiom of smooth neighborhood) and retaining only up to linear terms in ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,{}^{m}\underset{\sim}{M}$ and θ (for simplicity).

$${}^{m}\underset{\sim}{\alpha}^{i}={}^{m}\underset{\sim}{\alpha}^{i}\Big|_{\underset{\sim}{\Omega}}+\sum_{j=1}^{{}^{m}M}\frac{\partial^{m}\underset{\sim}{\alpha}^{i}}{\partial^{m}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}}\left({}^{m}\underset{\sim}{I}^{j}-{}^{m}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right)+\frac{\partial^{m}\underset{\sim}{\alpha}^{i}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right);\ i=0,1,\cdots,{}^{m}\underset{\sim}{N} \tag{82}$$

Substituting ${}^{m}\underset{\sim}{\alpha}^{0}$ and ${}^{m}\underset{\sim}{\alpha}^{i};\ i=1,2,\cdots,{}^{m}\underset{\sim}{N}$ from (82) in (80):

$$\begin{aligned}\boldsymbol{m}^{[\underset{\sim}{m}]}=&\left({}^{m}\underset{\sim}{\alpha}^{0}\Big|_{\underset{\sim}{\Omega}}+\sum_{j=1}^{{}^{m}M}\frac{\partial^{m}\underset{\sim}{\alpha}^{0}}{\partial^{m}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}}\left({}^{m}\underset{\sim}{I}^{j}-{}^{m}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right)+\frac{\partial^{m}\underset{\sim}{\alpha}^{0}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\right)\boldsymbol{I}\\&+\left({}^{m}\underset{\sim}{\alpha}^{i}\Big|_{\underset{\sim}{\Omega}}+\sum_{j=1}^{{}^{m}M}\frac{\partial^{m}\underset{\sim}{\alpha}^{i}}{\partial^{m}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}}\left({}^{m}\underset{\sim}{I}^{j}-{}^{m}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right)+\frac{\partial^{m}\underset{\sim}{\alpha}^{i}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\right){}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\end{aligned} \tag{83}$$

Collecting coefficients (defined in $\underline{\Omega}$) of $\boldsymbol{I}$, ${}^m\underset{\sim}{I}^j\boldsymbol{I}$, ${}^m\underset{\sim}{\boldsymbol{G}}^i$, ${}^m\boldsymbol{I}^j\ {}^m\underset{\sim}{\boldsymbol{G}}^i$, $\left(\theta-\theta|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}$ in (83), we can write (83) as follows:

$$\begin{aligned}\boldsymbol{m}^{[\underset{\sim}{m}]} = &\left({}^m\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM} \frac{\partial\left({}^m\underset{\sim}{\alpha}^0\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left(-{}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right)\right)\boldsymbol{I} + \sum_{j=1}^{{}^mM}\frac{\partial\left({}^m\underset{\sim}{\alpha}^0\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\ {}^m\underset{\sim}{I}^j\boldsymbol{I}\\ &+\sum_{i=1}^{{}^mN}\left({}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\right)\Big|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i\\ &+\sum_{i=1}^{{}^mN}\sum_{j=1}^{{}^mM}\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\left({}^m\underset{\sim}{I}^j\right)\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right)\\ &+\sum_{i=1}^{{}^mN}\frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i + \frac{\partial^m\underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{84}$$

If we define

$$\begin{aligned}m^0\Big|_{\underline{\Omega}} &= {}^m\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right)\\ {}^m\underset{\sim}{a}_j &= \frac{\partial^m\underset{\sim}{\alpha}^0}{\partial^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}}\\ {}^m\underset{\sim}{b}_i &= {}^m\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{{}^mM}\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial^m I^j}\Bigg|_{\underline{\Omega}}\left(-{}^m\underset{\sim}{I}^j\right)\Big|_{\underline{\Omega}}\\ {}^m\underset{\sim}{c}_{ij} &= \frac{\partial\left({}^m\underset{\sim}{\alpha}^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}}\\ {}^m\underset{\sim}{d}_i &= -\frac{\partial^m\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}}\ ;\ {}^m\alpha_{tm} = -\frac{\partial^m\underset{\sim}{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}\end{aligned} \tag{85}$$

then, using (85) in (84), we can write (84) in more compact form.

$$\begin{aligned}\boldsymbol{m}^{[\underset{\sim}{m}]} = m^0\Big|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{{}^mM}{}^m\underset{\sim}{a}_j\ {}^m\underset{\sim}{I}^j\boldsymbol{I} + \sum_{i=1}^{{}^mN}{}^m\underset{\sim}{b}_i\ {}^m\underset{\sim}{\boldsymbol{G}}^i + \sum_{i=1}^{{}^mN}\sum_{j=1}^{{}^mM}{}^m\underset{\sim}{c}_{ij}\ {}^m\underset{\sim}{I}^j\ {}^m\underset{\sim}{\boldsymbol{G}}^i\\ -\sum_{i=1}^{{}^mN}{}^m\underset{\sim}{d}_i\left(\theta-\theta|_{\underline{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i - {}^m\alpha_{tm}\left(\theta-\theta|_{\underline{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{86}$$

