Scientific Research Publishing

Journal of Mathematical Finance, 2024, 14, 397-416
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

On Discrete Risk Process with Stochastic Premiums and Dividends Modulated by Random Discount Rates

Enoch J. Dangbe, Andrzej Korzeniowski

Department of Mathematics, University of Texas at Arlington, Arlington, USA
Email: ejd8342@mavs.uta.edu, korzeniowski@uta.edu

How to cite this paper: Dangbe, E.J. and Korzeniowski, A. (2024) On Discrete Risk Process with Stochastic Premiums and Dividends Modulated by Random Discount Rates. *Journal of Mathematical Finance*, **14**, 397-416.
https://doi.org/10.4236/jmf.2024.144023

Received: August 14, 2024
Accepted: November 24, 2024
Published: November 27, 2024

Abstract

We extend the discrete time risk model studied by Korzeniowski [1] [2] via incorporating the effect of dividend payments subject to random discount factor. By applying generating functions technique, effective recursive formulas for the total expected discounted dividends prior to ruin are derived. Results are illustrated by examples representing various surplus process risk scenarios.

Keywords

Discrete Time Surplus Process, Random Premiums, Constant Dividend Barrier, Random Discount Factor, Total Expected Discounted Dividends Prior to Ruin, Generating Function Method

1. Introduction

Risk considerations in insurance and finance are often modelled by the classical Crámer-Lundberg surplus process [3]

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k \tag{1.1}$$

where $U(t)$ represents the capital available at time $t > 0$, given the initial capital $U(0) = u \geq 0$, after paying claims X_i which occurred at random times during the interval (0, *t*] according to a Poisson process $N(t)$. The premium income stream *ct* is deterministic with premium rate *c* per unit of time. $U(t)$ represents the risk reserve of a company at time *t*. The main objective is to calculate the odds that the company reserve will ever become negative, referred to as the probability of ultimate ruin.

We remark, that while it may be reasonable for an insurance company to conveniently collect premiums according to deterministic formula *ct*, given customers contractual obligation to pay premiums to receive coverage for their claims, it however may not be a reasonable assumption for many models of company revenue, due to the fact the future number of customers and their respective premium payments cannot be guaranteed. Furthermore, while it may be true "on average" that an insurance company receives premiums as a continuous stream, it is still possible that the total premiums collected by time *t* may be substantially smaller than *ct*, at some future times *t*.

As a remedy for model (1.1) limitations, Korzeniowski considered instead a discrete-time surplus process, where in addition, the premium income *ct* was replaced by a suitable stochastic component. Namely,

$$U_n = u + \sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} X_k,\ Y_k \in [0,\infty),\ X_k \in [0,\infty) \tag{1.2}$$

with *i.i.d.* random premiums Y_k and *i.i.d.* claims X_k respectively, for which the probability of ultimate ruin on infinite and finite time horizons were obtained.

The paper is organized as follows. Section 2 introduces our new model by incorporating dividend payments into (1.2) for non-negative integer-valued X_k, Y_k. In section 3 we establish a key generating function lemma for the total expected discounted dividend prior to ruin. Main results are presented in Section 4. Illustrating examples are given in Section 5, while Section 6 provides summary conclusions and direction for future research.

2. Model Description

Before introducing a central object of this paper, *i.e.*, total expected discounted dividends prior to ruin, we need to define a discount factor in the context of the interest rates subject to random fluctuations. Recall that for a constant interest rate $r > 0$ a unit amount of money at $t = 0$ is worth $M(t) = \mathrm{e}^{rt}$ at time *t*, which results in the discount factor $\frac{1}{M(t)} = \mathrm{e}^{-rt}$ that represents the present value of a unit amount of money in the future.

Since, in general, the value of money is subject to random fluctuations of the economy (notably due to varying interest rates), we shall assume that a unit amount of money at a future time *t* will be worth

$$M(t) = \mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)},\ E\left[M(t)\right] = \mathrm{e}^{rt} \tag{2.1}$$

given fixed interest rate $r > 0$ with volatility $\sigma > 0$, and the standard Brownian Motion $W(t)$ with $E\left[W(t)\right] = 0$, $Var\left[W(t)\right] = t$. Geometric Brownian Motion $M(t)$ is known to be a unique solution to the Stochastic Differential Equation (SDE) [4]:

$$\begin{cases} \mathrm{d}M(t) = rM(t)\mathrm{d}t + \sigma M(t)\mathrm{d}W(t),\ t \ge 0 \\ M(0) = 1 \end{cases} \tag{2.2}$$

Definition 2.3. A *Random Discount Factor*, assumed to be independent of the surplus process $U(t)$, is defined by

$$D(t)=\frac{1}{M(t)}=\mathrm{e}^{-\left(r-\frac{1}{2}\sigma^2\right)t-\sigma W(t)} \tag{2.4}$$

The corresponding expected discount factor is given by

$$E\left[D(t)\right]=E\left[\mathrm{e}^{-\left(r-\frac{1}{2}\sigma^2\right)t-\sigma W(t)}\right]=\mathrm{e}^{-\left(r-\sigma^2\right)t},\ \ 0<\sigma^2<r \tag{2.5}$$

Remark 2.6. In the absence of noise, *i.e.*, $\sigma=0$, $D(t)$ falls back to e^{-rt}.

In what follows, we utilize the *Random Discount Factor* and its expected value for discrete times $t=k=0,1,2,\cdots$

$$E\left[D(k)\right]=\mathrm{e}^{-\left(r-\sigma^2\right)k}=v^k,\ \ v\equiv v(r,\sigma)=\mathrm{e}^{-\left(r-\sigma^2\right)}<1 \tag{2.7}$$

where v represents an average discount factor per period.

Definition 2.8. Integer-valued surplus process is defined as follows:

$$U(t)=u+\sum\nolimits_{k=1}^{t}X_k-\sum\nolimits_{k=1}^{t}Y_k-\sum\nolimits_{k=0}^{t}d_k,\ \ t=0,1,2,\cdots \tag{2.9}$$

where $U(0)=u\in\mathbb{N}=\{0,1,2,\cdots\}$ is the initial capital, $X_k\in\mathbb{N}$ are *i.i.d.* premiums, $Y_k\in\mathbb{N}$ are *i.i.d.* claims, $d_k=\max\{U(k)-b,0\}$ are dividends for a given constant barrier $b\in\mathbb{N}$. In addition, the stream of premiums is assumed independent of the stream of claims. The distributions of $X_k\sim X$, $Y_k\sim Y$, are denoted by

$$\begin{aligned} f(n)&=P(X=n),\ \ n=0,1,\cdots,\ \ N<\infty,\\ g(n)&=P(Y=n),\ \ n=0,1,\cdots \end{aligned} \tag{2.10}$$

Furthermore, we assume that at any time t the surplus process rebalancing is done in the following order: premiums first, claims next, dividends last (if any). The ruin time of $U(t)$ is defined by

$$T_{u,b}=\inf\{k\ge 1\,|\,U(k)<0\} \tag{2.11}$$

By (2.4) the total discounted dividends prior to ruin reads

$$\sum\nolimits_{k=0}^{T_{u,b}}\mathrm{e}^{-\left(r-\frac{1}{2}\sigma^2\right)k-\sigma W(k)}d_k=\sum\nolimits_{k=0}^{T_{u,b}}D(k)d_k \tag{2.12}$$

Finally, the total expected discounted dividends prior to ruin, given the initial surplus u, is defined as follows:

$$\begin{aligned} V(u,b)&=E\left[\sum\nolimits_{k=0}^{T_{u,b}}D(k)d_k\,|\,U(0)=u\right]\\ &=E\left[\sum\nolimits_{k=0}^{T_{u,b}}E\left[D(k)\right]d_k\,|\,U(0)=u\right]\\ &=E\left[\sum\nolimits_{k=0}^{T_{u,b}}v^k d_k\,|\,U(0)=u\right] \end{aligned} \tag{2.13}$$

where the second equality is the result of $\{D(k)\}$ being independent of $\{U(k)\}$ (thus in particular independent of ruin time $T_{u,b}$ and dividends $\{d_k\}$, whereas the third equality comes by substitution from (2.7).

