International Journal of Modern Nonlinear Theory and Application, 2024, 13, 45-52
https://www.scirp.org/journal/ijmnta
ISSN Online: 2167-9487
ISSN Print: 2167-9479

Application of Stability Criteria for Complex-Valued Impulsive System by Lyapunov Function

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How to cite this paper: Yang, W. (2024) Application of Stability Criteria for Complex-Valued Impulsive System by Lyapunov Function. *International Journal of Modern Nonlinear Theory and Application*, **13**, 45-52.
https://doi.org/10.4236/ijmnta.2024.134004

Received: November 12, 2024
Accepted: December 14, 2024
Published: December 17, 2024

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Abstract

Stability criteria for the complex-valued impulsive system are applied widely in many fields, such as quantum systems, which have been studied in recent decades. In this paper, I investigate the Lyapunov control of finite dimensional complex-valued systems with impulsive control fields, where the studied complex-valued systems are governed by the Schrödinger equation and can be used in quantum systems. By one Lyapunov function based on state error and the invariant principle of impulsive systems, I study the convergence of complex-valued systems with impulsive control fields and propose new results for the mentioned complex-valued systems in the form of sufficient conditions. A numerical simulation to validate the proposed control method is provided.

Keywords

Complex-Valued Systems, Stability, Lyapunov Function, Impulsive Control Field

1. Introduction

The concept of stability theory is fundamental to the fields of control theory and system engineering [1]-[6]. In the stability problem, the state error of the system is required to converge to zero. For example, the authors in references [7]-[12] have investigated the stability problem for different systems.

The common setting adopted in the aforementioned works is always in real number fields, *i.e.*, the objective of the study is real-valued differential systems. The authors in [12]-[15] studied the stability problem of complex-valued differential systems. The complex-valued differential system has many potential applications in science and engineering, such as the quantum system, which is one of the classical complex-valued systems. There have been many theoretical and experimental breakthroughs in the stability study of quantum systems [16]-[21], and references therein indicate that quantum control has become an important area of research.

Impulsive dynamical systems are a special class of dynamical systems which exhibit continuous evolution typically described by ordinary differential equations and instantaneous state jumps or impulses. Nowadays, there have been various approaches for control of impulsive dynamical systems [8] [10] [11] [22], which indicates that increasing interest in the analysis and synthesis of impulsive systems, or impulsive control systems, due to their significance both in theory and applications.

Quantum control is a significant aspect of quantum information processing and quantum technologies. It is the cornerstone of quantum technology, extensively utilized in quantum computing and quantum communication. Various approaches for finite-time control of quantum systems have been developed.

In order to drive the system state to its target in less time, I add an impulsive control field beside the continuous control field and apply the impulsive control method to control quantum systems from the result in [3], when one control field with a given frequency, quantum systems governed by the Schrödinger equation can be described as impulsive dynamical systems.

In this paper, I apply the stability criteria for a complex-valued impulsive system to give a better option to drive the quantum state to its target. The Lyapunov function is presented based on state error. In Section 2, a complex-valued system with impulsive control fields is presented, and introduce the invariant principle of impulsive systems. In Section 3, I give one control field to drive quantum state based on one Lyapunov function and analyze the asymptotic stability of quantum systems with impulsive control fields. In order to justify the effectiveness of the proposed control fields, one simulation experiment is given in Section 4.

2. Notations and Definitions

Consider the impulsive dynamical system described by

$$\begin{cases} \dot{x}(t) = f_c(x(t)), & t \in (\tau_k, \tau_{k+1}); \\ \Delta x(t) = f_d(x(t)), & t = \tau_k. \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $f_c(x)$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$, the set $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset \mathbb{R}^+$ is an unbounded, closed, discrete subset of $\mathbb{R}^+$ which denotes the set of times when jumps occur, and $f_d : \mathbb{R}^n \to \mathbb{R}^n$ denotes the incremental change of the state at the time τ_k. In the n-dimensional complex space $\mathbb{C}^n$, I choose the most common norm $\|x\| := \sqrt{x^* x}$, where x is represented as a column vector $(x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, and x^* denotes its conjugate transpose.