In this constitutive theory (86) for $\boldsymbol{m}^{[\underset{\sim}{m}]}$, the material coefficients are defined by (85). The material coefficients can be functions of $\rho|_{\underline{\Omega}}, \theta|_{\underline{\Omega}}$ and ${}^m\underset{\sim}{I}^j\Big|_{\underline{\Omega}}$; $j=1,2,\cdots,{}^m\underset{\sim}{M}$. This constitutive theory requires $\left(2\left({}^m\underset{\sim}{N}\right)+\left({}^m\underset{\sim}{M}\right)\left({}^m\underset{\sim}{N}\right)+{}^m\underset{\sim}{M}+1\right)$ material coefficients. Simplified forms of (86) can be obtained by retaining desired generators and invariants in (86). A simplified, yet general constitutive theory resulting from (86) would be one in which $\boldsymbol{m}^{[\underset{\sim}{m}]}$ is a linear function of the components of the argument tensors. The constitutive theory will contain:

$$
\begin{gathered}
{}^{c\Theta}\boldsymbol{\varepsilon}_{[0]};\ {}^{c\Theta}\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\cdots,\underset{\sim}{n};\ \boldsymbol{m}^{[0]};\ \boldsymbol{m}^{[j]};\ j=1,2,\cdots,\underset{\sim}{m} \\
\operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right);\ \operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right);\ i=1,2,\cdots,\underset{\sim}{n} \\
\operatorname{tr}\left(\boldsymbol{m}^{[0]}\right);\ \operatorname{tr}\left(\boldsymbol{m}^{[k]}\right);\ k=1,2,\cdots,\underset{\sim}{m}-1
\end{gathered}
\tag{87}
$$

If we define new material coefficients, then (86) can be written as to:

$$
\begin{aligned}
&\boldsymbol{m}^{[\underset{\sim}{m}]}+\sum_{k=1}^{m}{}^{m}\lambda_k\boldsymbol{m}^{[k]} \\
&=m^0\big|_{\underline{\Omega}}\boldsymbol{I}+2\left({}^{m}\underset{\sim}{\mu}\right)\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)+{}^{m}\underset{\sim}{\lambda}\operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)\boldsymbol{I}+\sum_{i=1}^{\underset{\sim}{n}}2\left({}^{m}\underset{\sim}{\eta}_i\right)\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right) \\
&\quad+\sum_{i=1}^{n}{}^{m}\underset{\sim}{\kappa}_i\operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[i]}\right)\boldsymbol{I}+\sum_{k=0}^{\underset{\sim}{m}-1}\left({}^{m}\beta_k\right)\operatorname{tr}\left(\boldsymbol{m}^{[k]}\right)\boldsymbol{I}-{}^{m}\alpha_{tm}\left(\theta-\theta\big|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned}
\tag{88}
$$

in which

$$
{}^{m}\underset{\sim}{\boldsymbol{G}}^{1}={}^{c\Theta}\boldsymbol{\varepsilon}_{[0]};\ {}^{m}\underset{\sim}{\boldsymbol{G}}^{i+1}={}^{c\Theta}\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\cdots,\underset{\sim}{n}
\tag{89}
$$

${}^{m}\underset{\sim}{\mu},\ {}^{m}\underset{\sim}{\lambda}$ are similar to Lame's constants but for the moment tensor. ${}^{m}\underset{\sim}{\eta}_i,\ {}^{m}\underset{\sim}{\kappa}_i$ are first and second viscosities for micropolar dissipation corresponding to rotation gradient rates of order i. ${}^{m}\alpha_{tm}$ is thermal coefficient. ${}^{m}\underset{\sim}{\lambda}_k$ is a spectrum of relaxation times associated with $\boldsymbol{m}^{[k]};\ k=1,2,\cdots,m$. We refer to these as micropolar relaxation times. This constitutive theory has ordered rate mechanism of two dissipation physics (medium and micropolar) and similarly ordered rate mechanism of two relaxation or rheology physics (medium and micropolar).