The key feature of our generalized risk model is two-fold. First, unlike in classical Binomial Model and its extensions with fixed premium income of 1 per period, our premiums are random with range $\{0,1,\cdots,N\}$ which is a realistic assumption from an insurer's viewpoint, as premium income cannot be guaranteed. Second, unlike the existing models that consider quite arbitrary random discount factors associated with dividend payments, our model reflects the actual state of the varying economy. Specifically, the novelty of our discount factor model stems from the fact it ties the discount factor to the randomly varying future interest rates of the underlying economy. As a result, it provides more accurate predictions of the expected present value of the future dividend payments.

Remark 2.14. Other discrete time dividend models were considered in [5]-[9]. A somewhat similar to our model was a model introduced by Nie *et al.* [10] which considered premiums and claims modulated by two-state Markov chain under constant discount factor. By contrast, our accounting of dividends is modulated by a random discount factor associated with the varying interest rates in the underlying economy. Moreover, the premiums in [10] were limited to {0,1}. For more information about discrete risk models with dividends we refer the reader to [10] which contains additional references on the subject.

3. Recursive Formulas for *V*(*u*,*b*)

Since the barrier b is fixed, in order to simplify notation we set

$$W(u)\equiv V(u,b) \text{ with its generating function } \tilde{W}(s)=\sum_{k=0}^{\infty}s^kW(k) \quad (3.1)$$

Generating Function Lemma 3.2. Let $W(k)$ be a function for the total expected discounted dividends prior to ruin. Then $W(k)$ satisfies the following equation:

$$\begin{aligned}
&\left(s^N-s^Nvf(0)\tilde{g}(s)-s^{N-1}vf(1)\tilde{g}(s)-s^{N-2}vf(2)\tilde{g}(s)-vf(N)\tilde{g}(s)\right)\tilde{W}(s)\\
&=-vs^{N-1}\left[f(1)g(0)W(0)+f(2)\sum_{n\le 1}g(n)W(1-n)+\cdots+f(N)\sum_{n\le N-1}g(n)W(N-1-n)\right]\\
&\quad -vs^{N-2}\left[f(2)g(0)W(0)+f(3)\sum_{n\le 1}g(n)W(1-n)+\cdots+f(N)\sum_{n\le N-2}g(n)W(N-2-n)\right]\\
&\quad \cdots\\
&\quad -vs^{N-i}\left[f(i)g(0)W(0)+f(i+1)\sum_{n\le 1}g(n)W(1-n)+\cdots+f(N)\sum_{n\le N-i}g(n)W(N-i-n)\right]\\
&\quad \cdots\\
&\quad -vf(N)g(0)W(0)
\end{aligned} \quad (3.3)$$

Proof. We begin by derivation of difference equations for $W(u)$. Based on stationary and independent increments of $U(t)$, the surplus process starts afresh when conditioning on $U(t)$ at time $t=1$. As a result we have the following set of expression for $W(k)$, $k\in\mathbb{N}$.

$$\begin{aligned}
W(0)=vf(0)g(0)W(0)+vf(1)\sum_{n\le 1}g(n)W(1-n)\\
+vf(2)\sum_{n\le 2}g(n)W(2-n)+\cdots+vf(N)\sum_{n\le N}g(n)W(N-n)
\end{aligned}$$

$$W(1) = vf(0)\sum_{n\le 1} g(n)W(1-n) + vf(1)\sum_{n\le 2} g(n)W(2-n) + vf(2)\sum_{n\le 3} g(n)W(3-n) + \cdots + vf(N)\sum_{n\le N+1} g(n)W(N+1-n)$$

$$W(2) = vf(0)\sum_{n\le 2} g(n)W(2-n) + vf(1)\sum_{n\le 3} g(n)W(3-n) + vf(2)\sum_{n\le 4} g(n)W(4-n) + \cdots + vf(N)\sum_{n\le N+2} g(n)W(N+2-n)$$

$$\vdots$$

$$W(k) = vf(0)\sum_{n\le k} g(n)W(k-n) + vf(1)\sum_{n\le k+1} g(n)W(k+1-n) + vf(2)\sum_{n\le k+2} g(n)W(k+2-n) + \cdots + vf(N)\sum_{n\le N+k} g(n)W(N+k-n) \tag{3.4}$$

Aside from (3.4), due to dividend payment strategy, the following equations hold above barrier *b*.

$$W(k) = k - b + W(b) \text{ for } k = b+1, b+2, \cdots. \tag{3.5}$$

Multiplying both sides of (3.4) by $s^N s^k$ for $k = 0,1,2,\cdots$ we have

$$s^N \cdot s^0 W(0) = s^N vf(0)s^0 g(0)W(0) + s^{N-1}vf(1)s\sum_{n\le 1} g(n)W(1-n) + s^{N-2}vf(2)s^2\sum_{n\le 2} g(n)W(2-n) + \cdots + vf(N)s^N\sum_{n\le N} g(n)W(N-n)$$

$$s^N \cdot sW(1) = s^N vf(0)s\sum_{n\le 1} g(n)W(1-n) + s^{N-1}vf(1)s^2\sum_{n\le 2} g(n)W(2-n) + s^{N-2}vf(2)s^3\sum_{n\le 3} g(n)W(3-n) + \cdots + vf(N)s^{N+1}\sum_{n\le N+1} g(n)W(N+1-n)$$

$$s^N \cdot s^2 W(2) = s^N vf(0)s^2\sum_{n\le 2} g(n)W(2-n) + s^{N-1}vf(1)s^3\sum_{n\le 3} g(n)W(3-n) + s^{N-2}vf(2)s^4\sum_{n\le 4} g(n)W(4-n) + \cdots + vf(N)s^{N+2}\sum_{n\le N+2} g(n)W(N+2-n)$$

$$\vdots$$

$$s^N \cdot s^k W(k) = s^N vf(0)s^k\sum_{n\le k} g(n)W(k-n) + s^{N-1}vf(1)s^{k+1}\sum_{n\le k+1} g(n)W(k+1-n) + s^{N-2}vf(2)s^{k+2}\sum_{n\le k+2} g(n)W(k+2-n) + \cdots + vf(N)s^{N+k}\sum_{n\le N+k} g(n)W(N+k-n)$$

$$\vdots$$

Summing up for k from 0 to ∞ we have

$$\begin{aligned} s^N\tilde{W}(s) &= s^N vf(0)\tilde{g}(s)\tilde{W}(s) + s^{N-1}vf(1)\left(\tilde{g}(s)\tilde{W}(s) - g(0)W(0)\right) \\ &\quad + s^{N-2}vf(2)\left(\tilde{g}(s)\tilde{W}(s) - g(0)W(0) - s\sum_{n\le 1} g(n)W(1-n)\right) + \cdots \\ &\quad + vf(N)\left(\tilde{g}(s)\tilde{W}(s) - g(0)W(0) - s\sum_{n\le 1} g(n)W(1-n) - \cdots \right. \\ &\quad \left. - s^{N-1}\sum_{n\le N-1} g(n)W(N-1-n)\right) \end{aligned} \tag{3.6}$$

By grouping and moving terms involving $\tilde{W}(s)$ to the left-hand side we obtain

$$\begin{aligned} &\left(s^N - s^N vf(0)\tilde{g}(s) - s^{N-1}vf(1)\tilde{g}(s) - s^{N-2}vf(2)\tilde{g}(s) - vf(N)\tilde{g}(s)\right)\tilde{W}(s) \\ &= -vs^{N-1}\left[f(1)g(0)W(0) + f(2)\sum_{n\le 1} g(n)W(1-n) + \cdots + f(N)\sum_{n\le N-1} g(n)W(N-1-n)\right] \\ &\quad - vs^{N-2}\left[f(2)g(0)W(0) + f(3)\sum_{n\le 1} g(n)W(1-n) + \cdots + f(N)\sum_{n\le N-2} g(n)W(N-2-n)\right] \\ &\quad \cdots \\ &\quad - vs^{N-i}\left[f(i)g(0)W(0) + f(i+1)\sum_{n\le 1} g(n)W(1-n) + \cdots + f(N)\sum_{n\le N-i} g(n)W(N-i-n)\right] \\ &\quad \cdots \\ &\quad - vf(N)g(0)W(0) \end{aligned}$$

which completes the proof.

Corollary 3.7. By comparing coefficient of the left-hand side and the right-hand side in (3.3), we will derive the recursive formulas for $W(k)$ depending on distributions of premiums, claims and the boundary conditions (3.5) at barrier b.