Denote by $M_n(\mathbb{C})$ the space of $n \times n$ complex matrices with an inner product $(\cdot,\cdot): M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{C}$,

$$(a,b) = Tr(ab),$$

and the norm $\|a\|^2 = (a,a)$. As we know, quantum systems are typical complex systems, whose state variables are defined on the complex field. Consider the following n-level quantum system with two control fields, and set the Plank constant $\hbar = 1$:

$$i\left|\dot{\beta}(t)\right\rangle = \left(H_0 + f_1(t)H_1 + \sum_{k=1}^{\infty} f_2(t)H_2\delta(t-\tau_k)\right)\left|\beta(t)\right\rangle, \tag{2.2}$$

where the ket $\left|\beta(t)\right\rangle \in \mathbb{C}^n$ represents the state vector of quantum systems, which is right continuous, and the state vector evolves on or in a sphere with radius one, and I denote the set of quantum states by VS_n, and $\delta(\cdot)$ is the Dirac impulse. Physically, two states $\left|\beta_1\right\rangle$ and $\left|\beta_2\right\rangle$ that differ by a phase $\theta(t) \in R$, *i.e.*, $\left|\beta_1\right\rangle = \exp(i\theta(t))\left|\beta_2\right\rangle$, describe the same physical state in or on the sphere of $\mathbb{C}^n$. I denote the bra associated with the ket $\left|\beta(t)\right\rangle$ with $\left\langle\beta(t)\right|$. When the quantum system evolves freely under its own internal dynamics, *i.e.*, there is no external field implemented on the system, just the free Hamiltonian H_0 is introduced. $H_j\,(j=1,2)$ represents the interaction energy between the system and the external classical control fields $f_j(t)(j=1,2)$, and is called interaction Hamiltonian. $H_j\,(j=0,1,2)$ are all $n \times n$ self-adjoint operators in the n-dimensional Hilbert space $\mathcal{H}$ and assumed to be time-independent. In this paper, I set the first control function $f_1(t)$ is continuous, and the other one $f_2(t)$ only takes effect on quantum systems at the impulsive points *E*.

In quantum control, the target state is usually an eigenstate of the free Hamiltonian, and I set the target state $\left|\beta_f\right\rangle$ satisfies:

$$H_0\left|\beta_f\right\rangle = \lambda_f\left|\beta_f\right\rangle,$$

where λ_f is the eigenvalue of H_0 corresponding to $\left|\beta_f\right\rangle$.

By the same method in [4], I obtain that quantum systems (2.2) with impulsive control fields can be described as

$$\begin{cases} i\left|\dot{\beta}(t)\right\rangle = \left(H_0 + f_1(t)H_1\right)\left|\beta(t)\right\rangle, & t \neq \tau_k; \\ \Delta\left|\beta\right\rangle = f_2(t)H_2\left|\beta\left(\tau_k^-\right)\right\rangle, & t = \tau_k. \end{cases} \tag{2.3}$$

When taking non-trivial geometry about states, I add a second control ω corresponding to $\theta(t)$ into consideration [9], then investigate the following quantum systems

$$i\left|\dot{\beta}(t)\right\rangle = \left(H_0 + f_1(t)H_1 + \sum_{k=1}^{\infty} f_2(t)H_2\delta(t-\tau_k) + \omega I\right)\left|\beta(t)\right\rangle, \tag{2.4}$$

where I is the identity matrix. If the control field $f_2(t)$ only takes effect at the impulsive point *E*, the quantum systems with impulsive control fields are

$$\begin{cases} i\left|\dot{\beta}(t)\right\rangle = \left(H_0 + f_1(t)H_1 + \omega I\right)\left|\beta(t)\right\rangle, & t \neq \tau_k; \\ \Delta\left|\beta\right\rangle = f_2(t)H_2\left|\beta\left(\tau_k^-\right)\right\rangle, & t = \tau_k. \end{cases} \tag{2.5}$$

Subject to quantum systems (2.2) or (2.4), I focus on finding control fields

$f_1(t)$ and $f_2(\tau_k)$, such that the quantum systems with impulsive control field (2.3) or (2.5) are driven to target states. Firstly, I introduce the invariant principle of impulsive systems.