4.3. Constitutive Theory for $\boldsymbol{q}$

Considering

$$
\boldsymbol{q}=\boldsymbol{q}(\boldsymbol{g},\theta)
\tag{90}
$$

and following references [57] [58] we can derive the following constitutive theory for $\boldsymbol{q}$ using representation theorem.

$$
\boldsymbol{q}=-\kappa\boldsymbol{g}-\kappa_1(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g}-\kappa_2\left(\theta-\theta\big|_{\underline{\Omega}}\right)\boldsymbol{g}
\tag{91}
$$

κ,κ_1 and κ_2 are material coefficients. These can be functions of $(\boldsymbol{g}\cdot\boldsymbol{g})\big|_{\underline{\Omega}}$ and $\theta\big|_{\underline{\Omega}}$. $\boldsymbol{g}\cdot\boldsymbol{g}$ is invariant of argument tensor $\boldsymbol{g}$. Simplified form of (91), the Fourier heat conduction law is given by

$$
\boldsymbol{q}=-\kappa\boldsymbol{g}
\tag{92}
$$

5. Complete Mathematical Model

Complete mathematical model consisting of CBL (including BMM balance law) and the constitutive theories that are linear in the components of the argument tensors are given in the following using $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ as stress and moment tensors.

$$
\rho_0(\boldsymbol{x})=|J|\rho(\boldsymbol{x},t)
\tag{93}
$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \nabla \cdot \left(\boldsymbol{\sigma}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) = 0 \tag{94}$$

$$\nabla \cdot \left(\boldsymbol{m}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) + \boldsymbol{\epsilon} : \left(\boldsymbol{\sigma}^{[0]} \cdot \boldsymbol{J}^{\mathrm{T}} \right) + \rho_0 \boldsymbol{m}^b = 0 \tag{95}$$

$$\epsilon_{ijk} m_{ij}^{(0)} = 0 \tag{96}$$

$$\rho_0 \frac{De}{Dt} + \nabla \cdot \boldsymbol{q} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : {}^{c}{}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} = 0 \tag{97}$$

$$\frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{m}^{[0]} : \left({}^{c}{}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]} \right) \le 0 \,; \ (\text{Reduced form}) \tag{98}$$

$$\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]} + {}_a\boldsymbol{\sigma}^{[0]} \tag{99}$$

$${}_s\boldsymbol{\sigma}^{[0]} = {}^e_s\boldsymbol{\sigma}^{[0]} + {}^d_s\boldsymbol{\sigma}^{[0]} \tag{100}$$

Constitutive theories:

$$\begin{aligned} {}^e_s\boldsymbol{\sigma}^{[0]} &= |J| \, p(\rho,\theta) \left(\boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{J} \right)^{-1}; \ \text{compressible} \\ &= |J| \, p(\theta) \left(\boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{J} \right)^{-1}; \ \text{incompressible} \end{aligned} \tag{101}$$

$p(\theta)$ or $p(\rho,\theta)$ is known mechanical or thermodynamic pressure, equations of state.

$$\begin{aligned} {}^d_s\boldsymbol{\sigma}^{[m]} + \sum_{k=1}^{m} \lambda_k \left({}^d_s\boldsymbol{\sigma}^{[k]} \right) &= \sigma^0\big|_{\underset{\sim}{\Omega}} \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}_{[0]} + \lambda \operatorname{tr} \boldsymbol{\varepsilon}_{[0]} \boldsymbol{I} + \sum_{i=1}^{n} 2\eta_i \boldsymbol{\varepsilon}_{[i]} \\ &+ \sum_{i=1}^{n} \kappa_i \left(\operatorname{tr} \boldsymbol{\varepsilon}_{[i]} \right) I + \sum_{k=0}^{m-1} \operatorname{tr} \left({}^d_s\boldsymbol{\sigma}^{[k]} \right) \boldsymbol{I} - \alpha_{tm} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}} \right) \boldsymbol{I} \end{aligned} \tag{102}$$

in which ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1 = \boldsymbol{\varepsilon}_{[0]}$; ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i+1} = \boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \cdots, n$

$$\begin{aligned} \boldsymbol{m}^{[\underset{\sim}{m}]} + \sum_{k=1}^{m} {}^m\lambda_k \boldsymbol{m}^{[k]} &= m^0\big|_{\underset{\sim}{\Omega}} \boldsymbol{I} + 2\left({}^m\underset{\sim}{\mu} \right) \left({}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[0]} \right) + \left({}^m\underset{\sim}{\lambda} \right) \operatorname{tr} \left({}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[0]} \right) \boldsymbol{I} \\ &+ \sum_{i=1}^{n} 2\left({}^m\underset{\sim}{\eta_i} \right) \left({}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[i]} \right) + \sum_{i=1}^{n} \left({}^m\underset{\sim}{\kappa_i} \right) \operatorname{tr} \left({}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[i]} \right) \boldsymbol{I} \\ &+ \sum_{k=0}^{m-1} {}^m\beta_k \operatorname{tr} \left(\boldsymbol{m}^{[k]} \right) \boldsymbol{I} - {}^m\alpha_{tm} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}} \right) \boldsymbol{I} \end{aligned} \tag{103}$$

in which ${}^m\underset{\sim}{\boldsymbol{G}}^1 = {}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[0]}$; ${}^m\underset{\sim}{\boldsymbol{G}}^{i+1} = {}^{c}{}^{\Theta}\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \cdots, \underset{\sim}{n}$