4. Main Result

In what follows, we will determine the solutions for $W(k)$ in the case of $N = 3$ based on formula (3.3) of Generating Function Lemma.

Theorem 4.1. Given premiums $X_t \in \{0,1,2,3\}$ and barriers $b \in \{1,2,3,4,5\}$, the total expected discounted dividend $V(k,b) \equiv W(k)$ has the form

$$W(k) = c_1W_1(k) + c_2W_2(k) + c_3W_3(k),\ \ k = 1,2,\cdots,8 \tag{4.2}$$

based on the boundary conditions:

$$\begin{aligned} W(b+1) - W(b) &= 1 \\ W(b+2) - W(b) &= 2 \\ W(b+3) - W(b) &= 3 \end{aligned} \tag{4.3}$$

with c_1, c_2, c_3 satisfying the boundary conditions:

$$\begin{aligned} &c_1\left(W_1(b+1)-W_1(b)\right)+c_2\left(W_2(b+1)-W_2(b)\right)+c_3\left(W_3(b+1)-W_3(b)\right)=1 \\ &c_1\left(W_1(b+2)-W_1(b)\right)+c_2\left(W_2(b+2)-W_2(b)\right)+c_3\left(W_3(b+2)-W_3(b)\right)=2 \\ &c_1\left(W_1(b+3)-W_1(b)\right)+c_2\left(W_2(b+3)-W_2(b)\right)+c_3\left(W_3(b+3)-W_3(b)\right)=3 \end{aligned} \tag{4.4}$$

where $W_1(.),W_2(.),W_3(.)$ are particular solutions satisfying the initial conditions as follows:

$$\begin{array}{lll} W_1(0)=1 & W_1(1)=1 & W_1(2)=1 \\ W_2(0)=0 & W_2(1)=1 & W_2(2)=1 \\ W_3(0)=0 & W_3(1)=0 & W_3(2)=1 \end{array} \tag{4.5}$$

Proof. By applying formula (3.3) of Generating Function Lemma for $N = 3$ we have

$$\begin{aligned} &\left(s^3-s^3vf(0)\tilde{g}(s)-s^2vf(1)\tilde{g}(s)-svf(2)\tilde{g}(s)-vf(3)\tilde{g}(s)\right)\tilde{W}(s) \\ &=-vs^2\left[f(1)g(0)W(0)-f(2)\left(g(0)W(1)+g(1)W(0)\right)\right. \\ &\quad\left.-f(3)\left(g(0)W(2)+g(1)W(1)+g(2)W(0)\right)\right] \\ &\quad-vs\left[f(2)g(0)W(0)+f(3)\left(g(0)W(1)+g(1)W(0)\right)\right] \\ &\quad-vf(3)g(0)W(0) \end{aligned}$$

Expanding $\tilde{g}(s)$ and $\tilde{W}(s)$ up to 8 terms in s gives

$$\begin{aligned} &\left(s^3-s^3vf(0)\left[g(0)+sg(1)+s^2g(2)+s^3g(3)+s^4g(4)+s^5g(5)+\cdots\right]\right. \\ &-s^2vf(1)\left[g(0)+sg(1)+s^2g(2)+s^3g(3)+s^4g(4)+s^5g(5)+s^6g(6)+\cdots\right] \\ &-svf(2)\left[g(0)+sg(1)+s^2g(2)+s^3g(3)+s^4g(4)+s^5g(5)+s^6g(6)\right. \\ &\left.+s^7g(7)+\cdots\right]-vf(3)\left[g(0)+sg(1)+s^2g(2)+s^3g(3)+s^4g(4)\right. \\ &\left.\left.+s^5g(5)+s^6g(6)+s^7g(7)+s^8g(8)+\cdots\right]\right)\times\left(W(0)+sW(1)+s^2W(2)\right. \\ &\left.+s^3W(3)+s^4W(4)+s^5W(5)+s^6W(6)+s^7W(7)+s^8W(8)\right) \\ &=-vs^2\left[f(1)g(0)W(0)-f(2)\left(g(0)W(1)+g(1)W(0)\right)\right. \\ &\left.-f(3)\left(g(0)W(2)+g(1)W(1)+g(2)W(0)\right)\right]-vs\left[f(2)g(0)W(0)\right. \\ &\left.+f(3)\left(g(0)W(1)+g(1)W(0)\right)\right]-vf(3)g(0)W(0) \end{aligned}$$

resulting in

$$\begin{aligned} &\left(s^3\left(1-vf(0)g(0)\right)-s^4vf(0)g(1)-s^5vf(0)g(2)-s^6vf(0)g(3)\right. \\ &-s^7vf(0)g(4)-s^8vf(0)g(5)-\cdots-s^2vf(1)g(0)-s^3vf(1)g(1) \\ &-s^4vf(1)g(2)-s^5vf(1)g(3)-s^6vf(1)g(4)-s^7vf(1)g(5) \\ &-s^8vf(1)g(6)-\cdots-svf(2)g(0)-s^2vf(2)g(1)-s^3vf(2)g(2) \\ &-s^4vf(2)g(3)-s^5vf(2)g(4)-s^6vf(2)g(5)-s^7vf(2)g(6) \\ &-s^8vf(2)g(7)-\cdots-vf(3)g(0)-svf(3)g(1)-s^2vf(3)g(2) \\ &-s^3vf(3)g(3)-s^4vf(3)g(4)-s^5vf(3)g(5)-s^6vf(3)g(6) \\ &\left.-s^7vf(3)g(7)-s^8vf(3)g(8)-\cdots\right)\times\left(W(0)+sW(1)+s^2W(2)\right. \\ &\left.+s^3W(3)+s^4W(4)+s^5W(5)+s^6W(6)+s^7W(7)+s^8W(8)\right) \\ &=-vs^2\left[f(1)g(0)W(0)-f(2)\left(g(0)W(1)+g(1)W(0)\right)\right. \\ &\left.-f(3)\left(g(0)W(2)+g(1)W(1)+g(2)W(0)\right)\right]-vs\left[f(2)g(0)W(0)\right. \\ &\left.+f(3)\left(g(0)W(1)+g(1)W(0)\right)\right]-vf(3)g(0)W(0) \end{aligned}$$

Notice that the constant term $-vf(3)g(0)W(0)$, present on both sides, cancels out.

Comparing coefficients of s on both sides gives

$$\begin{aligned}&-svf(2)g(0)W(0)-svf(3)g(0)W(1)-svf(3)g(1)W(0)\\&=-vs\big(f(2)g(0)W(0)+f(3)g(0)W(1)+f(3)g(1)W(0)\big)\end{aligned}$$

and therefore they cancel out.

Similarly, comparing coefficients of s^2 gives

$$\begin{aligned}&-s^2vf(1)g(0)W(0)-s^2vf(2)g(0)W(1)-s^2vf(2)g(1)W(0)\\&-s^2vf(3)g(0)W(2)-s^2vf(3)g(1)W(1)-s^2vf(3)g(2)W(0)\\&=-vs^2\big(f(1)g(0)W(0)+f(2)g(0)W(1)+f(2)g(1)W(0)\\&\quad+f(3)g(0)W(2)+f(3)g(1)W(1)+f(3)g(2)W(0)\big)\end{aligned}$$

and they cancel out.