Lemma 2.1 [2] *Consider the impulsive dynamical system* (2.1), *assume* $\mathcal{D}_c \subset \mathcal{D}$ *is a compact positively invariant set with respect to* (2.1), *and assume that there exists a* C^1 *function* $V : \mathcal{D}_c \to \mathbb{R}$ *such that*

1) $\dot{V}(x(t)) \le 0, x \in \mathcal{D}_c, t \ne \tau_k$;
2) $V(x(\tau_k^-)) + f_d(x(\tau_k^-)) \le V(x(\tau_k^-)), x \in \mathcal{D}_c, t = \tau_k$;

Let

$$G \triangleq \left\{x \in \mathcal{D}_c : t \ne \tau_k, \dot{V}(x(t)) = 0\right\} \cup \left\{x \in \mathcal{D}_c : t = \tau_k, V(x(\tau_k^-)) + f_d(x(\tau_k^-)) = V(x(\tau_k^-))\right\},$$

and let $M \subset G$ denote the largest invariant set contained in *G*. If $x_0 \in \mathcal{D}_c$, then $x(t) \to M$ as $t \to \infty$.

3. Impulsive Control of Quantum System

In this section, I shall design an impulsive feedback controller for complex-valued quantum systems. As can be seen later, the proposed controller is not only simple and direct, but the bus can also stabilise the quantum systems so that all trajectories will converge to the target state. When the phase θ is considered, I choose the Lyapunov function based on the state error [9] [19].

Theorem 1 *For quantum system* (2.5), *if* H_0 *is non-degenerate, set the control fields* $\lambda_f + \omega = K_0 g_0\left(Im\left(\langle \beta_f | \beta(t)\rangle\right)\right)$, $f_1(t) = K_1 g_1\left(Im\left(\beta_f | H_1 | \beta(t)\right)\right)$, *and*

$$f_2(\tau_k) = \frac{-2Re\left(\left\langle \beta(\tau_k^-)\right| H_2 \left|\beta(\tau_k^-) - \beta_f\right\rangle\right)}{\sqrt{Tr\left(H_2^4\right)}},$$ *where constants* $K_j > 0\ (j = 0,1)$,

the image of function $y_j = g_j(x_j)$ *passes the origin of plane* $x_j - y_j$ *monotonically and lies in quadrant I or III, then quantum systems with impulses* (2.5) *converge to the largest invariant set* $VS_n \cap E_2$, *where* $E_2 = \left\{|\beta\rangle : \langle \beta_f | H_1 | \beta\rangle = 0, Im\left(\langle \beta_f | \beta\rangle\right) = 0\right\}$. *If all the states in* E_2 *are equivalent to the target state* $|\beta_f\rangle$, *then the systems will converge asymptotically to the target state* $|\beta_f\rangle$.

Proof. I choose a Lyapunov function based on state error,

$$V\left(|\beta(t)\rangle, t\right) = \left\langle \beta(t) - \beta_f \middle| \beta(t) - \beta_f \right\rangle. \tag{3.1}$$

When $t \ne \tau_k$, $\dot{V} = -(\lambda_f + \omega) Im\left(\langle \beta_f | \beta(t)\rangle\right) - f_1(t) Im\left(\langle \beta_f | H_1 | \beta(t)\rangle\right)$, the simple control field

$$\lambda_f + \omega = K_0 g_0\left(Im\left(\langle \beta_f | \beta(t)\rangle\right)\right), f_1 = K_1 g_1\left(Im\left(\langle \beta_f | H_1 | \beta(t)\rangle\right)\right), \tag{3.2}$$

I have

$$\begin{aligned}\dot{V}(t) &= -K_0 Im\left(\langle \beta_f | \beta(t)\rangle\right) g_0\left(Im\left(\langle \beta_f | \beta(t)\rangle\right)\right) \\ &\quad - K_1 Im\left(\langle \beta_f | H_1 | \beta(t)\rangle\right) g_1\left(Im\left(\langle \beta_f | H_1 | \beta(t)\rangle\right)\right) \\ &< 0 \quad (t \ne \tau_k).\end{aligned}$$