$$\boldsymbol{g} = \nabla \cdot \theta \tag{104}$$

$$\boldsymbol{q} = -\kappa \boldsymbol{g} \tag{105}$$

$$\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt} \tag{106}$$

The mathematical model (93)-(106) are a system of twenty two partial differential equations: BLM(3), BAM(3), FLT(1), constitutive theories for ${}^d_s\boldsymbol{\sigma}^{[0]}(6)$, $\boldsymbol{m}^{[0]}(6)$, $\boldsymbol{q}(3)$ in twenty two variables: $\boldsymbol{u}(3)$, ${}^d_s\boldsymbol{\sigma}^{[0]}(6)$, ${}_a\boldsymbol{\sigma}^{[0]}(3)$, $\boldsymbol{m}^{[0]}(6)$, $\theta(1)$ and $\boldsymbol{q}(3)$. Thus, this mathematical model consisting of conservation and balance laws (including BMM balance law) and constitutive theories has closure.

6. Summary and Conclusions

We have presented a finite deformation, finite strain nonlinear MPNCCT for a compressible thermoviscoelastic solid medium with dissipation and rheology based on classical rotations. The complete mathematical model consists of conservation and the balance laws (in Lagrangian description), including balance of moment of moments balance law, a new balance law necessary for all MPNCCT, and the constitutive theories derived using conjugate pairs in entropy inequality and the representation theorem. A summary of the work presented in this paper and some conclusions drawn from it are given in the following:

(1) Finite strain, finite deformation measures derived for 3M NCCT have been used in the derivation of the CBL and the constitutive theories as necessitated by the conjugate pairs in the entropy inequality.

(2) As pointed out in our recent papers, we distinguish between the deformation/strain measures and rigid rotations in the derivation of the constitutive theories. In MPNCCT, the microconstituents can only experience rigid rotations (evident from the deformation measures derived). These rotations (if considered as unknown dofs at a material point) cannot be added to the strain measures as done in Ref. [29]. This results in an incorrect strain tensor and invalid constitutive theories based on this strain tensor.

(3) The importance of additive decompositions $\boldsymbol{\sigma}^{[0]} = {}_s\boldsymbol{\sigma}^{[0]} + {}_a\boldsymbol{\sigma}^{[0]}$ and ${}_s\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]} + {}_s^d\boldsymbol{\sigma}^{[0]}$ has been pointed out in this paper as well as in References [58]. These are necessary to ensure that volumetric and distortional deformations are addressed correctly in the constitutive theories and that ${}_a\boldsymbol{\sigma}^{[0]}$ is not part of the constitutive tensor.

(4) Deformation measures derived clearly point out that in linear or nonlinear MPNCCT, rigid rotations of the microconstituents must be incorporated in the development of the theory. We must keep in mind that classical rotations ${}_c\boldsymbol{\Theta}$ already exist at the material and constitute a free field in the absence of microconstituents. Surana *et al.* [18] [20] [21] have presented details of linear MPNCCT based on classical rotations ${}_c\boldsymbol{\Theta}$ in the presence of microconstituents. This MPNCCT is thermodynamically and mathematically consistent and has closure when BMM is used as a balance law. Surana *et al.* [20] [21] have shown that the linear MPNCCT based on rotations (${}_c\boldsymbol{\Theta} + {}_\alpha\boldsymbol{\Theta}$) at the material points and a linear MPNCCT based on ${}_\alpha\boldsymbol{\Theta}$ at the material points (ignoring ${}_\alpha\boldsymbol{\Theta}$) are thermodynamically and mathematically inconsistent and suffer from lack of closure, hence are not considered here.

(5) Use of BMM balance law (Yang *et al.* [44], Surana *et al.* [13] [14]) is shown to be essential in linear MPNCCT. This is also true in case of nonlinear MPNCCT. In the absence of this balance law, the nonlinear MPNCCT is also nonphysical and suffers from lack of closure.

(6) All constitutive theories are derived using conjugate pairs in the entropy inequality and the representation theorem, hence they are mathematically consistent.

(7) The nonlinear MPNCCT for TVES with memory presented in this paper incorporates two nonlinear dissipation mechanisms: (a) The first one is due to viscosity of the medium, regular viscous dissipation. This is major source of damping in the TVE polymeric solids (b) The second dissipation mechanism is micropolar dissipation. This is due to interaction of the microconstituents with the viscous medium of the polymer. Both dissipation mechanisms are ordered rate mechanisms that consider dissipation mechanism to be dependent on rates of strain tensor up to order n and rates of rotation gradients tensor up to order n. The theory provides a spectrum of viscosities for the medium as well as for the micropolar dissipation.