Comparing coefficients of s^3 gives

$$\begin{aligned}&s^3\big(1-vf(0)g(0)\big)W(0)-s^3vf(1)g(0)W(1)-s^3vf(1)g(1)W(0)\\&-s^3vf(2)g(0)W(2)-s^3vf(2)g(1)W(1)-s^3vf(2)g(2)W(0)\\&-s^3vf(3)g(0)W(3)-s^3vf(3)g(1)W(2)-s^3vf(3)g(2)W(1)\\&-s^3vf(3)g(3)W(0)=0s^3\end{aligned}$$

$$\begin{aligned}&s^3\Big(\big(1-vf(0)g(0)\big)W(0)-vf(1)g(0)W(1)-vf(1)g(1)W(0)\\&-vf(2)g(0)W(2)-vf(2)g(1)W(1)-vf(2)g(2)W(0)\\&-vf(3)g(0)W(3)-vf(3)g(1)W(2)-vf(3)g(2)W(1)\\&-vf(3)g(3)W(0)\Big)=0s^3\end{aligned}$$

and we have

$$\begin{aligned}&\big(1-vf(0)g(0)\big)W(0)-vf(1)g(0)W(1)-vf(1)g(1)W(0)\\&-vf(2)g(0)W(2)-vf(2)g(1)W(1)-vf(2)g(2)W(0)\\&-vf(3)g(0)W(3)-vf(3)g(1)W(2)-vf(3)g(2)W(1)\\&-vf(3)g(3)W(0)=0\end{aligned}$$

Solving for $W(3)$ gives

$$\begin{aligned}W(3)=\frac{1}{vf(3)g(0)}\Big[&\big(1-vf(0)g(0)\big)W(0)-vf(1)g(0)W(1)\\&-vf(1)g(1)W(0)-vf(2)g(0)W(2)-vf(2)g(1)W(1)\\&-vf(2)g(2)W(0)-vf(3)g(1)W(2)-vf(3)g(2)W(1)\\&-vf(3)g(3)W(0)\Big]\end{aligned}$$

which can be written as

$$\begin{aligned}W(3)=\frac{1}{vf(3)g(0)}\Big[&W(0)-vf(0)g(0)W(0)-vf(1)\sum\nolimits_{n\le1}g(n)W(1-n)\\&-vf(2)\sum\nolimits_{n\le2}g(n)W(2-n)-vf(3)\sum\nolimits_{1\le n\le3}g(n)W(3-n)\Big]\end{aligned}\tag{4.6}$$

Comparing coefficient of s^4 gives

$$\begin{aligned}&s^4\left(1-vf(0)g(0)\right)W(1)-s^4vf(0)g(1)W(0)-s^4vf(1)g(0)W(2)\\&-s^4vf(1)g(1)W(1)-s^4vf(1)g(2)W(0)-s^4vf(2)g(0)W(3)\\&-s^4vf(2)g(1)W(2)-s^4vf(2)g(2)W(1)-s^4vf(2)g(3)W(0)\\&-s^4vf(3)g(0)W(4)-s^4vf(3)g(1)W(3)-s^4vf(3)g(2)W(2)\\&-s^4vf(3)g(3)W(1)-s^4vf(3)g(4)W(0)=0s^4\end{aligned}$$

$$\begin{aligned}&s^4\Big[\left(1-vf(0)g(0)\right)W(1)-vf(0)g(1)W(0)-vf(1)g(0)W(2)\\&-vf(1)g(1)W(1)-vf(1)g(2)W(0)-vf(2)g(0)W(3)\\&-vf(2)g(1)W(2)-vf(2)g(2)W(1)-vf(2)g(3)W(0)\\&-vf(3)g(0)W(4)-vf(3)g(1)W(3)-vf(3)g(2)W(2)\\&-vf(3)g(3)W(1)-vf(3)g(4)W(0)\Big]=0s^4\end{aligned}$$

and we have

$$\begin{aligned}&\left(1-vf(0)g(0)\right)W(1)-vf(0)g(1)W(0)-vf(1)g(0)W(2)\\&-vf(1)g(1)W(1)-vf(1)g(2)W(0)-vf(2)g(0)W(3)\\&-vf(2)g(1)W(2)-vf(2)g(2)W(1)-vf(2)g(3)W(0)\\&-vf(3)g(0)W(4)-vf(3)g(1)W(3)-vf(3)g(2)W(2)\\&-vf(3)g(3)W(1)-vf(3)g(4)W(0)=0\end{aligned}$$

Solving for $W(4)$ gives

$$\begin{aligned}W(4)=\frac{1}{vf(3)g(0)}&\Big[\left(1-vf(0)g(0)\right)W(1)-vf(0)g(1)W(0)\\&-vf(1)g(0)W(2)-vf(1)g(1)W(1)-vf(1)g(2)W(0)\\&-vf(2)g(0)W(3)-vf(2)g(1)W(2)-vf(2)g(2)W(1)\\&-vf(2)g(3)W(0)-vf(3)g(1)W(3)-vf(3)g(2)W(2)\\&-vf(3)g(3)W(1)-vf(3)g(4)W(0)\Big]\end{aligned}$$

which can be written as

$$\begin{aligned}W(4)=\frac{1}{vf(3)g(0)}&\Big[W(1)-vf(0)\textstyle\sum_{n\le 1}g(n)W(1-n)\\&-vf(1)\textstyle\sum_{n\le 2}g(n)W(2-n)-vf(2)\sum_{n\le 3}g(n)W(3-n)\\&-vf(3)\textstyle\sum_{1\le n\le 4}g(n)W(4-n)\Big]\end{aligned}\tag{4.7}$$

Comparing coefficients of s^5 gives

$$\begin{aligned}&s^5\left(1-vf(0)g(0)\right)W(2)-s^5vf(0)g(1)W(1)-s^5vf(0)g(2)W(0)\\&-s^5vf(1)g(0)W(3)-s^5vf(1)g(1)W(2)-s^5vf(1)g(2)W(1)\\&-s^5vf(1)g(3)W(0)-s^5vf(2)g(0)W(4)-s^5vf(2)g(1)W(3)\\&-s^5vf(2)g(2)W(2)-s^5vf(2)g(3)W(1)-s^5vf(2)g(4)W(0)\\&-s^5vf(3)g(0)W(5)-s^5vf(3)g(1)W(4)-s^5vf(3)g(2)W(3)\\&-s^5vf(3)g(3)W(2)-s^5vf(3)g(4)W(1)-s^5vf(3)g(5)W(0)=0s^5\end{aligned}$$

$$s^{5}\Big[\big(1-vf(0)g(0)\big)W(2)-vf(0)g(1)W(1)-vf(0)g(2)W(0)$$
$$-vf(1)g(0)W(3)-vf(1)g(1)W(2)-vf(1)g(2)W(1)$$
$$-vf(1)g(3)W(0)-vf(2)g(0)W(4)-vf(2)g(1)W(3)$$
$$-vf(2)g(2)W(2)-vf(2)g(3)W(1)-vf(2)g(4)W(0)$$
$$-vf(3)g(0)W(5)-vf(3)g(1)W(4)-vf(3)g(2)W(3)$$
$$-vf(3)g(3)W(2)-vf(3)g(4)W(1)-vf(3)g(5)W(0)\Big]=0s^{5}$$

and we have

$$\big(1-vf(0)g(0)\big)W(2)-vf(0)g(1)W(1)-vf(0)g(2)W(0)$$
$$-vf(1)g(0)W(3)-vf(1)g(1)W(2)-vf(1)g(2)W(1)$$
$$-vf(1)g(3)W(0)-vf(2)g(0)W(4)-vf(2)g(1)W(3)$$
$$-vf(2)g(2)W(2)-vf(2)g(3)W(1)-vf(2)g(4)W(0)$$
$$-vf(3)g(0)W(5)-vf(3)g(1)W(4)-vf(3)g(2)W(3)$$
$$-vf(3)g(3)W(2)-vf(3)g(4)W(1)-vf(3)g(5)W(0)=0$$

Solving for $W(5)$ gives

$$W(5)=\frac{1}{vf(3)g(0)}\Big[\big(1-vf(0)g(0)\big)W(2)-vf(0)g(1)W(1)$$
$$-vf(0)g(2)W(0)-vf(1)g(0)W(3)-vf(1)g(1)W(2)$$
$$-vf(1)g(2)W(1)-vf(1)g(3)W(0)-vf(2)g(0)W(4)$$
$$-vf(2)g(1)W(3)-vf(2)g(2)W(2)-vf(2)g(3)W(1)$$
$$-vf(2)g(4)W(0)-vf(3)g(1)W(4)-vf(3)g(2)W(3)$$
$$-vf(3)g(3)W(2)-vf(3)g(4)W(1)-vf(3)g(5)W(0)\Big]$$

which can be written as

$$W(5)=\frac{1}{vf(3)g(0)}\Big[W(2)-vf(0)\sum\nolimits_{n\le 2}g(n)W(2-n)$$
$$-vf(1)\sum\nolimits_{n\le 3}g(n)W(3-n)-vf(2)\sum\nolimits_{n\le 4}g(n)W(4-n) \tag{4.8}$$
$$-f(3)\sum\nolimits_{1\le n\le 5}g(n)W(5-n)\Big]$$