When $t = \tau_k$,

$$\begin{aligned}
&V\left(\left|\beta(\tau_k)\right\rangle,\tau_k\right)=V\left(\left|\beta(\tau_k^+)\right\rangle,\tau_k^+\right)\\
&=\left(\left\langle\beta(\tau_k^-)\right|\left(I+f_2(\tau_k)H_2\right)-\left\langle\beta_f\right|\right)\left(\left(I+f_2(\tau_k)H_2\right)\left|\beta(\tau_k^-)\right\rangle-\left|\beta_f\right\rangle\right)\\
&=\left\langle\beta(\tau_k^-)-\beta_f\middle|\beta(\tau_k^-)-\beta_f\right\rangle+2f_2(\tau_k)Re\left(\left\langle\beta(\tau_k^-)\right|H_2\left|\beta(\tau_k^-)-\beta_f\right\rangle\right)\\
&\quad+f_2^2(\tau_k)\left\langle\beta(\tau_k^-)\right|H_2^2\left|\beta(\tau_k^-)\right\rangle,
\end{aligned}\tag{3.3}$$

since $\left\langle\beta(\tau_k^-)\right|H_2^2\left|\beta(\tau_k^-)\right\rangle\le\left\|\beta(\tau_k^-)\right\|^2\left\|H_2^2\right\|\le\sqrt{Tr\left(H_2^4\right)}$, by the control field $f_2(\tau_k)=\dfrac{-2Re\left(\left\langle\beta(\tau_k^-)\right|H_2\left|\beta(\tau_k^-)-\beta_f\right\rangle\right)}{\sqrt{Tr\left(H_2^4\right)}}$, I have

$$\begin{aligned}
V\left(\left|\beta(\tau_k)\right\rangle,\tau_k\right)&=\left\langle\beta(\tau_k^-)-\beta_f\middle|\beta(\tau_k^-)-\beta_f\right\rangle-\frac{4Re^2\left(\left\langle\beta(\tau_k^-)\right|H_2\left|\beta(\tau_k^-)-\beta_f\right\rangle\right)}{\sqrt{Tr\left(H_2^4\right)}}\\
&\quad+\frac{4Re^2\left(\left\langle\beta(\tau_k^-)\right|H_2\left|\beta(\tau_k^-)-\beta_f\right\rangle\right)}{Tr\left(H_2^4\right)}\left\langle\beta(\tau_k^-)\right|H_2^2\left|\beta(\tau_k^-)\right\rangle\\
&\le V\left(\left|\beta(\tau_k^-)\right\rangle,\tau_k^-\right).
\end{aligned}\tag{3.4}$$

Using the control field $\lambda_f+\omega, f_1(t)$ (3.2), the largest invariant set of quantum systems with impulsive control fields (2.5) is $VS_n\cap E_2$ [9], where $E_2=\left\{|\beta\rangle:\left\langle\beta_f\right|H_1|\beta\rangle=0, Im\left(\left\langle\beta_f|\beta\right\rangle\right)=0\right\}$. From the invariant principle Lemma 2.1, the quantum systems with impulsive control fields (2.5) will converge to $VS_n\cap E_2$.

Thus I complete the proof.

4. Illustrative Examples

In order to illustrate the effectiveness of the proposed method in this paper, one numerical simulation was presented for a five-level quantum system, and the Fourth-Order Runge-Kitta method was used to solve the problem with a time step size of 0.05.

Example 1. Consider the five-level quantum system with internal Hamiltonian and the control Hamiltonians given as follows:

$$H_0=\begin{pmatrix}1&0&0&0&0\\0&1.1&0&0&0\\0&0&1.2&0&0\\0&0&0&1.4&0\\0&0&0&0&1.7\end{pmatrix}, H_1=\begin{pmatrix}0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&1\\1&0&0&1&0\end{pmatrix}, H_2=\begin{pmatrix}0&0&1&0&0\\0&0&1&0&1\\1&1&0&0&0\\0&0&0&0&1\\0&1&0&1&0\end{pmatrix}.$$