(8) The nonlinear MPNCCT presented in this paper also incorporates two mechanisms of rheology or memory, the first one is due to long chain polymer molecules and their Brownian motion in the viscous medium and the second one is due to interaction of microconstituents with the viscous medium and long chain polymer molecules, micropolar rheology or micropolar memory. Both rheology mechanisms are ordered rate mechanisms, thus providing a spectrum of relaxation times for the stress tensor ${}_s^d\boldsymbol{\sigma}^{[0]}$ as well as for the moment tensor $\boldsymbol{m}^{[0]}$.

(9) Appearance of deformation measures for nonlinear 3M NCCT as conjugate to ${}_s^d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{m}^{[0]}$ confirm their validity of derivation as well as consistency of derivation of the nonlinear MPNCCT presented in this paper for TVES with rheology.

(10) It is straight forward to confirm that a linear MPNCCT for TVES with memory is a subset of the nonlinear MPNCCT presented in this paper.

(11) Model problem studies using the nonlinear MPNCCT presented here for TVES with memory and comparison with the nonlinear classical continuum models for TVES with memory will be presented in a follow-up paper.

Acknowledgements

First author is grateful for his endowed professorships and the Department of Mechanical Engineering of the University of Kansas for providing financial support to the second author. The computational facilities provided by the Computational Mechanics Laboratory of the mechanical engineering departments are also acknowledged.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Solid Continua. *Journal of Thermal Engineering*, **1**, 446-459. https://doi.org/10.18186/jte.17430

[2] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Fluent Continua. *Journal of Thermal Engineering*, **1**, 460-475. https://doi.org/10.18186/jte.00314

[3] Surana, K.S., Reddy, J.N., Nunez, D. and Powell, M.J. (2015) A Polar Continuum Theory for Solid Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 77-106.

[4] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A Polar Continuum Theory for Fluent Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 107-146.

[5] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Constitutive Theories for Internal Polar Thermoelastic Solid Continua. *Journal of Pure and Applied Mathematics*: *Advances and Applications*, **14**, 89-150.

[6] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Ordered Rate Constitutive Theories for Internal Polar Thermofluids. *International Journal of Mathematics*, *Science*, *and Engineering Applications*, **9**, 51-116.

[7] Surana, K.S., Long, S.W. and Reddy, J.N. (2016) Rate Constitutive Theories of Orders n and ${}^1n$ for Internal Polar Non-Classical Thermofluids without Memory. *Applied Mathematics*, **7**, 2033-2077. https://doi.org/10.4236/am.2016.716165

[8] Surana, K.S., Mohammadi, F., Reddy, J.N. and Dalkilic, A.S. (2016) Ordered Rate Constitutive Theories for Non-Classical Internal Polar Thermoviscoelastic Solids Without Memory. *International Journal of Mathematics*, *Science*, *and Engineering Applications* (*IJMSEA*), **10**, 99-131.

[9] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) Non-classical Continuum Theory for Solids Incorporating Internal Rotations and Rotations of Cosserat Theories. *Continuum Mechanics and Thermodynamics*, **29**, 665-698. https://doi.org/10.1007/s00161-017-0554-1

[10] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) A Finite Deformation, Finite Strain Non-Classical Internal Polar Continuum Theory for Solids. *Mechanics of Advanced Materials and Structures*, **26**, 381-393.

[11] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Non-Classical Continuum Theories for Solid and Fluent Continua and Some Applications. *International Journal of Smart and Nano Materials*, **10**, 28-89. https://doi.org/10.1080/19475411.2018.1530700

[12] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Solids with Dissipation and Memory Incorporating Internal Rotations. *Polytechnica*, **1**, 19-35. https://doi.org/10.1007/s41050-018-0004-2

[13] Surana, K.S., Shanbhag, R. and Reddy, J.N. (2018) Necessity of Law of Balance of Moment of Moments in Non-Classical Continuum Theories for Solid Continua. *Meccanica*, **53**, 2939-2972. https://doi.org/10.1007/s11012-018-0851-1

[14] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Necessity of Law of Balance/Equilibrium of Moment of Moments in Non-Classical Continuum Theories for Fluent Continua. *Acta Mechanica*, **229**, 2801-2833. https://doi.org/10.1007/s00707-018-2143-1

[15] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Fluids with Internal Rotation Rates. *Applied Mathematics*, **9**, 907-939. https://doi.org/10.4236/am.2018.98063

[16] Surana, K.S., Joy, A.D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Thermoviscoelastic Solids without Memory Incorporating Internal and Cosserat Rotations. *Acta Mechanica*, **229**, 3189-3213. https://doi.org/10.1007/s00707-018-2163-x

[17] Surana, K.S. and Carranza, C.H. (2020) Dynamic Behavior of Thermoelastic Solid

Continua Using Mathematical Model Derived Based on Non-Classical Continuum Mechanics with Internal Rotations. *Meccanica*, **56**, 1345-1375. https://doi.org/10.1007/s11012-020-01221-2