Comparing coefficients of s^6 gives

$$s^{6}\big(1-vf(0)g(0)\big)W(3)-s^{6}vf(0)g(1)W(2)-s^{6}vf(0)g(2)W(1)$$
$$-s^{6}vf(0)g(3)W(0)-s^{6}vf(1)g(0)W(4)-s^{6}vf(1)g(1)W(3)$$
$$-s^{6}vf(1)g(2)W(2)-s^{6}vf(1)g(3)W(1)-s^{6}vf(1)g(4)W(0)$$
$$-s^{6}vf(2)g(0)W(5)-s^{6}vf(2)g(1)W(4)-s^{6}vf(2)g(2)W(3)$$
$$-s^{6}vf(2)g(3)W(2)-s^{6}vf(2)g(4)W(1)-s^{6}vf(2)g(5)W(0)$$
$$-s^{6}Vf(3)g(0)W(6)-s^{6}vf(3)g(1)W(5)-s^{6}vf(3)g(2)W(4)$$
$$-s^{6}vf(3)g(3)W(3)-s^{6}vf(3)g(4)W(2)-s^{6}vf(3)g(5)W(1)$$
$$-s^{6}vf(3)g(6)W(0)=0s^{6}$$

$$s^6\Big[\big(1-vf(0)g(0)\big)W(3)-vf(0)g(1)W(2)-vf(0)g(2)W(1)$$
$$-vf(0)g(3)W(0)-vf(1)g(0)W(4)-vf(1)g(1)W(3)-vf(1)g(2)W(2)$$
$$-vf(1)g(3)W(1)-vf(1)g(4)W(0)-vf(2)g(0)W(5)-vf(2)g(1)W(4)$$
$$-vf(2)g(2)W(3)-vf(2)g(3)W(2)-vf(2)g(4)W(1)-vf(2)g(5)W(0)$$
$$-Vf(3)g(0)W(6)-vf(3)g(1)W(5)-vf(3)g(2)W(4)-vf(3)g(3)W(3)$$
$$-vf(3)g(4)W(2)-vf(3)g(5)W(1)-vf(3)g(6)W(0)\Big]=0s^6$$

and we have

$$\big(1-vf(0)g(0)\big)W(3)-vf(0)g(1)W(2)-vf(0)g(2)W(1)$$
$$-vf(0)g(3)W(0)-vf(1)g(0)W(4)-vf(1)g(1)W(3)-vf(1)g(2)W(2)$$
$$-vf(1)g(3)W(1)-vf(1)g(4)W(0)-vf(2)g(0)W(5)-vf(2)g(1)W(4)$$
$$-vf(2)g(2)W(3)-vf(2)g(3)W(2)-vf(2)g(4)W(1)-vf(2)g(5)W(0)$$
$$-Vf(3)g(0)W(6)-vf(3)g(1)W(5)-vf(3)g(2)W(4)-vf(3)g(3)W(3)$$
$$-vf(3)g(4)W(2)-vf(3)g(5)W(1)-vf(3)g(6)W(0)=0$$

Solving for $W(6)$ gives

$$W(6)=\frac{1}{vf(3)g(0)}\Big[\big(1-vf(0)g(0)\big)W(3)-vf(0)g(1)W(2)$$
$$-vf(0)g(2)W(1)-vf(0)g(3)W(0)-vf(1)g(0)W(4)$$
$$-vf(1)g(1)W(3)-vf(1)g(2)W(2)-vf(1)g(3)W(1)$$
$$-vf(1)g(4)W(0)-vf(2)g(0)W(5)-vf(2)g(1)W(4)$$
$$-vf(2)g(2)W(3)-vf(2)g(3)W(2)-vf(2)g(4)W(1)$$
$$-vf(2)g(5)W(0)-vf(3)g(1)W(5)-vf(3)g(2)W(4)$$
$$-vf(3)g(3)W(3)-vf(3)g(4)W(2)-vf(3)g(5)W(1)$$
$$-vf(3)g(6)W(0)\Big]$$

which can be written as

$$W(6)=\frac{1}{vf(3)g(0)}\Big[W(3)-vf(0)\sum_{n\le 3}g(n)W(3-n)$$
$$-vf(1)\sum_{n\le 4}g(n)W(4-n)-vf(2)\sum_{n\le 5}g(n)W(5-n) \quad (4.9)$$
$$-vf(3)\sum_{1\le n\le 6}g(n)W(6-n)$$

Comparing coefficients of s^7 gives

$$s^7\big(1-vf(0)g(0)\big)W(4)-s^7vf(0)g(1)W(3)-s^7vf(0)g(2)W(2)$$
$$-s^7vf(0)g(3)W(1)-s^7vf(0)g(4)W(0)-s^7vf(1)g(0)W(5)$$
$$-s^7vf(1)g(1)W(4)-s^7vf(1)g(2)W(3)-s^7vf(1)g(3)W(2)$$
$$-s^7vf(1)g(4)W(1)-s^7vf(1)g(5)W(0)-s^7vf(2)g(0)W(6)$$
$$-s^7vf(2)g(1)W(5)-s^7vf(2)g(2)W(4)-s^7vf(2)g(3)W(3)$$
$$-s^7vf(2)g(4)W(2)-s^7vf(2)g(5)W(1)-s^7vf(2)g(6)W(0)$$
$$-s^7vf(3)g(0)W(7)-s^7vf(3)g(1)W(6)-s^7vf(3)g(2)W(5)$$
$$-s^7vf(3)g(3)W(4)-s^7vf(3)g(4)W(3)-s^7vf(3)g(5)W(2)$$
$$-s^7vf(3)g(6)W(1)-s^7vf(3)g(7)W(0)=0s^7$$

$$s^7\left[(1-vf(0)g(0))W(4)-vf(0)g(1)W(3)-vf(0)g(2)W(2)\right.$$
$$-vf(0)g(3)W(1)-vf(0)g(4)W(0)-vf(1)g(0)W(5)-vf(1)g(1)W(4)$$
$$-vf(1)g(2)W(3)-vf(1)g(3)W(2)-vf(1)g(4)W(1)-vf(1)g(5)W(0)$$
$$-vf(2)g(0)W(6)-vf(2)g(1)W(5)-vf(2)g(2)W(4)-vf(2)g(3)W(3)$$
$$-vf(2)g(4)W(2)-vf(2)g(5)W(1)-vf(2)g(6)W(0)-vf(3)g(0)W(7)$$
$$-vf(3)g(1)W(6)-vf(3)g(2)W(5)-vf(3)g(3)W(4)-vf(3)g(4)W(3)$$
$$\left.-vf(3)g(5)W(2)-vf(3)g(6)W(1)-vf(3)g(7)W(0)\right]=0s^7$$

and we have

$$(1-vf(0)g(0))W(4)-vf(0)g(1)W(3)-vf(0)g(2)W(2)$$
$$-vf(0)g(3)W(1)-vf(0)g(4)W(0)-vf(1)g(0)W(5)-vf(1)g(1)W(4)$$
$$-vf(1)g(2)W(3)-vf(1)g(3)W(2)-vf(1)g(4)W(1)-vf(1)g(5)W(0)$$
$$-vf(2)g(0)W(6)-vf(2)g(1)W(5)-vf(2)g(2)W(4)-vf(2)g(3)W(3)$$
$$-vf(2)g(4)W(2)-vf(2)g(5)W(1)-vf(2)g(6)W(0)-vf(3)g(0)W(7)$$
$$-vf(3)g(1)W(6)-vf(3)g(2)W(5)-vf(3)g(3)W(4)-vf(3)g(4)W(3)$$
$$-vf(3)g(5)W(2)-vf(3)g(6)W(1)-vf(3)g(7)W(0)=0$$