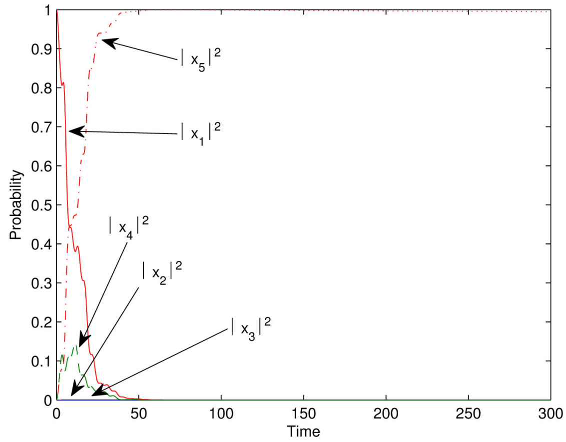
Let the initial state and the target state also be $\left|\beta_0\right\rangle=\begin{pmatrix}1&0&0&0&0\end{pmatrix}^{\mathrm{T}}$ and $\left|\beta_f\right\rangle=\begin{pmatrix}0&0&0&0&1\end{pmatrix}^{\mathrm{T}}$, respectively. The parameters are chosen as $K_1=0.15$, $K_2=0.001$. Set the state $\left|\beta(t)\right\rangle=\begin{pmatrix}x_1&x_2&x_3&x_4&x_5\end{pmatrix}^{\mathrm{T}}$, by the same control fields

$$\lambda_f + \omega = K_0 g_0 \left(Im\left(\left\langle \beta_f \middle| \beta(t) \right\rangle \right) \right),$$

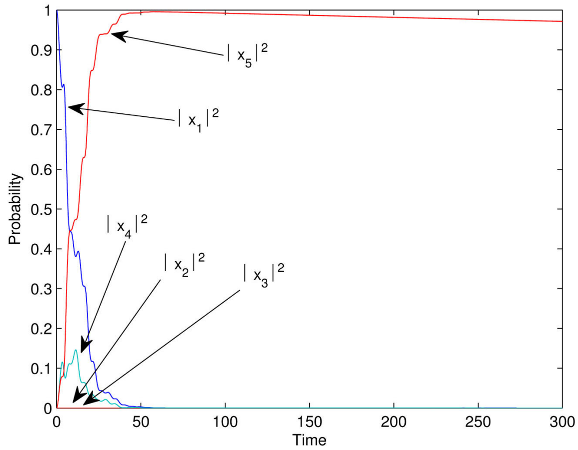
$$f_1(t) = K_1 g_1 \left(Im\left(\beta_f \middle| H_1 \middle| \beta(t) \right) \right),$$

$$f_2(\tau_k) = \frac{-2Re\left(\left\langle \beta(\tau_k^-) \middle| H_2 \middle| \beta(\tau_k^-) - \beta_f \right\rangle \right)}{\sqrt{Tr\left(H_2^4\right)}},$$

The simulation results are shown in Figure 1. In Figure 1(a), the population of the system with impulsive control field f_2 is shown, component $|x_5|$ increases to 1, and other components decrease to 0. The result shown in Figure 1(b) demonstrates the control performance without impulsive control, component $|x_5|$ increases to 1 firstly, then decreases a little. The final transition probability attains about 0.99942 in Figure 1(a), which is better than the one (about 0.99581) in Figure 1(b), and significantly, the control method with one impulsive control field can prevent the evolution from decaying.



(a)



(b)

Figure 1. (a) The population of the five-level system trajectory from $|\beta_0\rangle$ by control fields f_1, f_2 in Example 1; (b) The population of the five-level system trajectory from $|\beta_0\rangle$ without control field f_2 in Example 1.

5. Conclusion

In this paper, I have introduced the Lyapunov control method to complex-valued systems with impulsive control fields and given one kind of control field based on a Lyapunov function inspired by state error. The theoretical results have been verified by a numerical simulation to illustrate the effectiveness and advantages of the proposed method compared with existing results.

Acknowledgements

The authors would like to thank the editor and the reviewers for their constructive comments and suggestions, which improved the quality of the paper. This work is supported by the founding 20AR57 of Shanghai Dianji University.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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