[18] Surana, K.S. and Kendall, J.K. (2020) Existence of Rotational Waves in Non-Classical Thermoelastic Solid Continua Incorporating Internal Rotations. *Continuum Mechanics and Thermodynamics*, **32**, 1659-1683. https://doi.org/10.1007/s00161-020-00872-6

[19] Surana, K.S. and Long, S.W. (2020) Ordered Rate Constitutive Theories for Non-Classical Thermofluids Based on Convected Time Derivatives of the Strain and Higher Order Rotation Rate Tensors Using Entropy Inequality. *Entropy*, **22**, Article 443. https://doi.org/10.3390/e22040443

[20] Surana, K.S. and Carranza, C.H. (2022) Nonclassical Continuum Theories for Fluent Media Incorporating Rotation Rates and Their Thermodynamic Consistency. *ZAMM—Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, **103**, e202200079. https://doi.org/10.1002/zamm.202200079

[21] Surana, K.S. and Mathi, S.S.C. (2022) Thermodynamic Consistency of Nonclassical Continuum Theories for Solid Continua Incorporating Rotations. *Continuum Mechanics and Thermodynamics*, **35**, 17-59. https://doi.org/10.1007/s00161-022-01163-y

[22] Surana, K.S. and Kendall, J.K. (2022) Rotational Inertial Physics in Non-Classical Thermoviscous Fluent Continua Incorporating Internal Rotation Rates. *Applied Mathematics*, **13**, 453-487. https://doi.org/10.4236/am.2022.136030

[23] Surana, K.S. and Kendall, J.K. (2023) NCCT for Micropolar Solid and Fluid Media Based on Internal Rotations and Rotation Rates with Rotational Inertial Physics: Model Problem Studies. *Applied Mathematics*, **14**, 612-651. https://doi.org/10.4236/am.2023.149037

[24] Eringen, A.C. (1964) Simple microfluids. *International Journal of Engineering Science*, **2**, 205-217. https://doi.org/10.1016/0020-7225(64)90005-9

[25] Eringen, A.C. (1966) Mechanics of Micromorphic Materials. In: Görtler, H., Ed., *Applied Mechanics*, Springer, 131-138. https://doi.org/10.1007/978-3-662-29364-5_12

[26] Eringen, A. (1966) Theory of Micropolar Fluids. *Indiana University Mathematics Journal*, **16**, 1-18. https://doi.org/10.1512/iumj.1967.16.16001

[27] Eringen, A.C. (1968) Mechanics of Micromorphic Continua. In: Kröner, E., Ed., *Mechanics of Generalized Continua*, Springer, 18-35. https://doi.org/10.1007/978-3-662-30257-6_2

[28] Eringen, A.C. (1967) Linear Theory of Micropolar Viscoelasticity. *International Journal of Engineering Science*, **5**, 191-204. https://doi.org/10.1016/0020-7225(67)90004-3

[29] Eringen, A.C. (1968) Theory of Micropolar Elasticity. In: Liebowitz, H., Ed., *Fracture*, Academic Press, 621-729.

[30] Eringen, A.C. (1969) Micropolar Fluids with Stretch. *International Journal of Engineering Science*, **7**, 115-127. https://doi.org/10.1016/0020-7225(69)90026-3

[31] Bringen, A.C. (1970) Balance Laws of Micromorphic Mechanics. *International Journal of Engineering Science*, **8**, 819-828. https://doi.org/10.1016/0020-7225(70)90084-4

[32] Eringen, A.C. (1972) Theory of Thermomicrofluids. *Journal of Mathematical Analysis and Applications*, **38**, 480-496. https://doi.org/10.1016/0022-247x(72)90106-0

[33] Eringen, A.C. (1978) Micropolar Theory of Liquid Crystals. In: Johnson, J.F. and Porter, R.S., Eds., *Liquid Crystals and Ordered Fluids*, Springer, 443-474. https://doi.org/10.1007/978-1-4615-8888-7_30

[34] Eringen, A.C. (1990) Theory of Thermo-Microstretch Fluids and Bubbly Liquids. *International Journal of Engineering Science*, **28**, 133-143. https://doi.org/10.1016/0020-7225(90)90063-o

[35] Eringen, A.C. (1992) Balance Laws of Micromorphic Continua Revisited. *International Journal of Engineering Science*, **30**, 805-810. https://doi.org/10.1016/0020-7225(92)90109-t

[36] Eringen, A.C. (1992) Continuum Theory of Microstretch Liquid Crystals. *Journal of Mathematical Physics*, **33**, 4078-4086. https://doi.org/10.1063/1.529859

[37] Eringen, A.C. (1966) A Unified Theory of Thermomechanical Materials. *International Journal of Engineering Science*, **4**, 179-202. https://doi.org/10.1016/0020-7225(66)90022-x

[38] Eringen, A.C. (1972) Theory of Micromorphic Materials with Memory. *International Journal of Engineering Science*, **10**, 623-641. https://doi.org/10.1016/0020-7225(72)90089-4

[39] Eringen, A.C. (1999) Microcontinuum Field Theories I. Foundations and Solids. Springer.