Solving for $W(7)$ gives

$$\begin{aligned}W(7)=\frac{1}{vf(3)g(0)}\Big[&(1-vf(0)g(0))W(4)-vf(0)g(1)W(3)\\&-vf(0)g(2)W(2)-vf(0)g(3)W(1)-vf(0)g(4)W(0)\\&-vf(1)g(0)W(5)-vf(1)g(1)W(4)-vf(1)g(2)W(3)\\&-vf(1)g(3)W(2)-vf(1)g(4)W(1)-vf(1)g(5)W(0)\\&-vf(2)g(0)W(6)-vf(2)g(1)W(5)-vf(2)g(2)W(4)\\&-vf(2)g(3)W(3)-vf(2)g(4)W(2)-vf(2)g(5)W(1)\\&-vf(2)g(6)W(0)-vf(3)g(1)W(6)-vf(3)g(2)W(5)\\&-vf(3)g(3)W(4)-vf(3)g(4)W(3)-vf(3)g(5)W(2)\\&-vf(3)g(6)W(1)-vf(3)g(7)W(0)\Big]\end{aligned}$$

which can be written as

$$\begin{aligned}W(7)=\frac{1}{vf(3)g(0)}\Big[&W(4)-vf(0)\sum\nolimits_{n\le 4}g(n)W(4-n)\\&-vf(1)\sum\nolimits_{n\le 5}g(n)W(5-n)-vf(2)\sum\nolimits_{n\le 6}g(n)W(6-n)\\&-vf(3)\sum\nolimits_{1\le n\le 7}g(n)W(7-n)\Big]\end{aligned}\tag{4.10}$$

Comparing coefficients of s^8 gives

$$s^8(1-vf(0)g(0))W(5)-s^8vf(0)g(1)W(4)-s^8vf(0)g(2)W(3)$$
$$-s^8vf(0)g(3)W(2)-s^8vf(0)g(4)W(1)-s^8vf(0)g(5)W(0)$$
$$-s^8vf(1)g(0)W(6)-s^8vf(1)g(1)W(5)-s^8vf(1)g(2)W(4)$$
$$-s^8vf(1)g(3)W(3)-s^8vf(1)g(4)W(2)-s^8vf(1)g(5)W(1)$$
$$-s^8vf(1)g(6)W(0)-s^8vf(2)g(0)W(7)-s^8vf(2)g(1)W(6)$$

$$-s^8vf(2)g(2)W(5)-s^8vf(2)g(3)W(4)-s^8vf(2)g(4)W(3)$$
$$-s^8vf(2)g(5)W(2)-s^8vf(2)g(6)W(1)-s^8vf(2)g(7)W(0)$$
$$-s^8vf(3)g(0)W(8)-s^8vf(3)g(1)W(7)-s^8vf(3)g(2)W(6)$$
$$-s^8vf(3)g(3)W(5)-s^8vf(3)g(4)W(4)-s^8vf(3)g(5)W(3)$$
$$-s^8vf(3)g(6)W(2)-s^8vf(3)g(7)W(1)-s^8vf(3)g(8)W(0)=0s^8$$

$$\begin{aligned}&s^8\Big[(1-vf(0)g(0))W(5)-vf(0)g(1)W(4)-vf(0)g(2)W(3)\\&-vf(0)g(3)W(2)-vf(0)g(4)W(1)-vf(0)g(5)W(0)-vf(1)g(0)W(6)\\&-vf(1)g(1)W(5)-vf(1)g(2)W(4)-vf(1)g(3)W(3)-vf(1)g(4)W(2)\\&-vf(1)g(5)W(1)-vf(1)g(6)W(0)-vf(2)g(0)W(7)-vf(2)g(1)W(6)\\&-vf(2)g(2)W(5)-vf(2)g(3)W(4)-vf(2)g(4)W(3)-vf(2)g(5)W(2)\\&-vf(2)g(6)W(1)-vf(2)g(7)W(0)-vf(3)g(0)W(8)-vf(3)g(1)W(7)\\&-vf(3)g(2)W(6)-vf(3)g(3)W(5)-vf(3)g(4)W(4)-vf(3)g(5)W(3)\\&-vf(3)g(6)W(2)-vf(3)g(7)W(1)-vf(3)g(8)W(0)\Big]=0s^8\end{aligned}$$

and we have

$$\begin{aligned}&(1-vf(0)g(0))W(5)-vf(0)g(1)W(4)-vf(0)g(2)W(3)-vf(0)g(3)W(2)\\&-vf(0)g(4)W(1)-vf(0)g(5)W(0)-vf(1)g(0)W(6)-vf(1)g(1)W(5)\\&-vf(1)g(2)W(4)-vf(1)g(3)W(3)-vf(1)g(4)W(2)-vf(1)g(5)W(1)\\&-vf(1)g(6)W(0)-vf(2)g(0)W(7)-vf(2)g(1)W(6)-vf(2)g(2)W(5)\\&-vf(2)g(3)W(4)-vf(2)g(4)W(3)-vf(2)g(5)W(2)-vf(2)g(6)W(1)\\&-vf(2)g(7)W(0)-vf(3)g(0)W(8)-vf(3)g(1)W(7)-vf(3)g(2)W(6)\\&-vf(3)g(3)W(5)-vf(3)g(4)W(4)-vf(3)g(5)W(3)-vf(3)g(6)W(2)\\&-vf(3)g(7)W(1)-vf(3)g(8)W(0)=0\end{aligned}$$

Solving for $W(8)$ gives

$$\begin{aligned}W(8)=\frac{1}{vf(3)g(0)}\Big[&(1-vf(0)g(0))W(5)-vf(0)g(1)W(4)\\&-vf(0)g(2)W(3)-vf(0)g(3)W(2)-vf(0)g(4)W(1)\\&-vf(0)g(5)W(0)-vf(1)g(0)W(6)-vf(1)g(1)W(5)\\&-vf(1)g(2)W(4)-vf(1)g(3)W(3)-vf(1)g(4)W(2)\\&-vf(1)g(5)W(1)-vf(1)g(6)W(0)-vf(2)g(0)W(7)\\&-vf(2)g(1)W(6)-vf(2)g(2)W(5)-vf(2)g(3)W(4)\\&-vf(2)g(4)W(3)-vf(2)g(5)W(2)-vf(2)g(6)W(1)\\&-vf(2)g(7)W(0)-vf(3)g(1)W(7)-vf(3)g(2)W(6)\\&-vf(3)g(3)W(5)-vf(3)g(4)W(4)-vf(3)g(5)W(3)\\&-vf(3)g(6)W(2)-vf(3)g(7)W(1)-vf(3)g(8)W(0)\Big]\end{aligned}$$

which can be written as

$$\begin{aligned}W(8)=\frac{1}{vf(3)g(0)}\Big[&W(5)-vf(0)\textstyle\sum_{n\le5}g(n)W(5-n)\\&-vf(1)\textstyle\sum_{n\le6}g(n)W(6-n)-vf(2)\sum_{n\le7}g(n)W(7-n)\\&-vf(3)\textstyle\sum_{1\le n\le8}g(n)W(8-n)\Big]\end{aligned}\tag{4.11}$$

Recursive formulas $W(3)$ through $W(8)$ given by (4.6) - (4.11) show their dependence on $W(0)$, $W(1)$ and $W(2)$. Therefore, given three boundary conditions (4.3), it leads to a unique solution. The unique solution is represented as a linear combination of the particular solutions prescribed by (4.5).

Theorem 4.12. Given premiums $X_t \in \{0,1,2\}$ and barriers $b \in \{1,2,3,4,5\}$, the total expected discounted dividend $V(k,b) \equiv W(k)$ has the form

$$W(k) = c_1 W_1(k) + c_2 W_2(k),\ \ k = 1,2,\cdots,7 \tag{4.13}$$

based on the boundary conditions:

$$\begin{aligned} W(b+1) - W(b) &= 1 \\ W(b+2) - W(b) &= 2 \end{aligned} \tag{4.14}$$

with c_1, c_2 satisfying the boundary conditions:

$$\begin{aligned} c_1\left(W_1(b+1) - W_1(b)\right) + c_2\left(W_2(b+1) - W_2(b)\right) &= 1 \\ c_1\left(W_1(b+2) - W_1(b)\right) + c_2\left(W_2(b+2) - W_2(b)\right) &= 2 \end{aligned} \tag{4.15}$$

where $W_1(.), W_2(.)$ are particular solutions satisfying the initial conditions as follows:

$$\begin{aligned} W_1(0) &= 1 \qquad W_1(1) = 1 \\ W_2(0) &= 0 \qquad W_2(1) = 1 \end{aligned} \tag{4.16}$$

Proof. Similarly to the proof of Theorem 4.1, applying (3.3) with $N = 2$, we determine recursive formulas as follows:

$$\begin{aligned} W(2) = \frac{1}{vf(2)g(0)}\Big[& W(0) - vf(0)g(0)W(0) \\ & - vf(1)\textstyle\sum_{n\le 1} g(n)W(1-n) - vf(2)\sum_{1\le n\le 2} g(n)W(2-n)\Big] \end{aligned} \tag{4.17}$$

$$\begin{aligned} W(3) = \frac{1}{vf(2)g(0)}\Big[& W(1) - vf(0)\textstyle\sum_{n\le 1} g(n)W(1-n) \\ & - vf(1)\textstyle\sum_{n\le 2} g(n)W(2-n) - vf(2)\sum_{1\le n\le 3} g(n)W(3-n)\Big] \end{aligned} \tag{4.18}$$

$$\begin{aligned} W(4) = \frac{1}{vf(2)g(0)}\Big[& W(2) - vf(0)\textstyle\sum_{n\le 2} g(n)W(2-n) \\ & - vf(1)\textstyle\sum_{n\le 3} g(n)W(3-n) - vf(2)\sum_{1\le n\le 4} g(n)W(4-n)\Big] \end{aligned} \tag{4.19}$$

$$\begin{aligned} W(5) = \frac{1}{vf(2)g(0)}\Big[& W(3) - vf(0)\textstyle\sum_{n\le 3} g(n)W(3-n) \\ & - vf(1)\textstyle\sum_{n\le 4} g(n)W(4-n) - vf(2)\sum_{1\le n\le 5} g(n)W(5-n)\Big] \end{aligned} \tag{4.20}$$

$$\begin{aligned} W(6) = \frac{1}{vf(2)g(0)}\Big[& W(4) - vf(0)\textstyle\sum_{n\le 4} g(n)W(4-n) \\ & - vf(1)\textstyle\sum_{n\le 5} g(n)W(5-n) - vf(2)\sum_{1\le n\le 6} g(n)W(6-n)\Big] \end{aligned} \tag{4.21}$$

$$\begin{aligned} W(7) = \frac{1}{vf(2)g(0)}\Big[& W(5) - vf(0)\textstyle\sum_{n\le 5} g(n)W(5-n) \\ & - vf(1)\textstyle\sum_{n\le 6} g(n)W(6-n) - vf(2)\sum_{1\le n\le 7} g(n)W(7-n)\Big] \end{aligned} \tag{4.22}$$

Recursive formulas $W(2)$ through $W(7)$ given by (4.17) - (4.22) show their dependence on $W(0)$ and $W(1)$. Therefore, given two boundary conditions (4.14), it leads to a unique solution. The unique solution is represented as a linear combination of the particular solutions prescribed by (4.16).

Theorem 4.23. Given premiums $X_t \in \{0,1\}$ and barriers $b \in \{1,2,3,4,5\}$, the total expected discounted dividend $V(k,b) \equiv W(k)$ has the form

$$W(k) = \frac{1}{W_1(b+1) - W_1(b)} W_1(k), \quad k = 0,1,\cdots,b+1$$

with boundary condition

$$W(b+1) - W(b) = 1 \tag{4.24}$$

where $W_1(k)$ is a particular solution with $W_1(0) = 1$.

Proof. Similarly to the proof of Theorem 4.1, applying (3.3) with $N = 1$, we determine recursive formulas as follows

$$W(1) = \frac{1}{vf(1)g(0)}\left[W(0) - vf(0)g(0)W(0) - vf(1)g(1)W(0)\right] \tag{4.25}$$

$$\begin{aligned} W(2) = \frac{1}{vf(1)g(0)}\Big[&W(1) - vf(0)\textstyle\sum_{n\le 1} g(n)W(1-n) \\ &- vf(1)\textstyle\sum_{1\le n\le 2} g(n)W(2-n)\Big] \end{aligned} \tag{4.26}$$

$$\begin{aligned} W(3) = \frac{1}{vf(1)g(0)}\Big[&W(2) - vf(0)\textstyle\sum_{n\le 2} g(n)W(2-n) \\ &- vf(1)\textstyle\sum_{1\le n\le 3} g(n)W(3-n)\Big] \end{aligned} \tag{4.27}$$

$$\begin{aligned} W(4) = \frac{1}{vf(1)g(0)}\Big[&W(3) - vf(0)\textstyle\sum_{n\le 3} g(n)W(3-n) \\ &- vf(1)\textstyle\sum_{1\le n\le 4} g(n)W(4-n)\Big] \end{aligned} \tag{4.28}$$

$$\begin{aligned} W(5) = \frac{1}{vf(1)g(0)}\Big[&W(4) - vf(0)\textstyle\sum_{n\le 4} g(n)W(4-n) \\ &- vf(1)\textstyle\sum_{1\le n\le 5} g(n)W(5-n)\Big] \end{aligned} \tag{4.29}$$

$$\begin{aligned} W(6) = \frac{1}{vf(1)g(0)}\Big[&W(5) - vf(0)\textstyle\sum_{n\le 5} g(n)W(5-n) \\ &- vf(1)\textstyle\sum_{1\le n\le 6} g(n)W(6-n)\Big] \end{aligned} \tag{4.30}$$

Recursive formulas $W(1)$ through $W(6)$ given by (4.25) - (4.30) show their dependence on $W(0)$. Therefore, given the boundary condition it leads to a unique solution $W(k)$ given by (4.24).

5. Illustrating Examples

We will present several concrete scenarios in the order of theorems above. All examples below utilize the discount factor $v = 0.94$ given by formula (2.7) which corresponds to an interest rate $r = 0.065$ with volatility $\sigma = 0.056$.

Example 5.1

Table 1 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N = 3$, $p = 2/5$, $n = 0, 1, 2, 3$. Claim Y is binomial $b(N,p)$ with $g(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N=8$, $p=\frac{1}{8}$, $n=0,1,2,3,4,5,6,7,8$.

The loading factor $\theta = EX - EY = 6/5 - 1 = 0.20$.

Table 1. $V(u,b)$.

	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$
$u = 0$	1.6929	1.5980	1.4399	1.2564	1.0753
$u = 1$	2.6613	2.5160	2.2666	1.9776	1.6925
$u = 2$	3.6613	3.4406	3.1062	2.7095	2.3188
$u = 3$	4.6613	4.4406	3.9727	3.4739	2.9722
$u = 4$	5.6613	5.4406	4.9727	4.3018	3.6902
$u = 5$	6.6613	6.4406	5.9727	5.3018	4.4956

Example 5.2

Table 2 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N = 3$, $p = 2/5$, $n = 0, 1, 2, 3$. Claim Y is geometric ($p = 1/2$) with $g(n)=q^n p$ for $n=0,1,2,\cdots$.

The loading factor $\theta = EX - EY = 6/5 - 1 = 0.20$.

Table 2. $V(u,b)$.

	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$
$u = 0$	2.1393	1.9888	1.8046	1.6079	1.4137
$u = 1$	3.0824	2.8720	2.6052	2.3210	2.0408
$u = 2$	4.0824	3.7736	3.4315	3.0561	2.6871
$u = 3$	5.0824	4.7736	4.3014	3.8408	3.3759
$u = 4$	6.0824	5.7736	5.3014	4.6875	4.1310
$u = 5$	7.0824	6.7736	6.3014	5.6875	4.9613

Example 5.3

Table 3 gives values for $V(u,b)$ where premium X is conditional geometric with $f(n)=P(X=n \mid X\le 3)$, $0\le n\le 3$. Namely, $f(0)=\frac{8}{15}$, $f(1)=\frac{4}{15}$, $f(2)=\frac{2}{15}$ and $f(3)=\frac{1}{15}$, for X geometric ($p = 1/2$). Claim Y is geometric ($p = 2/3$) with $g(n)=q^n p$ for $n=0,1,2,\cdots$.

The loading factor $\theta = EX - EY = 11/15 - 1/2 = 0.2333$.

Table 3. $V(u,b)$.

	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
$u=0$	2.0162	1.9108	1.7296	1.5224	1.3164
$u=1$	3.0311	2.8765	2.6059	2.2918	1.9822
$u=2$	4.0311	3.7993	3.4491	3.0357	2.6237
$u=3$	5.0311	4.7993	4.3105	3.8032	3.2895
$u=4$	6.0311	5.7993	5.3105	4.6266	4.0122
$u=5$	7.0311	6.7993	6.3105	5.6266	4.8133

Example 5.4

Table 4 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N=2$, $p=\frac{3}{5}$, $n=0,1,2$. Claim Y is binomial $b(N,p)$ with $g(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N=7$, $p=\frac{1}{7}$, $n=0,1,2,3,4,5,6,7$.