[40] Eringen, A. and Ryan, M. (2002) Microcontinuum Field Theories II: Fluent Media. *Applied Mechanics Reviews*, **55**, B15. https://doi.org/10.1115/1.1445333

[41] Toupin, R.A. (1962) Elastic Materials with Couple-stresses. *Archive for Rational Mechanics and Analysis*, **11**, 385-414. https://doi.org/10.1007/bf00253945

[42] Eremeyev, V.A. and Pietraszkiewicz, W. (2015) Material Symmetry Group and Constitutive Equations of Micropolar Anisotropic Elastic Solids. *Mathematics and Mechanics of Solids*, **21**, 210-221. https://doi.org/10.1177/1081286515582862

[43] Eremeyev, V.A., Lebedev, L.P. and Altenbach, H. (2013) Foundations of Micropolar Mechanics. Springer.

[44] Yang, F., Chong, A.C.M., Lam, D.C.C. and Tong, P. (2002) Couple Stress Based Strain Gradient Theory for Elasticity. *International Journal of Solids and Structures*, **39**, 2731-2743. https://doi.org/10.1016/s0020-7683(02)00152-x

[45] Smith, G.F. (1965) On Isotropic Integrity Bases. *Archive for Rational Mechanics and Analysis*, **18**, 282-292. https://doi.org/10.1007/bf00251667

[46] Smith, G.F. (1970) On a Fundamental Error in Two Papers of C.-C. Wang "on Representations for Isotropic Functions, Parts I and II". *Archive for Rational Mechanics and Analysis*, **36**, 161-165. https://doi.org/10.1007/bf00272240

[47] Smith, G.F. (1971) On Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. *International Journal of Engineering Science*, **9**, 899-916. https://doi.org/10.1016/0020-7225(71)90023-1

[48] Spencer, A.J.M. (1971) Theory of Invariants. In: Eringen, A.C., Ed., *Mathematics*, Elsevier, 239-353. https://doi.org/10.1016/b978-0-12-240801-4.50008-x

[49] Spencer, A.J.M. and Rivlin, R.S. (1958) The Theory of Matrix Polynomials and Its Application to the Mechanics of Isotropic Continua. *Archive for Rational Mechanics and Analysis*, **2**, 309-336. https://doi.org/10.1007/bf00277933

[50] Spencer, A.J.M. and Rivlin, R.S. (1959) Further Results in the Theory of Matrix Polynomials. *Archive for Rational Mechanics and Analysis*, **4**, 214-230. https://doi.org/10.1007/bf00281388

[51] Wang, C.C. (1969) On Representations for Isotropic Functions, Part I. *Archive for Rational Mechanics and Analysis*, **33**, 249-267. https://doi.org/10.1007/bf00281278

[52] Wang, C.C. (1969) On Representations for Isotropic Functions, Part II. *Archive for Rational Mechanics and Analysis*, **33**, 268-287. https://doi.org/10.1007/bf00281279

[53] Wang, C.C. (1970) A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of My Papers on Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **36**, 166-197. https://doi.org/10.1007/bf00272241

[54] Wang, C.C. (1971) Corrigendum to My Recent Papers on "Representations for Isotropic Functions". *Archive for Rational Mechanics and Analysis*, **43**, 392-395. https://doi.org/10.1007/bf00252004

[55] Zheng, Q.S. (1993) On the Representations for Isotropic Vector-Valued, Symmetric Tensor-Valued and Skew-Symmetric Tensor-Valued Functions. *International Journal of Engineering Science*, **31**, 1013-1024. https://doi.org/10.1016/0020-7225(93)90109-8

[56] Zheng, Q.S. (1993) On Transversely Isotropic, Orthotropic and Relative Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. Part I: Two Dimensional Orthotropic and Relative Isotropic Functions and Three Dimensional Relative Isotropic Functions. *International Journal of Engineering Science*, **31**, 1399-1409. https://doi.org/10.1016/0020-7225(93)90005-f

[57] Surana, K.S. (2015) Advanced Mechanics of Continua. CRC/Taylor and Francis.