The loading factor $\theta=EX-EY=6/5-1=0.20$.

Table 4. $V(u,b)$.

	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
$u=0$	1.5327	1.5048	1.3474	1.1650	0.9864
$u=1$	2.3900	2.3478	2.1004	1.8166	1.5379
$u=2$	3.3900	3.3253	2.9818	2.5767	2.1820
$u=3$	4.3900	4.3253	3.8513	3.3366	2.8232
$u=4$	5.3900	5.3253	4.8513	4.1681	3.5364
$u=5$	6.3900	6.3253	5.8513	5.1681	4.3455

Example 5.5

Table 5 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for $N=2$, $p=\frac{3}{5}$, $n=0,1,2$. Claim Y is geometric ($p=1/2$) with $g(n)=q^n p$ for $n=0,1,2,\cdots$.

The loading factor $\theta=EX-EY=6/5-1=0.20$.

Table 5. $V(u,b)$.

	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
$u=0$	1.9099	1.8914	1.7117	1.5180	1.3276
$u=1$	2.7135	2.6882	2.4298	2.1559	1.8852
$u=2$	3.7135	3.6756	3.3320	2.9529	2.5833
$u=3$	4.7135	4.6756	4.2075	3.7400	3.2682
$u=4$	5.7135	5.6756	5.2075	4.5924	4.0254
$u=5$	6.7135	6.6756	6.2075	5.5924	4.8617

Example 5.6

Table 6 gives values for $V(u,b)$ where premium X is conditional geometric with $f(n)=P(X=n\,|\,X\le 2)$, $0\le n\le 2$ Namely, $f(0)=\frac{4}{7}$, $f(1)=\frac{2}{7}$ and $f(2)=\frac{1}{7}$ for X geometric (p = 1/2). Claim Y is geometric (p = 2/3) with $g(n)=q^n p$ for $n=0,1,2,\cdots$.

The loading factor $\theta=EX-EY=4/7-1/2=0.0714$.

Table 6. $V(u,b)$.

	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
$u=0$	1.2077	0.9978	0.7968	0.6240	0.4838
$u=1$	2.0160	1.6779	1.3363	1.0475	0.8119
$u=2$	3.0160	2.4632	1.9754	1.5449	1.1984
$u=3$	4.0160	3.4632	2.7229	2.1439	1.6593
$u=4$	5.0160	4.4632	3.7229	2.8734	2.2388
$u=5$	6.0160	5.4632	4.7229	3.8734	2.9581

Example 5.7

Table 7 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for N = 1, p = 3/5, n = 0, 1. Claim Y is binomial $b(N,p)$ with $g(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for N = 6, p = 1/12, n = 0, 1, 2, 3, 4, 5, 6.

The loading factor $\theta=EX-EY=3/5-1/2=0.10$.

Table 7. $V(u,b)$.

	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$
$u=0$	1.1217	0.9030	0.6917	0.5177	0.3835
$u=1$	1.9925	1.6040	1.2288	0.9196	0.6812
$u=2$	2.9925	2.4091	1.8455	1.3811	1.0230
$u=3$	3.9925	3.4091	2.6115	1.9545	1.4477
$u=4$	4.9925	4.4091	3.6115	2.7029	2.0020
$u=5$	5.9925	5.4091	4.6115	3.7029	2.7428

Example 5.8

Table 8 gives the values for $V(u,b)$ where premium X is binomial $b(N,p)$ with $f(n)=\binom{N}{n}p^n(1-p)^{N-n}$ for N = 1, p = 3/5, n = 0, 1. Claim Y is geometric (p = 2/3) with $g(n)=q^n p$ for $n=0,1,2,\cdots$.

The loading factor $\theta=EX-EY=3/5-1/2=0.10$.

Table 8. $V(u,b)$.

	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$
$u = 0$	1.3136	1.0725	0.8451	0.6529	0.4991
$u = 1$	2.1800	1.7798	1.4024	1.0835	0.8284
$u = 2$	3.1800	2.5963	2.0458	1.5805	1.2083
$u = 3$	4.1800	3.5963	2.8337	2.1893	1.6738
$u = 4$	5.1800	4.5963	3.8337	2.9618	2.2644
$u = 5$	6.1800	5.5963	4.8337	3.9618	3.0289

Remark. 5.9. We note that across Tables 1-8 the following patterns emerge. For each fixed barrier b the total expected discounted dividends prior to ruin increases with increasing initial capital u. This confirms our intuition, since the closer one starts from the barrier the higher the chance of crossing the barrier-resulting in the dividend payout. On the other hand, for each fixed initial capital u the total expected discounted dividends prior to ruin decreases with increasing barrier b. Again, this agrees with our intuition, since the farther away one starts from the barrier the smaller the chance of crossing the barrier-resulting in the dividend payout.

6. Conclusion

We have introduced a discrete time risk model featuring random premiums subject to dividend payments upon crossing a fixed barrier. The novelty of our approach stems from incorporating random discounts rates that are directly correlated to the varying interest rates of the underlying economy. By employing generating functions, we derive effective recursive formulas for the total expected discounted dividends prior to ruin. Our algorithm for effective computing of the total expected discounted dividends can directly benefit a decision-making process in the real-world risk management scenario, particularly in the insurance industry. Namely, given a prediction of the future interest rates, historical distributions of premium received, and claims paid, the algorithm provides a tool for the analysis of dividends payment in reference to a pre-selected barrier that triggers dividend payout when surplus process crosses the barrier. Depending on the company objective, a decision will be based on calculations outcome for various choices of initial capital u and barrier b. The robustness of the model in reference to initial capital, barrier height, and random interest rate parameters (which determine the discount factor) along with a generalization where an insurer has an option to redirect a portion of dividend payouts into an investment portfolio, will be addressed in our forthcoming paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Korzeniowski, A. (2022) Discrete Time Risk Model Financed by Random Premiums. *Journal of Mathematical Finance*, **12**, 126-137. https://doi.org/10.4236/jmf.2022.121008

[2] Korzeniowski, A. (2023) Ruin Probability for Risk Model with Random Premiums. *Journal of Mathematical Finance*, **13**, 171-179. https://doi.org/10.4236/jmf.2023.132011

[3] Dickson, D.C.M. (2017) Insurance Risk and Ruin. 2nd Edition, Cambridge University Press.

[4] Mikosch, T. (1998) Elementary Stochastic Calculus, with Finance in View. World Scientific Publishing Co. Pte. Ltd. https://doi.org/10.1142/9789812386335

[5] Yu, W. (2013) Randomized Dividends in a Discrete Insurance Risk Model with Stochastic Premium Income. *Mathematical Problems in Engineering*, **2013**, Article ID: 579534. https://doi.org/10.1155/2013/579534

[6] Deng, Y., Liu, J., Huang, Y., Li, M. and Zhou, J. (2017) On a Discrete Interaction Risk Model with Delayed Claims and Stochastic Incomes under Random Discount Rates. *Communications in Statistics—Theory and Methods*, **47**, 5867-5883. https://doi.org/10.1080/03610926.2017.1406518

[7] Yu, W. (2013) Randomized Dividends in a Discrete Insurance Risk Model with Stochastic Premium Income. *Mathematical Problems in Engineering*, **2013**, Article ID: 579534. https://doi.org/10.1155/2013/579534

[8] Bao, Z. and Liu, H. (2012) The Compound Binomial Risk Model with Delayed Claims and Random Income. *Mathematical and Computer Modelling*, **55**, 1315-1323. https://doi.org/10.1016/j.mcm.2011.10.009

[9] Liu, C. and Zhang, Z. (2015) On a Discrete Risk Model with Delayed Claims and a Randomized Dividend Strategy. *Advances in Difference Equations*, **2015**, Article No. 284. https://doi.org/10.1186/s13662-015-0614-4

[10] Nie, C., Chen, M., Liu, H. and Yu, W. (2020) On a Discrete Markov-Modulated Risk Model with Random Premium Income and Delayed Claims. *Mathematical Problems in Engineering*, **2020**, Article ID: 3042543. https://doi.org/10.1155/2020/3042543