[58] Surana, K.S. (2021) Classical Continuum Mechanics, Second Edition. CRC/Taylor and Francis

Appendix

Nomenclature

$\bar{\boldsymbol{x}}$, $\bar{x}_i$, $\{\bar{x}\}$: deformed Coordinates

$\boldsymbol{x}$, x_i, $\{x\}$: undeformed Coordinates

ρ_0: reference density

ρ: density in Lagrangian description

$\bar{\rho}$: density in Eulerian description

η: specific entropy in Lagrangian description

$\bar{\eta}$: specific entropy in Eulerian description

e: specific internal energy in Lagrangian description

$\bar{e}$: specific internal energy in Eulerian description

${}_i\boldsymbol{\Theta}$, ${}_i\Theta_i$, $\{{}_i\Theta\}$: internal or classical rotations in Lagrangian description

${}_e\boldsymbol{\Theta}$, ${}_e\Theta_i$, $\{{}_e\Theta\}$: external or Cosserat or microrotations in Lagrangian description

${}_t\boldsymbol{\Theta}$, ${}_t\Theta_i$, $\{{}_t\Theta\}$: total rotations in Lagrangian description

$\boldsymbol{J}$: deformation gradient tensor in Lagrangian description

${}_s\boldsymbol{J}$: symmetric part of deformation gradient tensor in Lagrangian description

${}_a\boldsymbol{J}$: skew-symmetric part of deformation gradient tensor in Lagrangian description

${}^d\boldsymbol{J}$: displacement gradient tensor in Lagrangian description

${}^d_s\boldsymbol{J}$: symmetric part of displacement gradient tensor in Lagrangian description

${}^d_a\boldsymbol{J}$: skew-symmetric part of displacement gradient tensor in Lagrangian description

${}^\Theta\boldsymbol{J}$: rotation gradient tensor in Lagrangian description

${}^{c\Theta}\boldsymbol{J}$: internal rotation gradient tensor in Lagrangian description

${}^{c\Theta}_{s}\boldsymbol{J}$: symmetric part of internal rotation gradient tensor in Lagrangian description

${}^{c\Theta}_{a}\boldsymbol{J}$: skew-symmetric part of internal rotation gradient tensor in Lagrangian description

${}^{{}^r_c\bar{\Theta}}_{s}\bar{\boldsymbol{J}}$: symmetric part of gradient of internal rotation rate tensor in Eulerian description

${}^{c\Theta}_{s}\dot{\boldsymbol{J}}$: rate of symmetric part of gradient of internal rotation tensor in Lagrangian description

${}^r_c\bar{\boldsymbol{\Theta}}$: internal rotation rate tensor in Eulerian description

${}_c\dot{\boldsymbol{\Theta}}$: internal rotation rate tensor in Lagrangian description

$\boldsymbol{q}$, q_i, $\{q\}$: heat vector in Lagrangian description

$\bar{\boldsymbol{q}}$, $\bar{q}_i$, $\{\bar{q}\}$: heat vector in Eulerian description

$\boldsymbol{v}$, v_i, $\{v\}$: velocities in Lagrangian description

$\bar{\boldsymbol{v}}$, $\bar{v}_i$, $\{\bar{v}\}$: velocities in Eulerian description

$\boldsymbol{u}$, u_i, $\{u\}$: displacements in Lagrangian description

$\bar{\boldsymbol{u}}$, $\bar{u}_i$, $\{\bar{u}\}$: displacements in Eulerian description

$\boldsymbol{P}$: average stress in Lagrangian description

$\bar{\boldsymbol{P}}$: average stress in Eulerian description

$\boldsymbol{M}$: average moment in Lagrangian description

$\bar{\boldsymbol{M}}$: average moment in Eulerian description

$\boldsymbol{\sigma}$, σ_{ij}, $[\sigma]$: Cauchy stress tensor in Lagrangian description

$\bar{\boldsymbol{\sigma}}$, $\bar{\sigma}_{ij}$, $[\bar{\sigma}]$: Cauchy stress tensor in Eulerian description

${}_s\boldsymbol{\sigma}$: symmetric part of Cauchy stress tensor tensor

${}_a\boldsymbol{\sigma}$: anti-symmetric part of Cauchy stress tensor tensor

${}_s^d\boldsymbol{\sigma}$: deviatoric part of the symmetric Cauchy stress tensor tensor

${}_s^e\boldsymbol{\sigma}$: equilibrium part of the symmetric Cauchy stress tensor tensor

θ: temperature in Lagrangian description

$\bar{\theta}$: temperature in Eulerian description

k : thermal conductivity in Lagrangian

p : thermodynamic or Mechanical Pressure in Lagrangian description

$\bar{p}$: thermodynamic or Mechanical Pressure in Eulerian description

$\boldsymbol{g}$, g_i, $\{g\}$: temperature gradient tensor in Lagrangian description

$\bar{\boldsymbol{g}}$, $\bar{g}_i$, $\{\bar{g}\}$: temperature gradient tensor in Eulerian description

$\bar{\boldsymbol{L}}$: velocity gradient tensor in Eulerian description

$\bar{\boldsymbol{D}}$: symmetric part of velocity gradient tensor in Eulerian description

Abbreviations

CBL	Conservation and Balance Laws
CCM	Classical Continuum Mechanics
CCT	Classical Continuum Theory
NCCT	Nonclassical Continuum Theory
NCCM	Nonclassical Continuum Mechanics